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Chaotic Dynamics in Contests

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Abstract

Under a myopic best-reply dynamic, efforts in repeated contests may exhibit chaotic behavior. This may help explain, e.g., why experimental data often show nonconvergence to one-shot equilibrium efforts. *Journal of Economic Literature* Classification Numbers: C61, C72, D72, D74, D83. Keywords: contests, dynamics, chaos.

1 Introduction

Contests, in which players expend effort or resources in order to increase their probability of winning a prize, have long been a popular subject of economic experiments (see, e.g., Dechenaux *et al* [2]). In general, in experiments where participants play the same game repeatedly, very often play quickly converges to a stable pattern, more often than not resembling the play of some static equilibrium. This is typically not the case in contests, however.

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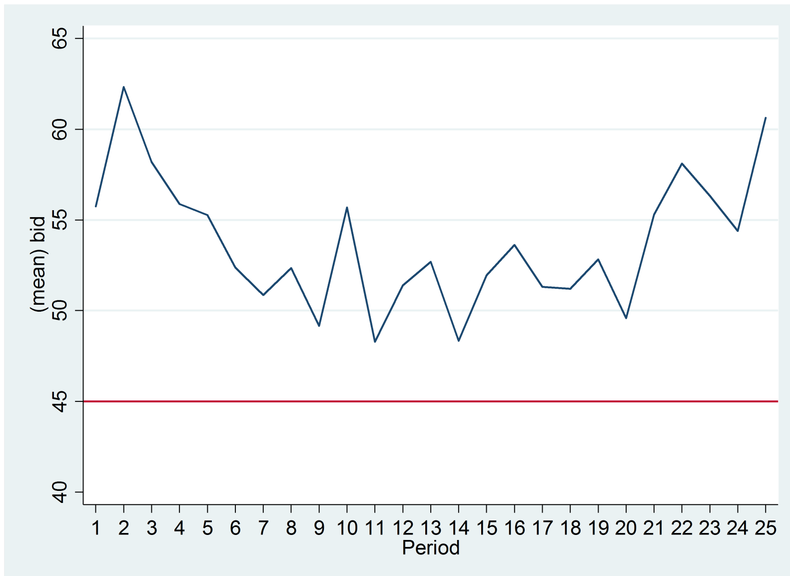

Figure 1: Average bids per period in a contest experiment (from Chowdhury *et al* [1])

Figure 1 shows the average bids in 25 periods of 44 participants in a contest experiment reported by Chowdhury *et al* [1]. The subjects played the game repeatedly in groups of 2, under a Tullock lottery contest success function where a contestant's probability of winning was the ratio of their bid to the total of bids. The line at 45 represents the predicted equilibrium individual bid, on the assumption that players are risk neutral and value only monetary rewards.

The picture is representative of behavior in experimental contests. Not only do participants on average overbid relative to the prediction; the bids also wander around wildly without converging to anything. In this paper, we shall consider the idea that what we see is, at least partly, the result of chaotic dynamics inherent in the nonlinear contest model when a simple learning rule is used by players.

When experimental game results are analyzed, frequently the implications of the fact that participants are actually playing a repeated game are overlooked. Repetition of the same game allows, potentially, for learning to take place. Learning processes are more likely to easily converge in some types of games than in others. Because of the highly nonlinear nature of popular contest models, even very simple learning rules may fail to converge, and instead lead to chaotic behavior. Here we shall consider a simple best-reply dynamics, where players are assumed to play their best response to the opponent's action in the previous period.

This paper owes a great deal to Puu [4], who studies chaotic dynamics in an equivalent model of Cournot duopoly with isoelastic demand.

2 Chaos: a very brief introduction

Chaos, in the mathematical sense, is present when, loosely speaking, the time series generated by a deterministic dynamical system look random. Very simple nonlinear systems may exhibit such behavior, as is illustrated by the logistic map

$$y_t = r\,y_{t-1}(1 - y_{t-1}),$$

where $y \in [0,1]$ and $r > 0$ is a parameter.

