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The Ramsey growth model is almost a paradigm in the study of optimal economic growth. Recently, Ramsey type models have been used to address questions of welfare measurement and sustainability e.g. suggestions on how to correct the Net National Product (NNP) so that the depletion of environmental resources are taken properly into account. The contributors include Weitzman (1976), Hartwick (1977, 1990), Mäler (1991), Aronsson & Löfgren (1993) and Asheim (1994). This means that a lot is known about the properties of these kinds of models. The purpose of this note is to introduce a result near at hand, but to my knowledge not previously known, on the relationship between investment decisions along the optimal path and future consumption. The result has some interest for the proper understanding of ”Hartwick rule” on sustainable development, and perhaps also for sustainability in general.

The paper is structured as follows. We start by introducing a slightly generalized version of the Ramsey growth model; we use a more general discount factor. We then proceed to derive the main result.

The paper ends with a discussion on the relationship between sustainability, Hartwick rule, and our result.

1 The model and the main results

The Ramsey growth model is here represented by:

$$\underset{c(t)}{Max} \int_0^{\infty} u(c(t))\mu(t)dt \qquad (1)$$

$$\text{s.t. } y(t) = f(k(t)) = c(t) + \frac{dk(t)}{dt} \qquad (2)$$

$$k(0) = k_0 > 0 \qquad (3)$$

where $y = f(k(t))$ is net output at time *t*, which is determined by the capital stock *k*(*t*). The ”capital stock” can be comprehended as a vector of different variations of capital, some of them representing natural resource stocks. The production function is assumed to be twice continuously differentiable, increasing and strictly concave in the capital stock. *c*(*t*) is consumption at time *t*, *dk*/*dt* is net investment, while $\mu(t)$ is a continuously differentiable generalized discount factor with $\mu(0) = 1$ and $\mu(\infty) = 0$. The utility function $u(\cdot)$ is assumed to be increasing and strictly concave in consumption at time *t*.

The present value Hamiltonian of the optimal growth problem can be written

$$H = u(c(t))\mu(t) + \lambda(t)k(t) \tag{4}$$

where $\lambda(t)$ is the co-state variable, which represents the shadow price of capital in terms of utility. If the Hamiltonian is continuously differentiable with respect to time one can, by making use of the necessary conditions for an optimal path, show the following result:

Theorem: If the Hamiltonian is continuously differentiable along the optimal path $\left[c^*(t)k^*(t)\right]$, then it holds that

$\dfrac{dH^*}{dt} = \dfrac{\partial H^*}{\partial t}$ along the optimal path.

If the above mathematical result is used in the present context we obtain:

$$\frac{dH^*}{dt} = \dot{\mu}(t)u\left[c^*(t)\right] \tag{5}$$

Integration (5) forwards yields:

$$H^*(t) = -\int_t^{\infty} \dot{\mu}(s)u\left[c^*(s)\right]ds \tag{6}$$

By integrating the right hand side of (6) partially we obtain after having used $\mu(\infty) = 0$

$$\begin{aligned} H^*(t) &= -\left[\mu(s)u\left(c^*(s)\right)\right]_t^{\infty} + \int_t^{\infty} \mu(s)u_c\left(c^*(s)\right)\dot{c}^*(s)ds \\ &= u\left(c^*(s)\right)\mu(s) + \int_t^{\infty} \mu(s)u_c\left(c^*(t)\right)\dot{c}^*(s)ds \end{aligned} \tag{7}$$

After netting out terms in both members we are left with:

$$\begin{aligned} \dot{k}^*(t) &= \frac{1}{\lambda(t)}\int_t^{\infty} \mu(s)u_c\left(c^*(s)\right)\dot{c}^*(s)ds = \\ &= \frac{1}{\lambda(t)}\int_t^{\infty} \mu(s)\lambda(s)c^*(s)ds \end{aligned} \tag{8}$$

where we have used that the marginal utility of consumption $u_c(\cdot)$, along an optimal path equals the co-state variable $\lambda(\cdot)$; the marginal value of an extra unit of capital. The division by

$\lambda(t)$ converts the equation into real terms (the division is, of course, not equally clear cut if $\dot{k}(t)$ has a vector interpretation). Equation (8) can be given the following interpretation:

Investment along an optimal path is positive (negative) in the interval (t, t+dt), if, and only if, the discounted value of the sum of future changes in consumption along the optimal path is positive (negative).

Another interpretation is that the right hand side of equation (8) is the (maximum) net future real willingness to pay for the investment.

<u>The relationship to Hartwick's rule</u>

Hartwick (1987) shows that in a closed economy with a constant population and a stationary technology steering along an optimal path with $\lambda(t)\dot{k}^*(t) = 0$ for all t, the utility level is constant and equal to the maximum sustainable level. More specifically:

<u>Hartwick's rule</u>: *Let* $\left(c^*(t), k^*(t)\dot{k}^*(t)\right)$ *be an optimal path for the above Ramsey problem, and for each t,* $\lambda(t)\dot{k}^*(t) = 0,$ then $u\left(c^*(t)\right)$ *is constant for all t.*

Note that since $u\left(c^*(t)\right)$ is constant and $c^*(t)$ is optimal, $u\left(c^*(t)\right)$ represents the maximum sustainable utility level.

Dixit et al. (1980) show that $\lambda(t)\dot{k}^*(t) = 0$ is also necessary for the utility level to be constant along an optimal path, or:

> <u>Dixit-Hammond-Hoel</u>: *If the utility level is constant along an optimal path* $(c^*, k^*, \dot{k}^{*)})$, *then* $\lambda(t)\dot{k}_t^* = 0$ *for all t.*

This means that $\lambda(t)\dot{k}^*(t) = 0$ is equivalent to $u\left(c^*(t)\right)$ being equal to the maximum sustainable utility level for all *t*.

A reasonable requirement on an indicator of sustainability is that it should equal utility (consumption) if utility (consumption) happens to be constant and at any time equal to the maximum sustainable level. In the above model the value of the Hamiltonian along is such an indicator. As Asheim (1994) has shown the Hamiltonian is not in general an exact indicator of sustainability, except in the case with only one capital good. This case is fairly uninteresting, since in order to sustain resource stocks must be substituted for man .....

Our rule in equation (8) is a vehicle capital to prove both the necessity and sufficiency of $\lambda(t)\dot{k}^{*}(t) = 0$ for all *t*, for $u\left(c^{*}(t)\right)$ being equal to the maximum sustainable level for all *t*. Loosely speaking, $\mu(s)\lambda(s) > 0$ for all finite *t* so $\dot{c}^{*}(t) = 0$ for all *t* implies $c^{*}(t)$ and $u\left(c^{*}(t)\right)$ constant and equal to the maximum sustainable level for all *t*, and also $\lambda(t)k^{*}(t) = 0$ for all *t*. Conversely, $\lambda(t)k^{*}(t) = 0$ for all *t* implies $\dot{c}^{*}(t) = 0$ and $u\left(c^{*}(t)\right)$ constant for all *t*.

In addition, our rule yields an exact relationship between investment at time *t* and the future optimal consumption plans. In particular, the rule shows that $\lambda(t)\dot{k}^{*}(t) = 0$ at some *t* in the context of a competitive economy is no indication of sustainability. It only shows that the weighted sum of future changes in consumption, the weights being the discounted value of the future marginal utilities, equals zero.

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