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Risky Taxes, Budget Balance Preserving Spreads and Precautionary Savings

Torbjörn Becker

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Risky Taxes, Budget Balance Preserving Spreads and Precautionary Savings*

Torbjörn Becker[✉]

Abstract

The paper analyzes the effects on consumption from changes in the riskiness of taxes. It starts by reinterpreting the Sandmo [1970] paper on general capital income risk to the case of risky capital taxation. In his framework the concept of a mean preserving spread (MPS) is used for the risk analysis. In connection with risky taxes it is however possible to explicitly connect the tax risk with the government's budget constraint. In this paper the concept of a budget balance preserving spread (BBPS) is developed and used for the analysis of stochastic taxes. The paper is concluded with a comparison of the effects that a MPS and a BBPS has on consumption decisions. It is shown that the comparative statics results for a BBPS could be different from the results obtained with a MPS.

Keywords: tax risk, precautionary savings

JEL: E21, H31, D81, E62

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✉ Stockholm School of Economics, Box 6501, 113 83 Stockholm, Sweden, Phone: +46 (8)736 92 29, Internet: NETB@HHS.SE.

1. INTRODUCTION

Since, at least, von Neumann and Morgenstern's [1944] expected utility maximization, economics has not only dealt seriously with choice under certainty, but also with decision making in an uncertain world. Pratt [1964] and Arrow [1971] then introduced the concepts of absolute and relative risk aversion, (which makes it possible to compare the taste or aversion for risk between different utility functions with a unit free measure that is independent of affine transformations of the utility function). These concepts have proved to be very useful in the study of decision making under uncertainty.

When we use expected utility maximization for intertemporal choice, we need to distinguish between temporal and timeless risk, as discussed in for example Drèze and Modigliani [1972] and Machina [1984]. The concept of risk aversion is specified in the case of timeless risk, where no time passes before the uncertainty is resolved. One of the results from Drèze and Modigliani is that a temporal uncertain gamble will never be preferred to a timeless one if the outcomes have the same distribution. In, for example, Sandmo [1970] the assumption of decreasing temporal risk aversion is used (with time separable utility, this implies a positive third derivative of the utility function), as motivated by for example Arrow [1971], in addition to the standard assumptions of timeless risk aversion (which implies a negative second derivative of the utility function).

The effect of uncertain future income on consumption/savings decisions are analyzed in Leland [1968] and Sandmo [1970]. Leland shows, by using a Taylor expansion, that pure risk aversion will not in itself give rise to precautionary savings, but assumptions on the third derivative of the utility function can ensure a precautionary savings motive. Sibley [1975] generalizes Leland's two period analysis to a multiperiod model, and shows that the condition for precautionary savings is still a positive third derivative of the utility function. Sandmo analyzes a more straightforward case, where a specific parameter is introduced that describes the risk, or, equivalently, variance in his model. This simplification with respect to the type of change in risk considered leads to a more direct analysis of the conditions needed for the utility function. In the case of income risk, it is for example enough to state that individuals have decreasing temporal risk aversion in order to find precautionary savings as a result of increasing risk. Where increasing risk is defined as a mean preserving spread, i.e. the expected outcome is not affected, but the variance of the outcome is. An analysis of the conditions of the utility function in combination with more general assumptions about distributional changes for deriving clear-cut comparative statics results could be found in Ormiston and Schlee [1992]. (The reader interested in comparative statics for non-expected utility[1]is referred to Machina [1989].)

Since Leland's [1968] paper, the notion of precautionary savings has received attention both in theoretical and empirical work. In the theoretical world, Kimball [1990] formalizes the concepts of precautionary savings, and shows how the theory of risk aversion can be translated into a theory of "prudence", where prudence is a measure of the strength of the precautionary savings motive, rather than the strength of disliking risk. The latter concept can be summarized by the familiar measure of absolute risk aversion defined as $a \equiv -u''/u'$. In an analogous way, the measure of absolute prudence is defined as $\eta \equiv -u'''/u''$, where u is the utility of future consumption. Kimball shows that most theorems about risk aversion can be applied to precautionary savings by simply substituting $-u'$ for u. He also notes that the measure of absolute prudence will be larger(smaller) than the measure of absolute risk aversion when absolute risk aversion is declining(increasing). (This is for example the case for time separable iso-elastic utility, where we also always have a positive coefficient of prudence if we assume that individuals are risk averse.)

The important aspect of the theory of prudence is that we can analyze how the consumption/savings choice will change in response to changes in risk. We can thus derive conditions on distributional changes in combination with conditions on the utility function in terms of prudence that are needed for clear-cut results of comparative statics exercises that involve risk. For example, in the case of capital income risk, there will be two effects that affect the consumption choice. First, a

[1]Expected utility analysis has the implication of linearity in probabilities, while non-expected utility analysis instead assumes that preferences over probability distributions is smooth, and could locally be approximated by expected utility analysis.

substitution effect that says that if capital income becomes more risky, so does second period consumption, and thus the agent will shift consumption away from the second period. However, there is an additional effect, labeled income effect, that says that with increased uncertainty about second period income implied by increased uncertainty about the return on savings, the agent faces a risk of having a very low consumption in the second period, which implies a very high marginal utility in that period. Since the expected utility from consumption in the first and second period should be the same, the agent will save more to avoid having a very high marginal utility in the second period. With a mean preserving spread in capital income, there are thus two effects on first period consumption with opposite signs. However, Ormiston and Schlee [1992] tell us that with a coefficient of relative prudence that is greater(smaller) than two, the income(substitution) effect will dominate and first period consumption will decrease(increase). In the case of iso-elastic utility, this condition is equivalent to Sandmo's [1970] condition that the coefficient of relative risk aversion is greater than one.

There is empirical evidence that the inclusion of precautionary savings helps explain observed deviations from models without precautionary savings. For example, in studies of the permanent income hypothesis, there have been features of data that could not be well explained with models that do not allow for precautionary savings, most noted, underspending of the elderly (see Mirer [1979]), excess growth (see Deaton [1986]), excess smoothness (see Campbell and Deaton [1989]) and excess sensitivity (see Flavin [1981]) of consumption. To define the empirical puzzles in more detail; *underspending of the elderly* is based on cross-section data that shows that the elderly do not dissave during retirement, *excess sensitivity* says that consumption moves ”too much” in response to anticipated changes in labor income, *excess growth* implies that aggregate consumption grows over time in a way that cannot be explained by the real interest rate being greater than the rate of time preference, and finally *excess smoothness* says that consumption responds ”too little” to unanticipated changes in labor income.

Zeldes [1989] simulates a model with preferences that display constant relative risk aversion, implying that households are prudent, and assumes that labor income is an i.i.d. process. He can then conclude that the above empirical puzzles can be explained by using a model with preferences that display a precautionary savings motive instead of the standard models of the permanent income hypothesis that lack this feature. Furthermore, Caballero [1990] derives a theoretical model which he combines with different assumptions about the stochastic process that governs labor income, and concludes that he can explain a large part of the empirical puzzles in the US data discussed above. In Caballero [1991] he also shows that about 60 percent of US wealth accumulation can be due to precautionary savings.