Figure 2 shows the logistic map with $r = 2.95$. The parameter r controls the height of the parabola. For low values of r, iteration of the map from arbitrary starting points will ultimately lead to convergence to the unique interior stationary point, as illustrated in Figure 3.

With $r = 4$, however, as in Figure 4, a typical time series will look like in Figure 5, an unpredictable jumping up and down all over the unit interval that is best described as a series of draws from a beta distribution.

To better understand how the transition to chaos happens, one may consider a so-called *bifurcation diagram*. Figure 6 shows, for every value of r, the corresponding set of points that are reached after $1,000$ iterations. Hence we see that at r slightly above 3, a stable cycle of period 2 appears, which as r is increased further splits into an orbit of period 4, and so on until chaos is reached.

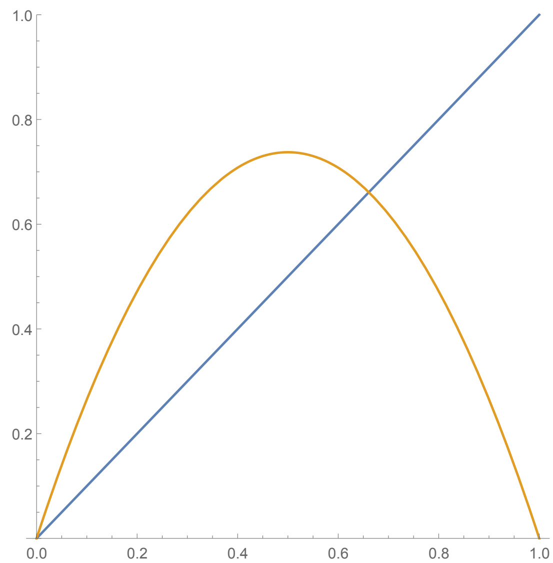

Figure 2: Logistic map with $r = 2.95$

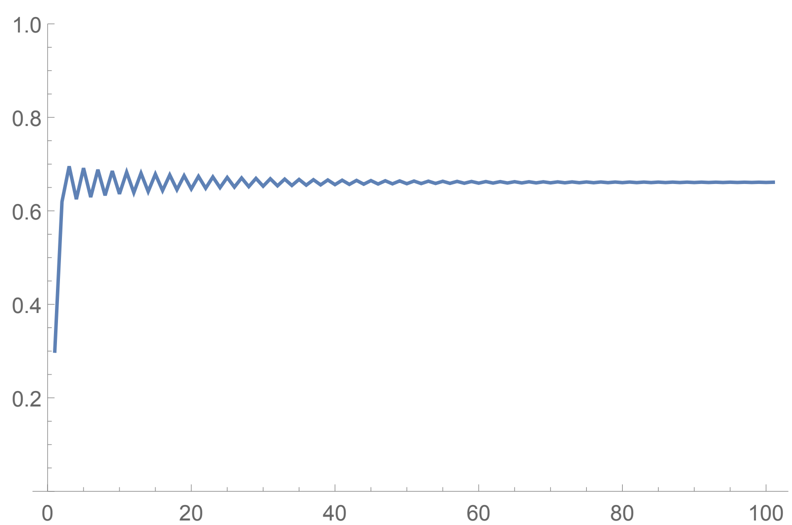

Figure 3: Iterating the logistic map with $r = 2.95$

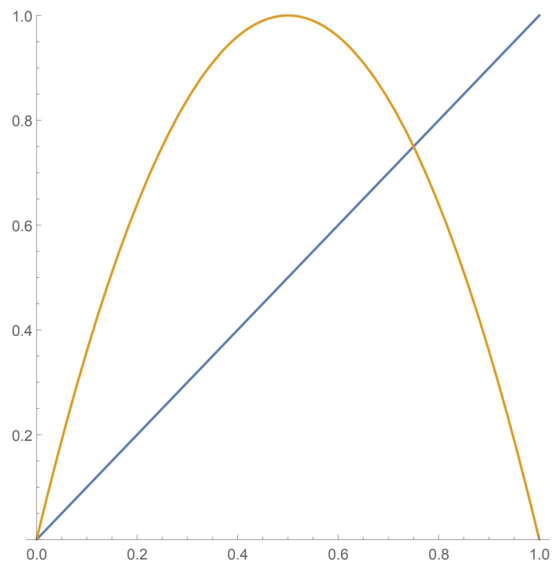

Figure 4: Logistic map with $r = 4$

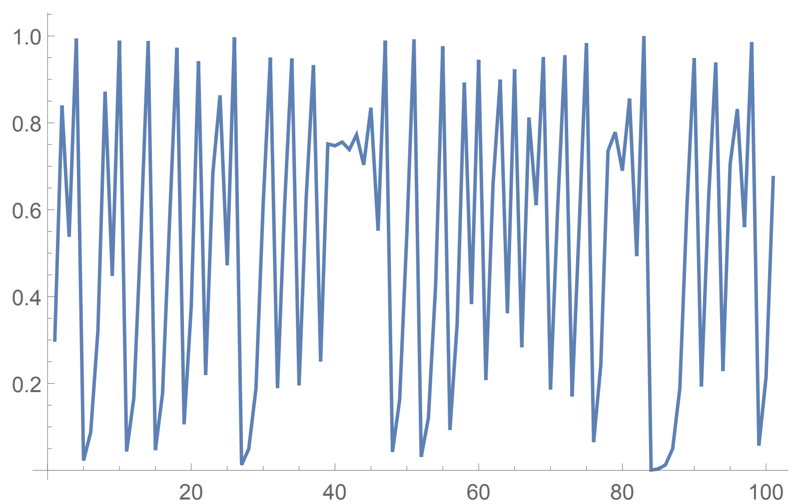

Figure 5: Iterating the logistic map with $r = 4$

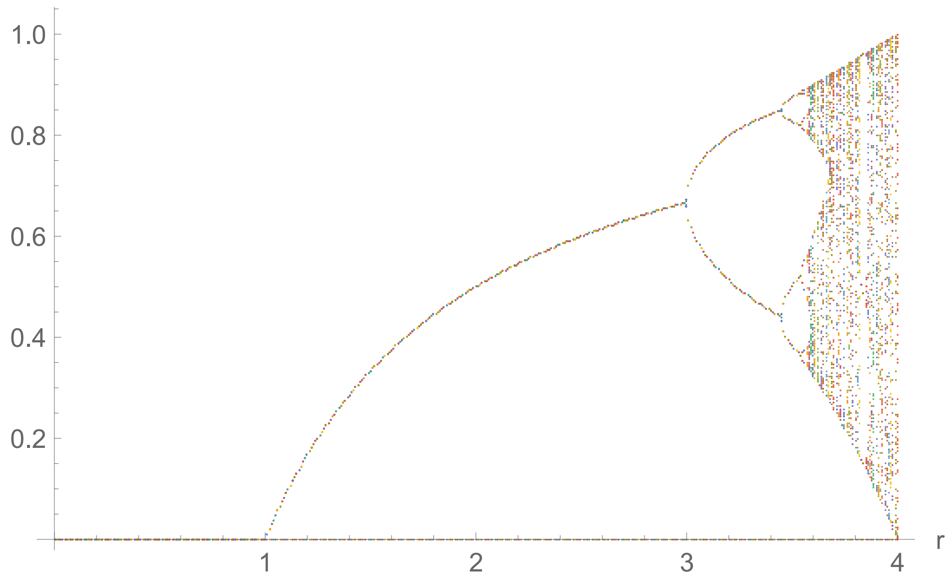

Figure 6: Bifurcation diagram for the logistic map

For more on chaos, see, e.g., Devaney [3]. In the following, we shall see how chaos may arise in iterated two-player contests.