However, in a micro based study estimating the coefficient of prudence by using data from the Consumer Expenditure Survey, Dynan [1993] concludes that the coefficient of prudence is very small, and actually so small that it is not consistent with normal assumptions about households’ aversion of risk. Taken seriously, this would raise the question why micro and macro data yield so different results. In this paper I will not try to answer that, and since this model is aimed at macro phenomena, the importance of the precautionary motive will be stressed.

In this paper as well as in many of the above papers, the portfolio nature of savings analyzed in the finance literature is neglected. There are of course models that combine the consumption decision with portfolio decisions, the early references being Samuelson [1969], who analyzes consumption and portfolio decision in discrete time, and Merton [1969], who uses continuous time. This type of analysis then answers the question of what amount of risky and riskless assets to hold in the portfolio when risk changes. The results are determined by assumptions made on the shape of the utility function, and the conditions on the parameters in the utility function are closely related to the conditions derived in Sandmo’s [1970] study of capital income risk without a portfolio decision. Drèze and Modigliani [1972] also discuss consumption and portfolio decision, and the separability between these decisions.

Risk in connection with taxation has been discussed since, at least, Domar and Musgrave [1944]. In Stiglitz [1969] the effects of different taxes on risk taking are discussed. More precisely, the amount of savings put into a risky asset rather than a safe one under different tax policies is analyzed. Another aspect from this part of the literature is that capital income taxation that is certain reduces the variance of net asset returns, since the uncertain gross returns are multiplied by a number smaller than one. If the tax system is then set up such that it also makes transfers in the case of negative returns, taxes act as an insurance or risk reducer. In the extreme, if the gross returns are symmetrically distributed around zero, the ”tax” will of course only have the characteristics of an insurance and not a tax, since the variance is reduced but the tax revenue is zero. In more sensible cases, the expected gross return is above zero, and loss offsets are not total, and thus the tax system will work partly as an insurance and partly as a revenue raiser for the government. The insurance aspect could of course be one part of the tax system, but by assuming that the gross return is deterministic, we will abstract from this issue here.

In this paper, the focus will be on the stochastic features of the *tax system* rather than on the stochastic features of the market. This approach might require some motivation. The first part of the motivation is due to the increasing part of disposable income that is contingent on government policies. In Sweden, for example, the public sector turns over about 70 percent of GDP, a large part of the turnover being transfers to the private sector. Combining this observation with the observation that the tax and transfer system is consistently being ”reformed”, implies that a large part of the households’ income is contingent on public policies that keep changing over time. In other words, a large part of the fluctuation in disposable income is due to the public sector rather than the market.

Furthermore, in real business cycle models, including stochastic features of government policy has helped to improve the performance of these models. If we read out the headings in McGrattan [1994], these say: ”The standard [real business cycle] model’s predictions improved slightly with indivisible labor and significantly with fiscal shocks”, which further motivates studying a stochastic tax system. Finally, in the institutional economics literature, the value of stable rules and regulations has been stressed, and this paper can be a formalization of a certain aspect of the rule system, namely how agents are taxed. However, the aim of the paper is not to provide a normative analysis of the tax system, but rather a positive analysis of the effects that uncertain taxes can have on private consumption.

There have been some previous papers taking the view that taxes rather than the market are uncertain, for example Chan [1983], and Alm [1988]. In Chan, lump-sum taxes are analyzed with a government budget constraint, while Alm analyzes different aspects of taxation more in the spirit of optimal taxation, including uncertainty, but ignoring the government's budget constraint.

This paper analyzes uncertain *distortionary* taxes when the government's budget constraint is explicitly considered. It is, however, not in the spirit of the optimal taxation literature, but more in the spirit of general choice under uncertainty as discussed above. The framework is adopted from Sandmo's [1970] paper, where we have an explicit parameter representing changes in risk. Analyzing risk in this set-up facilitates the interpretation of the results, and simplifies the algebra, but the price to pay is that, in reality, riskiness could be reflected by other aspects than changes in the variance (see for example Rothschild and Stiglitz [1970, 1971] for definitions of increasing risk, Huang and Litzenberger [1988] for an overview of stochastic dominance, and Whitmore [1970] for the definition of third degree stochastic dominance).

The reason for including the government's budget constraint in the analysis is that with distortionary taxes, households can affect the tax base and thus also the expected value of the uncertain tax rate. In the cases where we are interested in models with uncertain distortionary taxation, it is shown that the comparative statics results are affected by the choice of mean preserving spread (MPS) or budget balance preserving spread (BBPS) analysis. If we instead analyze models with lump-sum taxes (which is the analog of Sandmo's [1970] analysis of income risk) as in Becker [1994], MPS and BBPS will obviously give the same results, since then, by definition, consumers cannot affect the tax base.

To summarize, the analysis of risky choice has a fairly long history in the economics literature, and in many cases the tax system is not (explicitly) considered. In the cases where the tax system is considered, however, the focus is with few exceptions on market induced risk, with the potential for taxes to act as an insurance against risk.

This paper has a different starting point, namely that the tax system itself is the origin of uncertainty, and that its riskiness affects households' consumption decision. Further, one natural starting point when analyzing risk, namely mean preserving spreads, is now compared with budget balance preserving spreads, to take into account that the risk source is the government and the government has a budget constraint to obey. The budget balance preserving spread analysis is thus a more natural risk concept when we analyze risk created by the government rather than by the market.

It is shown in the paper that the BBPS risk concept modifies the assumptions needed on the utility function in order to derive clear comparative statics results with respect to risk and consumption. In particular, it is shown that, for *small* enough values of the relative *risk aversion* coefficient, as well as for values greater than one, the *precautionary savings* motive will dominate in the case of capital income uncertainty. In the usual mean preserving spread analysis this is the case only when the coefficient is larger than one.

The paper is organized as follows: in Section 2, two experiments with uncertain taxes are conducted. The first uses a mean preserving spread in the tax rate, while the second uses a budget balance preserving spread. In Subsection 2.4 the results of the two experiments are compared. The paper ends with a section containing a summary and some conclusions. Finally, in the appendix, changing a deterministic tax rate is analyzed, as well as Sandmo's original analysis of general capital income uncertainty within the notation and solution framework used in this paper.

2. Two concepts of tax risk

The general structure of the problem is that we study an economy that exists for two periods, with a government sector and an infinite number of households. In the first part, the government is simply an institution that keeps the mean of the expected tax rate constant, which is analogous to Sandmo's [1970] analysis of capital income risk, while in the second part the government obeys a budget constraint. In both cases, the households are expected utility maximizers that have labor income in the first period, and save in order to consume in the second period. Gross returns on savings are exogenous and known, while the tax rate on capital income is stochastic. In the problems below we will solve a two-equation system for two unknowns. The first equation is the representative household's first order condition, and the second equation is the mean preserving spread or the government's budget constraint, alternatively. In the case of a mean preserving spread, the system of equations approach is not really needed, since the system could be analyzed recursively, while in the second part the equations are interdependent. Using the system of equations approach also in the first case is motivated by the fact that we then derive the same type of conditions for both models, which facilitates comparisons.

2.1 HOUSEHOLDS' PROBLEM

Households are rational economic units that maximize their expected utility subject to a budget constraint. There are many (or an infinite number of) identical households that all solve the problem

$$\begin{aligned} &\max_{c_1,c_2}\; E\left[U\left(c_1,c_2\right)\right]=\int_{\Gamma}U\left(c_1,c_2\right)f\left(\gamma^i\right)d\gamma^i \\ &\text{s.t.}\quad c_1=y_1-\tau_1-a_1 \\ &\qquad\;\; c_2=(1+r-r(p\gamma^i+\theta))(y_1-c_1-\tau_1)\ , \end{aligned} \tag{1}$$

where U (.) is a von Neumann-Morgenstern utility function with $U' > 0$ and $U'' < 0$, i.e. agents are assumed to be risk averse, c_t is period t consumption, y_1 is an exogenous (labor) income that the individual has only in the first period, τ_1 is a lump-sum tax in the first period, r is the certain gross interest rate, a_1 is the savings from period one to two, p is the multiplicative policy parameter, γ^i is the individual's realized draw from the tax rate distribution Γ, and finally θ is the additive tax parameter. Since we are analyzing capital income tax rates, we only allow for realizations of the total tax rate, $p\gamma^i+\theta$, that are between zero and one, and therefore the average tax rate will also be between zero and one. We can interpret the relative magnitudes of p and θ as the relative weights of the stochastic and deterministic parts of the tax system, respectively. For example, a large *p* and a small θ indicate a relatively risky tax system in the sense that tax rates will have a large spread.

The assumption that we are analyzing an infinite number of households makes the probability distribution of tax rates independent of the individual's choice of consumption. In the mean preserving spread analysis, this is never a problem, but in the budget balance preserving spread analysis, we have a feed-back to the tax rates from household behavior. To avoid that the household internalizes this feed-back, we assume that there are many households, so the feed-back effect from the behavior of a particular household is negligible.

The first and second order conditions (FOC and SOC respectively) to this problem are

FOC: $$F^2 \equiv E\left[U_1 - R^i U_2\right] = 0 \qquad (2)$$

SOC: $$S \equiv E\left[U_{11} - 2R^i U_{12} + {R^i}^2 U_{22}\right] < 0 \ , \qquad (3)$$

where $R^i \equiv \left(1 + r - \left(p\gamma^i + \theta\right)r\right)$ is the individual-specific realized net rate of return on capital that is stochastic due to the stochastic tax rate. In this model this is the only source of uncertainty. The first order condition states that the expected marginal utility of consumption in period one and two should be equal. The FOC is the second equation in the system of equations we will analyze.

2.2 MEAN PRESERVING SPREADS IN TAX RATES

This section uses the framework of Sandmo [1970] to analyze precautionary savings in response to stochastic taxes when using a MPS. In this experiment, the government is only choosing tax rates such that the mean tax rate on capital is preserved, but the spread is altered for different choices of *p*. In the following sections the government will be defined by a budget restriction rather than from solely being a risk experimenter that determines the spread of returns[2].

The tax system consists of one standard tax rate, γ, that is stochastic, one multiplicative policy variable, *p*, and one additive deterministic policy variable θ. Together these variables define the expected tax rate, $E\left[p\gamma^i + \theta\right] = p\bar{\gamma} + \theta$, that is assumed to be constant, since we analyze a mean preserving spread in this section. This specification is equivalent to the formulation used in Sandmo's [1970] framework presented in the appendix. Since the parameter *p* is multiplicative, it affects both the spread and the mean of the total tax rate. θ, on the other hand, is additive, and only affects the mean. The restriction could be formalized as

$$F_M^1 \equiv p\bar{\gamma} + \theta - \text{constant} = 0 \,, \tag{4}$$

which will be the first equation of our two-equation system (subscript M for **M**PS). This condition could be totally differentiated as it is, to show how θ has to change for a given change in *p*, which is how Sandmo proceeds. Here we will use a more tedious system of equation approach in order to derive the analog of the equations obtained in the following experiment, where we have to take into account changes in the tax base. The tax parameter *p* is used for the comparative statics exercise, while θ and c_1 are the two endogenous variables that ensure budget balance for the government and that the representative household's first order condition is satisfied. Combining the FOC and the mean preserving spread condition yields the following system of equations

$$\begin{cases} F_M^1 = 0 \\ F^2 = 0 \,. \end{cases} \tag{5}$$

The system is totally differentiated and solved for the partial derivatives of interest (for a formal derivation, see the appendix). We start by investigating how the additive

[2]I leave it to the reader to decide which of these assumptions about the government is most closely related to real world observations.

tax parameter has to change in response to an increase in the multiplicative tax parameter in order to satisfy the mean preserving spread condition

$$\frac{\partial\theta}{\partial p} = -\bar{\gamma} \ , \tag{6}$$

which says that if p is increased, the average tax will rise with $\bar{\gamma}$ and thus the additive tax parameter has to change equally much in the other direction. This condition could of course equally well be obtained by total differentiation of the first equation, which is the way it is done in Sandmo. Again, this unnecessarily tedious approach is used to conform to the steps needed in the following sections.

Now we want to investigate how consumption changes in response to a mean preserving spread. We have that

$$\frac{\partial c_1}{\partial p} = -\frac{1}{S} E\left[\Phi\left(\gamma^i - \bar{\gamma}\right)\right], \tag{7}$$

where $\Phi \equiv r\left(U_2 - a_1 U_{12} + R^i a_1 U_{22}\right)$. This is proportional to the expression describing how consumption changes in response to increased capital income risk in Sandmo's original paper, which is presented in the appendix. The difference between this expression and Sandmo's is due to the fact that we here analyze a stochastic capital income *tax rate* and not simply a stochastic capital income. The sign determination is equivalent to the capital income risk case, since a higher realized value of the tax rate means a lower capital income, while a higher realized value in the original case leads to a higher capital income. Thus the minus sign in front of the present expression.

The intuition for the sign determination could be thought of as follows. An increase in p makes second period consumption more risky, implying that households would like to substitute their second period consumption to first period consumption, thus the *positive substitution effect*. At the same time, with increasing p, households face the risk of having a very low second period income, which creates an incentive for precautionary savings, thus the *negative income effect*. In the following discussion we will continue to associate the label substitution effect with the component involving the first derivative, and the label income effect with the terms including the second derivatives of the utility function, although, as will be shown, the results associated with some cases of a dominating substitution effect will differ.