3 Chaos in contests

3.1 Static equilibrium

Consider a 2-player contest where if players expend x_1 and x_2, respectively, Player i's probability of winning is

$$p_i(x_1, x_2) = \begin{cases} x_i/(x_1 + x_2) & \text{if } x_1 + x_2 > 0 \\ 1/2 & \text{otherwise.} \end{cases}$$

This popular *contest success function* was introduced by Tullock [5]. Let Player 1's valuation be $v > 0$, and Player 2's 1. Supposing the players are both risk neutral, and that effort x_i comes at unit cost, Player 1's objective function is then

$$u_1 = p_1(x_1, x_2)v - x_1,$$

and that of Player 2 is

$$u_2 = p_2(x_1, x_2) - x_2.$$

Consider first equilibrium in the one-shot game when both players are rational and the structure of the game is common knowledge among them. We note that there cannot be an equilibrium in which nobody expends anything. For suppose your opponent expends nothing. Then you win with probability one-half if you also expend nothing. If you expend an arbitrarily small positive amount, however, you win with probability one. Hence there must be some such small amount—in particular, less than one half of the your valuation of winning—that would be profitable to expend. There is therefore no equilibrium in which nobody expends anything.

We are therefore justified in considering just the ratio part of the success function. Player 1's first order condition for a best reply, given Player 2's effort, is then

$$\frac{\partial u_1}{\partial x_1} = \frac{x_2}{(x_1 + x_2)^2} v - 1 = 0,$$

and that of Player 2 is

$$\frac{\partial u_2}{\partial x_1} = \frac{x_1}{(x_1 + x_2)^2} - 1 = 0.$$

Hence the equilibrium efforts will be

$$x_1 = \frac{v^2}{(1+v)^2}$$

and

$$x_2 = \frac{v}{(1+v)^2}.$$

Aggregate effort in equilibrium is therefore $v/(1+v)$.

In the experimental literature on contests (see, e.g., the survey in Dechenaux *et al* [2]), it is often implicitly or explicitly assumed that all contestants have the same valuation of winning, typically the monetary value of the contested prize, and the one-shot equilibrium efforts computed above would be called the "game-theoretical prediction" of how play would proceed. Normally, however, experimental subjects play the contest more than one time. There seems to be no *prima facie* reason to assume that all players have the same valuation, and that they would play the one-shot equilibrium in a repeated game, and as we shall see in the following, relaxing these assumptions can lead to very different predictions—ones that are more in line with actual observed behavior.

3.2 Best-reply dynamics

Now suppose the contest is played repeatedly, and

- the players choose their initial efforts arbitrarily,
- and then each play their best response to the opponent's effort in the previous period.

We then get the dynamical system

$$x_1^{t+1} = \sqrt{v x_2^t} - x_2^t$$

and

$$x_2^{t+1} = \sqrt{x_1^t} - x_1^t,$$

where the superscripts on efforts now denote time periods.

Note that the one-shot equilibrium efforts from before constitute a stationary state of this system. That is, if the players were ever to reach the one-shot equilibrium, they would thereafter stay there forever. And, indeed, that is what happens for values of v sufficiently close to 1. If valuations are the same, as in the example in Figure 7, convergence to the one-shot equilibrium efforts is rapid.

Similarly, with $v = 3.415$, as in Figure 8, convergence takes a little longer but is still fast. In both these cases, convergence happens because the stationary state is globally *asymptotically stable*, meaning that from any positive starting point the system will eventually end up at the stationary state.