2.3 BUDGET BALANCE PRESERVING SPREADS IN TAX RATES

We are now ready to investigate the effects of explicitly incorporating the government's intertemporal budget constraint into the analysis of risky taxes. The tax system consists of lump-sum taxes in the first period, as well as government consumption in both periods. In the second period, there is a capital income tax rate that is stochastic from the individuals' perspective, but certain from the government's point of view, since there are many (or an infinite number of) identical households that pay taxes. Compared to the risk analysis above, the government now obeys a budget constraint, rather than a condition stating that the mean of the tax rate should be constant. In an analysis of risky taxes, the spread concept used in this section yields a more natural condition on the tax distribution. We write the budget constraint as

$$\bar{g} = g_1 + \frac{g_2}{1+r} = \tau_1 + \frac{E(\tau_2)}{1+r} = \tau_1 + (p\bar{\gamma} + \theta) r (y_1 - c_1 - \tau_1) / (1+r) \ , \tag{8}$$

where $\bar{g}$ is the present value of the total resources extracted by the government[3], with g_t extracted in period t. Taxes in the second period, τ_2, are collected as a capital income tax, with the tax rate $p\bar{\gamma} + \theta$. Rewrite this, and now define the function F_B^1 (B for **B**BPS) as

$$F_B^1 \equiv (p\bar{\gamma} + \theta)(y_1 - c_1 - \tau_1) - (\bar{g} - \tau_1)(1+r)/r = 0 \ , \tag{9}$$

[3]In general this waste of resources is not a very good approximation of government activity, but for the purpose of keeping the analysis at a more transparent level, this characterization could perhaps be justified. A more generous interpretation of the present set-up is to consider the government sector activity as fixed, and with some additive separability assumption, this activity could be ignored in the present analysis.

A notational point of importance is again the distinction between the individual's realized value of the stochastic tax rate, denoted by γ^i, and drawn from a distribution of tax rates Γ, defined for positive numbers between zero and one, and the aggregate variable that the government faces, $\bar{\gamma}$, which is no longer a stochastic variable since we assume that there are sufficiently many individuals for the law of large numbers to work. The total tax rate on capital income for a particular individual is $\left(p\gamma^i + \theta\right)$, with $E\left[p\gamma^i + \theta\right] = p\bar{\gamma} + \theta$, which is the tax rate that appears in the government's budget constraint.

The experiment in this section is again to investigate how consumption changes in response to a change in *p*, but now the government's budget balance has to be preserved, rather than the mean of the tax rate. This is achieved by changing the additive tax parameter, θ. What we are analyzing here is thus not the ”ordinary” mean preserving spread of the previous subsection, but rather a budget balance preserving spread of the tax rate.

The potential for this experiment to differ from the above analysis is that we have to take into account changes in the tax base in response to changes in risk. This implies that if the tax base actually changes, the mean of the tax rate will have to change in order to satisfy the government's budget constraint. However, this has the implication that for some levels of government spending in combination with some utility functions, there might not exist a tax rate that is between zero and one that satisfies the government's budget constraint. In the above section, it was sufficient to make the assumption that the tax rate has to be between zero and one. With a budget balance preserving spread, we can, however, start out in a situation where the government's budget is balanced, and end up in a situation where a tax rate equal to one is not sufficient for balancing the budget, due to the endogenous changes in the tax base as a response to changes in the tax risk. Only the cases where tax rates between zero and one is sufficient for achieving budget balance will be analyzed in this paper. However, it is still of importance to realize that the government could actually end up violating its budget constraint in this model, due to changes in the tax risk, without having changed its consumption.

If we use the household's first order condition and the government's budget constraint, we get the following system of equations

$$\begin{cases} F_B^1 = 0 \\ F^2 = 0\ . \end{cases} \tag{10}$$

The system is totally differentiated, and we can then solve for the partial derivatives of interest. We start by investigating how the additive tax parameter has to change in order to satisfy the government budget constraint. The derivative can be written as

$$\frac{\partial\theta}{\partial p} = -\bar{\gamma}\,\frac{a_1 S + pE\left[\gamma^{\,i}\Phi\right]}{|J|} - \theta\,\frac{E\left[\gamma^{\,i}\Phi\right]}{|J|}\ , \tag{11}$$

where $\Phi \equiv r\left(U_2 - a_1 U_{12} + R^i a_1 U_{22}\right)$, and $|J| = a_1 S + \left(p\bar{\gamma} + \theta\right)E[\Phi] \neq 0$ (which is the condition that the Jacobian determinant has to be non-zero in order to use the implicit function theorem.) The sign of this derivative will in general be negative, since if we raise the expected value of the stochastic tax rate, the value of the deterministic tax rate can, in general, be decreased, to maintain budget balance for the government. There is, however, a possibility that the dervative is positive, which is basically to say that we are to the right of the maximum of the Laffer curve. In such a case, an increase in *p*, and thus the risky part of the tax, have very strong effects on the consumption choice of the household. This implies that the tax base then erode to the extent that the deterministic tax rate has to increase to compensate for the reduced tax base to maintain budget balance for the government. We will discuss the circumstances where this situation might appear after we have investigated how consumption will change.

To investigate how consumption changes in response to a budget preserving spread, we have the derivative

$$\frac{\partial c_1}{\partial p} = \frac{\bar{\gamma} a_1 E[\Phi] - a_1 E[\gamma^i \Phi]}{|J|} = -\frac{a_1}{|J|} E\left[\Phi\left(\gamma^i - \bar{\gamma}\right)\right] \; . \tag{12}$$

The expression is (as expected) very similar to the result with a mean preserving spread, with the exception that we now have $a_1/|J|$ in the first factor rather than simply $1/S$. Again, the sign will be determined by the strength of the income and substitution effects, with the complication that we now do not always know that the sign of the denominator is negative, since the denominator can be positive in the case of ”dominating” substitution effects and a small enough coefficient of relative risk aversion. The specific conditions determining the sign of the derivative between first period consumption and the multiplicative tax parameter will be discussed below.

2.4 Comparing BBPS with MPS

A question of interest at this point is to establish how the above results from the mean preserving spread analysis compare to the budget balance preserving spread in more detail. In the case of a mean preserving spread of the return to capital, we know that the consumption response to increased risk could be divided into three cases: first, the substitution effect could dominate, secondly, the income effect could dominate and, finally, the effects could be equal and thus leave consumption unchanged.

The BBPS analysis could differ from the MPS analysis due to changes in the tax base, and thus to endogenous changes in the mean tax rate. We therefore know that in the case where the income and substitution effects cancel, the analysis of an intertemporal budget balance preserving spread is equivalent to Sandmo’s mean preserving analysis, since then the tax base remains unchanged, and thus the average tax rate has to be kept constant. This could be seen more formally in equations (11) and (12), which then reduce to the same expressions as in Sandmo’s case, namely equations (6) and (7).

The two interesting cases to analyze are thus when either the income or substitution effect dominates, to use Sandmo's original labels. Start by writing out some terms in (12) and reshuffle a little to obtain

$$\frac{\partial c_1}{\partial p} = \frac{1}{S + \frac{p\bar{\gamma} + \theta}{a_1} E[\Phi]} E\left[\Phi\left(\bar{\gamma} - \gamma^i\right)\right] . \tag{13}$$

The sign is determined by two parts, firstly

$$E[\Phi] = E\left[r\left(\underbrace{U_2}_{+} \underbrace{-a_1\left(U_{12} - R^i U_{22}\right)}_{-}\right)\right], \tag{14}$$

where the first term is positive and the second is negative,[4] and secondly

$$E\left[\Phi\left(\bar{\gamma} - \gamma^i\right)\right] = rE\left[U_2\left(\bar{\gamma} - \gamma^i\right) - a_1\left(U_{12} - R^i U_{22}\right)\left(\bar{\gamma} - \gamma^i\right)\right] . \tag{15}$$

In Sandmo's MPS case, the sign of the derivative is determined by the second factor, since the denominator is always negative in his case. The conclusion is then that if the substitution effect dominates the derivative is positive, and if the income effect dominates the derivative is negative. For the BBPS, we have to sort out the relationship between $E\left[\Phi\left(\bar{\gamma} - \gamma^i\right)\right]$ and $|J|$. A proof of these signs for the case of time separable iso-elastic utility is presented in the appendix, to show how the size of the relative risk aversion coefficient enters the analysis. For a proof with a general utility function, see the appendix in Sandmo [1970].

The three cases are then:

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[4]The second term is negative if we assume that $\partial c_1 / \partial y_1 > 0$.

- Income and substitution effects cancel out

  $$\Rightarrow E\left[\Phi\left(\bar{\gamma}-\gamma^{i}\right)\right]=0 \quad \Rightarrow \quad |J|<0.$$

- Income effect dominates substitution effect

  $$\Rightarrow E\left[\Phi\left(\bar{\gamma}-\gamma^{i}\right)\right]>0 \quad \Rightarrow \quad |J|<0.$$

- Substitution effect dominates income effect

  $$\Rightarrow E\left[\Phi\left(\bar{\gamma}-\gamma^{i}\right)\right]<0 \quad \Rightarrow \quad |J|>0 \text{ if } -a_1 S<\left(p\bar{\gamma}+\theta\right)E[\Phi],$$

  $$|J|<0 \text{ if } -a_1 S>\left(p\bar{\gamma}+\theta\right)E[\Phi].$$

Obviously the MPS and BBPS are equivalent in the first case, since then the BBPS analysis reduces to the MPS analysis. The intuition is, of course, that if we do not have a change in consumption patterns the tax base does not change, so the tax rate has to have the same mean also in the BBPS analysis.

The conclusion for a dominating income effect in the MPS analysis is that the derivative is negative. This is still valid, since then E[Φ] is negative and leaves the negative sign of the denominator $|J|$ unchanged, which combined with a positive numerator generates a negative derivative.

However, the magnitude of the derivative will be smaller in the BBPS analysis compared to the MPS analysis. The explanation is that as households save more, the tax base is larger and the expected tax rate on capital is thus reduced. This changes the relative price on first and second period consumption, but in a deterministic way through the additive tax parameter. From the appendix we know that for a deterministic increase in the net return on capital, due to a decrease in θ, and with what we here label a dominating income effect, first period consumption increases. This latter effect counteracts the initial effect of reduced first period consumption, but does not alter the sign of the derivative.

Finally, we have the case where the substitution effect dominates, which in the MPS analysis makes the derivative positive. In the BBPS this is no longer obvious. Instead it is possible to show that when the risk aversion coefficient is small enough, we will have a negative sign on the derivative also when the so-called substitution effect dominates. This will be the case when the Jacobian determinant is positive (which implies that $-a_1 S < \left(p\bar{\gamma} + \theta\right)E[\Phi]$). The conditions on the utility function and tax parameters needed to fulfill this condition can be found in the appendix. (It is then perhaps a misuse of language to call this a dominating substitution effect, but I will anyway keep Sandmo's original labels.)

The intuition for this is perhaps not so obvious, but one interpretation is that when the net return on capital becomes low, households do not want to save as much, which at the same time makes them more concerned about the spread in second period income and thus the spread in second period consumption. In other words, households face the risk of having a very low consumption in the second period if the spread increases, implying that they take precautionary actions against this by saving more, thus reducing first period consumption.

Finally, we also note that it is only when the substitution effect dominates that we can be to the right of the maximum of the Laffer curve, since we then know that first period consumption increases, implying that the tax base is smaller. However, this does not necessarily lead to the conclusion that the determininstic tax increases when the stochastic tax increases. It merely indicates that the total expected tax rate has to increase, i.e. the derivative between θ and p has to be greater than $-\bar{\gamma}$, and potentially, it can then also become positive. A formal condition of when the derivative is positive can be found in the appendix.

Two graphical summaries of the signs of the two derivatives for different values of the coefficient of relative risk aversion for MPS and BBPS are presented in Figure 1 and 2.

Mean preserving spread analysis

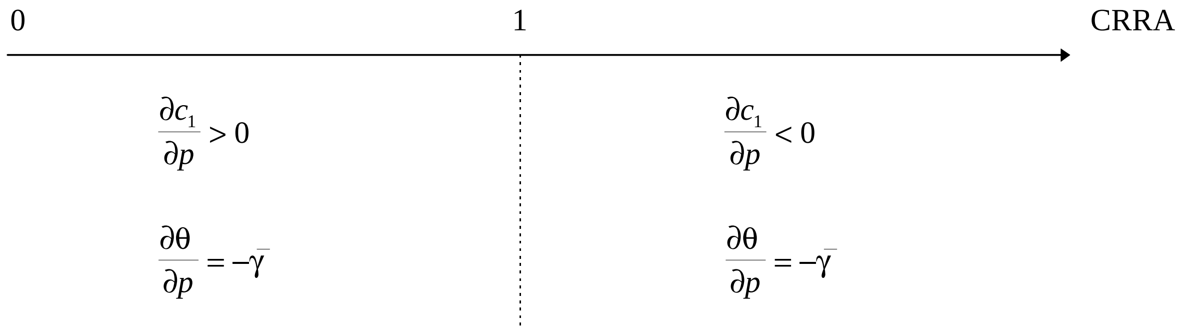

Figure 1. *The sign and magnitude of the derivatives for different values of the coefficient of relative risk aversion (CRRA) for a mean preserving spread in tax rates.*