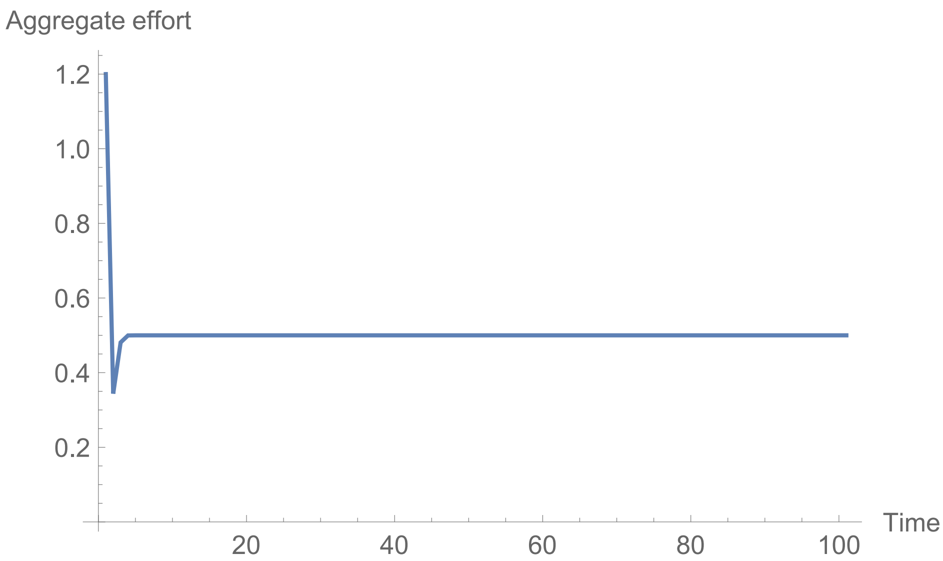

Figure 7: Aggregate effort time series with $\nu = 1$

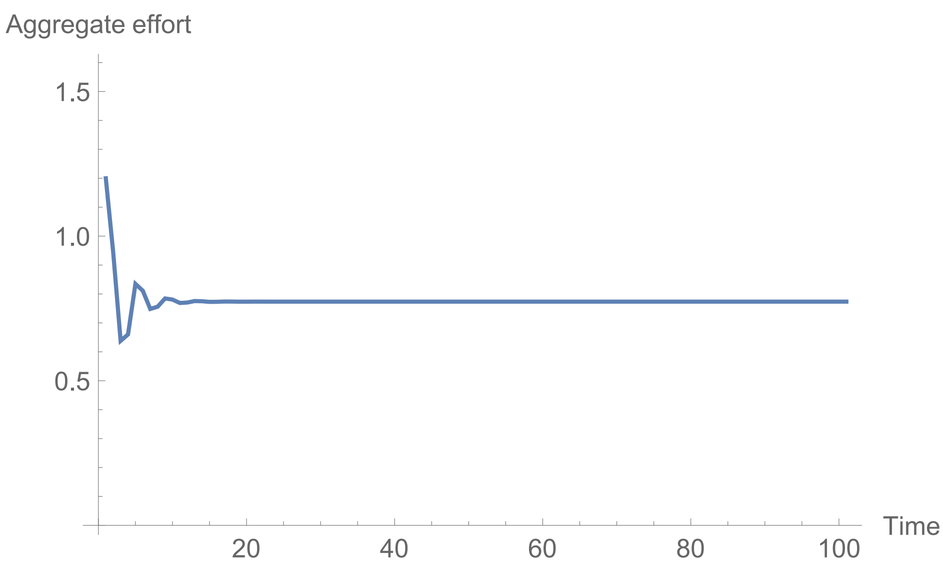

Figure 8: Aggregate effort time series with $\nu = 3.415$

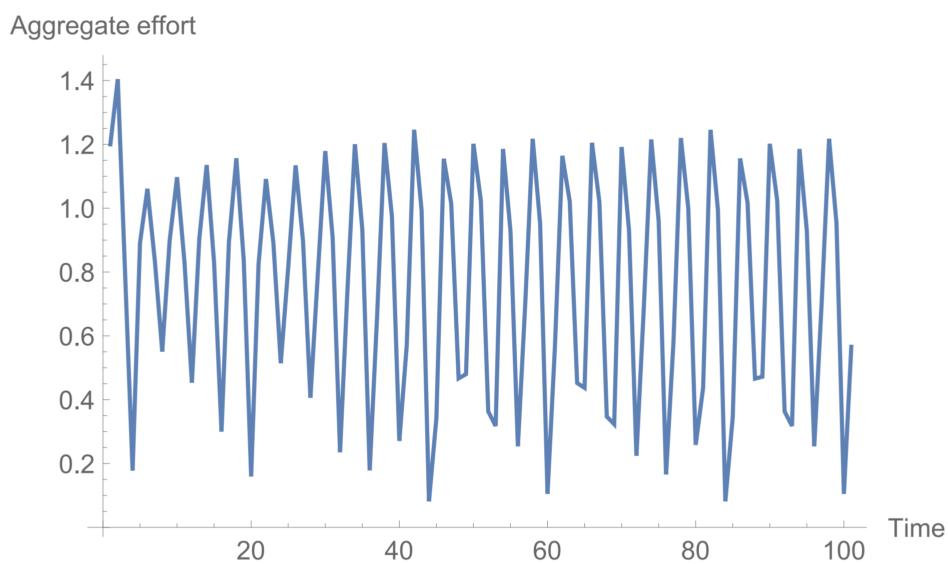

Figure 9: Aggregate effort time series with $v = 6.2484$

In Figure 9, however, with $v = 6.2484$, convergence never happens, and we instead observe chaotic behavior. So somewhere in between, the stationary state becomes unstable.

To see where the transition to instability occurs, we consider the Jacobian of the system, which is

$$J := \begin{pmatrix} 0 & (v/(2\sqrt{v x_2^t})) - 1 \\ (1/(2\sqrt{x_1^t})) - 1 & 0 \end{pmatrix}.$$

Evaluated at the stationary state, the Jacobian is

$$J^\star := \begin{pmatrix} 0 & (v/(2\sqrt{v^2/(1+v)^2})) - 1 \\ (v/(2\sqrt{v^2/(1+v)^2})) - 1 & 0 \end{pmatrix}.$$

The stationary state is asymptotically stable if the absolute values of both eigenvalues of $J^\star$ are strictly less than 1, or, equivalently, if we have that