Budget balance preserving spread analysis

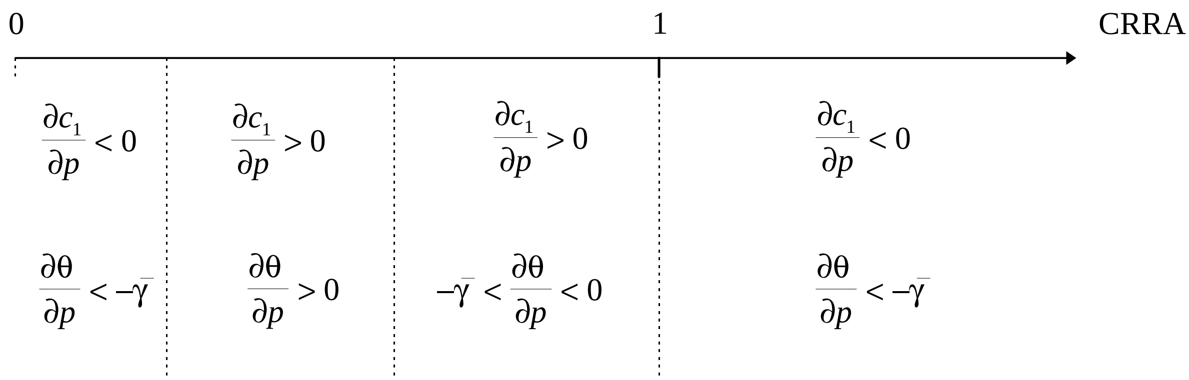

Figure 2. *The sign and magnitude of the derivatives for different values of the coefficient of relative risk aversion (CRRA) for a budget balance preserving spread in tax rates.*

3. SUMMARY AND CONCLUSIONS

This paper analyzes intertemporal consumption behavior in response to changes in risky capital income taxation. The framework used is adopted from Sandmo [1970], but instead of only using the risk concept of mean preserving spreads, the concept of budget balance preserving spreads is developed. The model is a simple two period model with a government sector and an infinite number of households. Risk is introduced in the model by assuming that individuals face a stochastic tax rate on capital income. In the case of a mean preserving spread of the tax rate, the effects on consumption are determined by the strength of the income and substitution effects; thus savings could either increase or decrease when additional risk is introduced. Using a time separable iso-elastic utility function, this is equivalent to assuming that the coefficient of relative risk aversion is greater (dominating income effect) or smaller (dominating substitution effect) than one.

When instead a BBPS is analyzed, the income effect will dominate also for small values of the risk aversion coefficient (or with high average tax rates), since the individual is then exposed to a high risk even before the tax risk is increased. This leads to the, at first, not so intuitive conclusion that the precautionary savings will take place both when the risk aversion coefficient is small enough and when it is greater than one. The crucial factor distinguishing the two cases of MPS and BBPS is that in the latter case the mean of the tax rate has to change in response to consumption changes, since the tax base then changes and the government has an intertemporal budget constraint to satisfy, which in turn affects the consumption choice.

The present analysis has focused on risk created by the tax system rather than the market/production side of the economy. This is of course not the full picture of risk in the real world, but it might serve as a starting point for understanding how risk created by the government could be different from (unspecified) market risk.

There are several ways in which the present analysis could be extended. One way could be to introduce uncertainty with respect to both gross returns and tax rates, in order to investigate how assumptions about the correlation between these would modify the results. Another extension would be to investigate portfolio decisions for the household, with different assets facing different tax and return risks.

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5. APPENDIX

5.1 LIST OF VARIABLES

θ additive tax parameter

p multiplicative tax parameter for comparative statics exercises

γ^i the stochastic tax variable, the superscript denotes that it is individual specific

$E[\gamma^i] = \bar{\gamma}$, (non-stochastic) is what the government uses,

since there is a large (enough) number of households

g_t government consumption in period t = 1,2

τ_1 first period lump-sum tax

U von Neumann-Morgenstern utility function

U_{ij} partial derivative with respect to elements i, j

c_t private consumption in period t = 1,2

y_t income in period t = 1,2

r interest rate given from world market

a_1 household savings ($a_1 \equiv y_1 - c_1 - \tau_1$)

$f(x)$ density function of variable x defined over X

5.2 THE DERIVATION OF THE PARTIAL DERIVATIVES

The analysis in the following subsections starts from the points in the main text where the systems of equations are presented, and derives the partial derivatives.

5.2.1 THE MEAN PRESERVING SPREAD CASE

Totally differentiate the system in (5) to get

$$\begin{bmatrix} \frac{\partial F_M^1}{\partial \theta} & \frac{\partial F_M^1}{\partial c_1} & \frac{\partial F_M^1}{\partial p} \\ \frac{\partial F^2}{\partial \theta} & \frac{\partial F^2}{\partial c_1} & \frac{\partial F^2}{\partial p} \end{bmatrix} \begin{bmatrix} d\theta \\ dc_1 \\ dp \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad . \tag{16}$$

The Jacobian matrix for this system is

$$\begin{bmatrix} \frac{\partial F_M^1}{\partial \theta} & \frac{\partial F_M^1}{\partial c_1} & \frac{\partial F_M^1}{\partial p} \\ \frac{\partial F^2}{\partial \theta} & \frac{\partial F^2}{\partial c_1} & \frac{\partial F^2}{\partial p} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \bar{\gamma} \\ E[\Phi] & S & E[\gamma^i \Phi] \end{bmatrix}, \tag{17}$$

where $\Phi \equiv r(U_2 - a_1(U_{12} - R^i U_{22}))$, and now with $R^i \equiv 1 + r - r(p\gamma^i + \theta)$.

In order to use the implicit function theorem we have to check that the Jacobian determinant of the endogenous variables is non-zero. Or

$$|J| = \begin{vmatrix} \frac{\partial F_M^1}{\partial \theta} & \frac{\partial F_M^1}{\partial c_1} \\ \frac{\partial F^2}{\partial \theta} & \frac{\partial F^2}{\partial c_1} \end{vmatrix} = S \neq 0 \quad , \tag{18}$$

which is satisfied if we have a well defined optimization, because then $S < 0$, since this is the second order condition for a maximum.

Proceed by dividing both equations in the differentiated system by ∂p to obtain the system of equations [5]

$$\begin{bmatrix} \frac{\partial F_M^1}{\partial \theta} & \frac{\partial F_M^1}{\partial c_1} \\ \frac{\partial F^2}{\partial \theta} & \frac{\partial F^2}{\partial c_1} \end{bmatrix} \begin{bmatrix} \frac{\partial \theta}{\partial p} \\ \frac{\partial c_1}{\partial p} \end{bmatrix} = \begin{bmatrix} -\frac{\partial F_M^1}{\partial p} \\ -\frac{\partial F^2}{\partial p} \end{bmatrix}. \tag{19}$$

[5] The interpretation of the derivatives will be as partials rather than totals, since the system contains more exogenous variables than p, thus the ”∂ ” rather than ”d” in the remaining analysis.

From this we can solve for the derivatives of interest by using Cramer's rule. First we have the derivative of the additive tax parameter with respect to the multiplicative tax parameter

$$\frac{\partial\theta}{\partial p}=\frac{\begin{vmatrix}-\dfrac{\partial F_M^1}{\partial p} & \dfrac{\partial F_M^1}{\partial c_1}\\ -\dfrac{\partial F^2}{\partial p} & \dfrac{\partial F^2}{\partial c_1}\end{vmatrix}}{|J|}=-\bar{\gamma}\ . \tag{20}$$

Finally, we have the partial derivative between first period consumption and the multiplicative tax parameter

$$\frac{\partial c_1}{\partial p}=\frac{\begin{vmatrix}\dfrac{\partial F_M^1}{\partial\theta} & -\dfrac{\partial F_M^1}{\partial p}\\ \dfrac{\partial F^2}{\partial\theta} & -\dfrac{\partial F^2}{\partial p}\end{vmatrix}}{|J|}=-\frac{1}{S}E\left[\Phi\left(\gamma^i-\bar{\gamma}\right)\right]. \tag{21}$$

5.2.2 THE BUDGET PRESERVING SPREAD CASE

Totally differentiate the system in (10) to get

$$\begin{bmatrix}\dfrac{\partial F_B^1}{\partial\theta} & \dfrac{\partial F_B^1}{\partial c_1} & \dfrac{\partial F_B^1}{\partial p}\\ \dfrac{\partial F^2}{\partial\theta} & \dfrac{\partial F^2}{\partial c_1} & \dfrac{\partial F^2}{\partial p}\end{bmatrix}\begin{bmatrix}d\theta\\ dc_1\\ dp\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\ . \tag{22}$$

The Jacobian matrix for this system is

$$\begin{bmatrix}\dfrac{\partial F_B^1}{\partial\theta} & \dfrac{\partial F_B^1}{\partial c_1} & \dfrac{\partial F_B^1}{\partial p}\\ \dfrac{\partial F^2}{\partial\theta} & \dfrac{\partial F^2}{\partial c_1} & \dfrac{\partial F^2}{\partial p}\end{bmatrix}=\begin{bmatrix}a_1 & -\left(p\bar{\gamma}+\theta\right) & \bar{\gamma}a_1\\ E[\Phi] & S & E\left[\gamma^i\Phi\right]\end{bmatrix}, \tag{23}$$

where $\Phi \equiv r\left(U_2 - a_1\left(U_{12} - R^i U_{22}\right)\right)$.

In order to use the implicit function theorem we have to check that the Jacobian determinant of the endogenous variables is non-zero. Or

$$|J| = \begin{vmatrix} \frac{\partial F_B^1}{\partial \theta} & \frac{\partial F_B^1}{\partial c_1} \\ \frac{\partial F^2}{\partial \theta} & \frac{\partial F^2}{\partial c_1} \end{vmatrix} \neq 0 \ . \tag{24}$$

We have that

$$|J| = \begin{vmatrix} a_1 & -\left(p\bar{\gamma} + \theta\right) \\ E[\Phi] & S \end{vmatrix} \ , \tag{25}$$

which yields the condition

$$|J| = a_1 S + \left(p\bar{\gamma} + \theta\right) E[\Phi] \neq 0 \ . \tag{26}$$

Proceed by dividing both equations in the differentiated system by ∂p to obtain the system of equations

$$\begin{bmatrix} \frac{\partial F_B^1}{\partial \theta} & \frac{\partial F_B^1}{\partial c_1} \\ \frac{\partial F^2}{\partial \theta} & \frac{\partial F^2}{\partial c_1} \end{bmatrix} \begin{bmatrix} \frac{\partial \theta}{\partial p} \\ \frac{\partial c_1}{\partial p} \end{bmatrix} = \begin{bmatrix} -\frac{\partial F_B^1}{\partial p} \\ -\frac{\partial F^2}{\partial p} \end{bmatrix} \ . \tag{27}$$

From this we can again solve for the derivatives of interest by using Cramer's rule. First we have the effect on the additive tax parameter

$$\frac{\partial\theta}{\partial p} = \frac{\begin{vmatrix} -\dfrac{\partial F_B^1}{\partial p} & \dfrac{\partial F_B^1}{\partial c_1} \\ -\dfrac{\partial F^2}{\partial p} & \dfrac{\partial F^2}{\partial c_1} \end{vmatrix}}{|J|} \quad . \tag{28}$$

We get

$$\frac{\partial\theta}{\partial p} = -\bar{\gamma}\,\frac{a_1 S + pE\left[\gamma^i \Phi\right]}{|J|} - \theta\,\frac{E\left[\gamma^i \Phi\right]}{|J|} \quad . \tag{29}$$

Secondly, we have the effect on first period consumption

$$\frac{\partial c_1}{\partial p} = \frac{\begin{vmatrix} \dfrac{\partial F_B^1}{\partial \theta} & -\dfrac{\partial F_B^1}{\partial p} \\ \dfrac{\partial F^2}{\partial \theta} & -\dfrac{\partial F^2}{\partial p} \end{vmatrix}}{|J|} \quad . \tag{30}$$

This can be written as

$$\frac{\partial c_1}{\partial p} = \frac{\bar{\gamma} a_1 E[\Phi] - a_1 E\left[\gamma^i \Phi\right]}{|J|} = -\frac{a_1}{|J|}\,E\left[\Phi\left(\gamma^i - \bar{\gamma}\right)\right] . \tag{31}$$

5.3 CONDITION FOR $|J| > 0$ WITH TIME-SEPARABLE ISO-ELASTIC UTILITY

We want to investigate if the Jacobian determinant could be larger than zero, or

$$|J| = a_1 E\left[U_{11} - 2R^i U_{12} + R^{i^2} U_{22}\right] + \left(p\bar{\gamma} + \theta\right) r E\left[U_2 - a_1\left(U_{12} - R^i U_{22}\right)\right] > 0 . \tag{32}$$

To obtain a more transparent condition than we could with a general utility function, we will use a time separable iso-elastic utility function, with the relative risk aversion equal to α, or $U(c_1,c_2)=u(c_1)+\beta u(c_2)$, with $u(c)=c^{1-\alpha}/(1-\alpha)$. The condition then becomes

$$a_1 E\left[-\alpha c_1^{-\alpha-1}-R^{i^2}\alpha\beta c_2^{-\alpha-1}\right]+\left(p\bar{\gamma}+\theta\right)r\beta E\left[c_2^{-\alpha}-a_1 R^i \alpha c_2^{-\alpha-1}\right]>0\ . \tag{33}$$