$$2 > 1 + \det J^\star > |\operatorname{tr} J^\star|,$$

i.e., if we have that

$$3 - 2\sqrt{2} < v < 3 + 2\sqrt{2}.$$

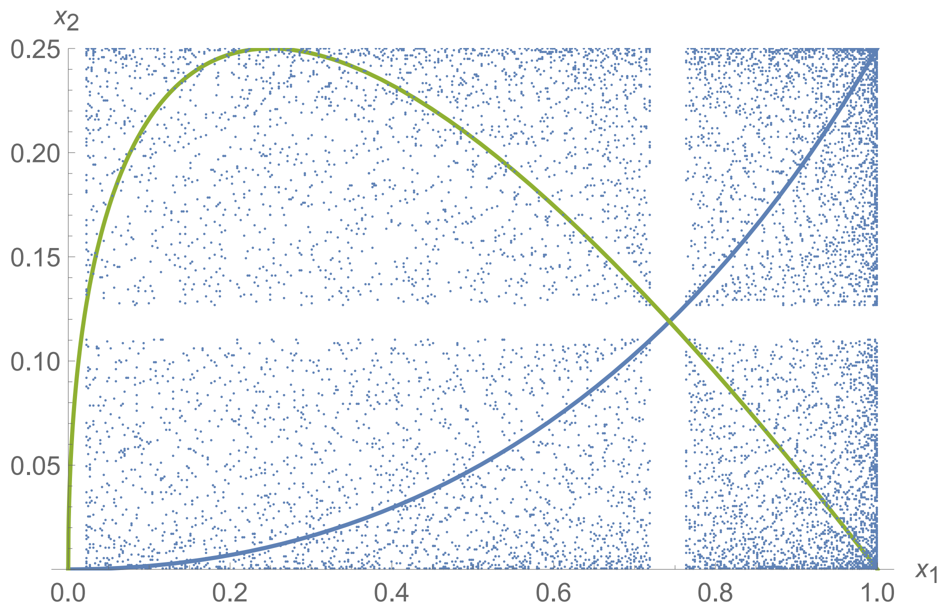

Figure 10: Points visited in (x_1, x_2)-space with $v = 6.2484$

Figure 10, which shows the phase space, was computed in the following fashion. Starting from an arbitrary initial pair of efforts, the system was iterated 10,000 times with $v = 6.2484$. The first 1,000 points were dropped; the rest were plotted. The two best reply curves are also in the figure.

The system is here moving around in a complicated attractor that occupies all of the phase space except for a cross-like shape with the one-shot equilibrium located at the intersection of its beams. In essence, the players will here play anything but the one-shot equilibrium.

The bifurcation diagram in Figure 11 was realized as follows. For every value of v, in steps of .01, 1,000 iterates of aggregate effort were generated, the first 100 dropped, and the rest plotted on the vertical axis.

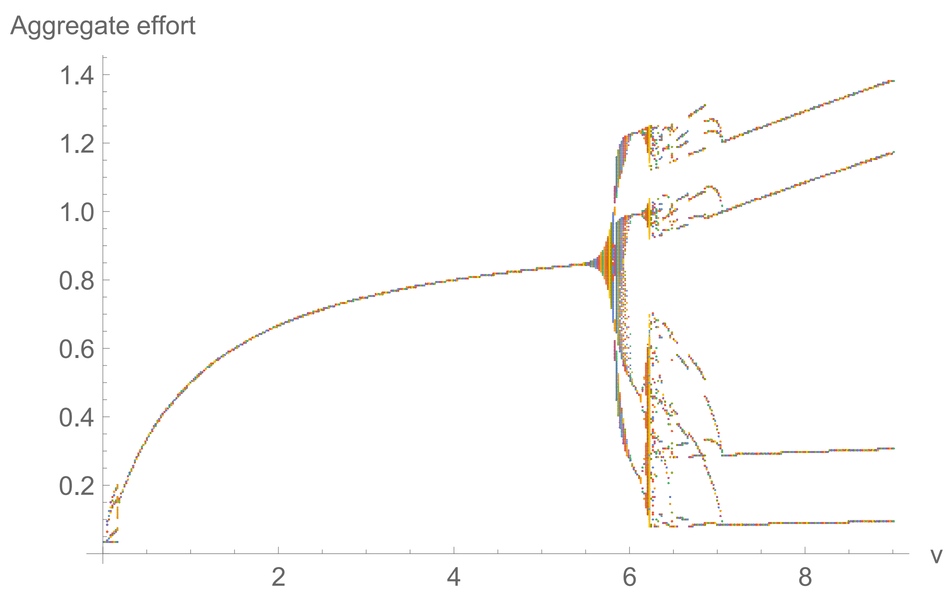

Figure 11: Bifurcation diagram for aggregate effort

4 Concluding remarks

No claim is made, or should be inferred, that the technical contribution of this paper is very great, as similar observations have already been made about an equivalent economic model in Puu [4]. But the analysis suggests, more generally, that interpreting experimental results, in particular when participants repeatedly play the same game, relative to predictions from a one-shot model may be too simplistic. In particular, in highly nonlinear games, such as the contests studied here, many learning rules may never converge to an equilibrium. In this paper we have looked at what is perhaps the simplest such learning rule, that of myopically playing a best reply to whatever happened in the immediately previous period of play. The field is of course open for investigation of other such rules.

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