Use $c_2 = R^i a_1$, the FOC: $c_1^{-\alpha}=\beta E\left[R^i c_2^{-\alpha}\right]$, and then define $\bar{c}_2 \equiv a_1 \bar{R}$, where $\bar{R} \equiv E\left[R^i\right]$ and $\bar{c}_g \equiv \bar{c}_2 / c_1$ to get

$$E\left[-\alpha\beta\bar{c}_g R^i c_2^{-\alpha}-\alpha\beta\bar{R}R^i c_2^{-\alpha}\right]+\bar{R}\left(p\bar{\gamma}+\theta\right)rE\left[\beta c_2^{-\alpha}-\alpha\beta c_2^{-\alpha}\right]>0\ , \tag{34}$$

or equivalently

$$-\alpha\beta\left(\bar{c}_g+\bar{R}\right)E\left[R^i c_2^{-\alpha}\right]+(1-\alpha)\beta\left(p\bar{\gamma}+\theta\right)r\bar{R}E\left[c_2^{-\alpha}\right]>0\ . \tag{35}$$

The first order condition tells us that $E\left[R^i c_2^{-\alpha}\right]>0$, so for $\alpha>1$, the condition can never be fulfilled, and thus we know that for a dominating income effect, the Jacobian determinant will be negative. If, however, $\alpha<1$, the condition can be fulfilled. The first term is still negative in this case, but the second term will be positive. To get a general idea of the magnitude of the two terms, we can use the covariance between the net return on capital and the marginal utility in the second period. We know that the covariance is negative for a risk averse agent, since with a higher return, second period income and consumption is higher, and thus the marginal utility is lower. The covariance is equal to $Cov\left(R^i,c_2^{-\alpha}\right)=E\left[R^i c_2^{-\alpha}\right]-\bar{R}E\left[c_2^{-\alpha}\right]<0$, and thus we know that $\bar{R}E\left[c_2^{-\alpha}\right]>E\left[R^i c_2^{-\alpha}\right]$. If we assume that these two factors are instead equal, we obtain a sufficient, but not necessary condition on α that is

$$\alpha<\frac{r\left(p\bar{\gamma}+\theta\right)}{\bar{c}_g+\bar{R}+r\left(p\bar{\gamma}+\theta\right)}<1\ . \tag{36}$$

The condition contains three variables that are functions of α, and we cannot derive a condition in terms of exogenous parameters with less than deriving an explicit solution to the agent's problem, which we cannot do in general. However, we know that all the included variables are positive, and thus that the expression is positive and less than one. We have therefore showed that there exists an α smaller than one and greater than zero that make the Jacobian determinant positive.

5.4 PROOF OF SIGN OF $E\left[\Phi\left(\bar{\gamma}-\gamma^i\right)\right]$ AND $E[\Phi]$

I will restrict the proof to the case of dominating income effect, but the analysis is completely analogous for the case of dominating substitution effect. In section 2.4 it is stated that for the case of dominating income effect, $E\left[\Phi\left(\bar{\gamma}-\gamma^i\right)\right]>0$ while $E[\Phi]<0$. Sandmo [1970] presents a general proof of the first inequality for a general utility function, but here we also show the connection between the two. In this proof, we will restrict the analysis to a time separable iso-elastic utility function to display how the coefficient of relative risk aversion enters the analysis. Start with the second inequality

$$E[\Phi] \equiv rE\left[U_2 - a_1\left(U_{12} - R^i U_{22}\right)\right] < 0 \ . \tag{37}$$

Use a time separable iso-elastic utility function and $c_2 = a_1 R^i$ to get

$$\beta r E\left[(1-\alpha)c_2^{-\alpha}\right] < 0 \ , \tag{38}$$

which will be satisfied if $\alpha > 1$. We will now show that this is consistent with the first inequality being positive, or

$$E\left[\Phi\left(\bar{\gamma}-\gamma^i\right)\right] \equiv rE\left[\left(U_2 - a_1\left(U_{12} - R^i U_{22}\right)\right)\left(\bar{\gamma}-\gamma^i\right)\right] > 0 \ . \tag{39}$$

With the same utility function we have

$$\beta r E\left[(1-\alpha)c_2^{-\alpha}\left(\bar{\gamma}-\gamma^i\right)\right] , \tag{40}$$

which requires some additional analysis before we know the sign. For $\alpha > 1$ we have that

$$\left(1-\alpha\right)c_2^{-\alpha} > \left\{\left(1-\alpha\right)c_2^{-\alpha}\right\}_{\bar{\gamma}} \quad \text{if } \bar{\gamma} > \gamma^i \ . \tag{41}$$

Further $\left(\bar{\gamma}-\gamma^i\right)>0$ if $\bar{\gamma}>\gamma^i$, so multiplying both sides of the equation will not change the inequality, and thus

$$\left(1-\alpha\right)c_2^{-\alpha}\left(\bar{\gamma}-\gamma^i\right) > \left\{\left(1-\alpha\right)c_2^{-\alpha}\right\}_{\bar{\gamma}}\left(\bar{\gamma}-\gamma^i\right) \quad \text{if } \bar{\gamma} > \gamma^i \ . \tag{42}$$

Taking expectations on both sides and noting that the factor in curly braces on the left hand side is non-stochastic we have

$$E\left[\left(1-\alpha\right)c_2^{-\alpha}\left(\bar{\gamma}-\gamma^i\right)\right] > \left\{\left(1-\alpha\right)c_2^{-\alpha}\right\}_{\bar{\gamma}} E\left[\bar{\gamma}-\gamma^i\right] = 0 \quad \text{if } \bar{\gamma} > \gamma^i \ . \tag{43}$$

To show that the left hand side is positive, it is enough to show that the right hand side is positive (or non-negative). If we write out the second factor we get $\bar{\gamma}-E\left[\gamma^i\right]=0$. We have thus shown that the right hand side is zero, implying that the left hand side is positive, which is what we wanted to prove.

The proof was constructed for $\bar{\gamma}>\gamma^i$, but will hold also for a realized tax rate that is greater than the expected, since then the inequality in (41) is reversed, but will be reversed back after multiplying with $\left(\bar{\gamma}-\gamma^i\right)<0$, so the inequality in (42) will still be valid.

5.5 Condition for $\partial\theta/\partial p > 0$

Start by writing the derivative as

$$\frac{\partial\theta}{\partial p} = -\frac{a_1 S\bar{\gamma} + \left(p\bar{\gamma}+\theta\right)E\left[\gamma^i\Phi\right]}{a_1 S + \left(p\bar{\gamma}+\theta\right)E\left[\Phi\right]} \ , \tag{44}$$

which will always be negative if the numerator and denominator have the same sign. This will for example be the case when the income effect dominates, since then $E[\Phi]<0$, which implies that both the numerator and denominator is negative.

However, when the substitution effect dominates, we have a possibility that the derivative is positive, if the tax base has eroded enough. This is the case when the numerator is positive, while the numerator is still negative. To make the numerator positive, the second term has to dominate the first, which is always negative. At the same time, the second term in the denominator is not allowed to dominate the first. Furthermore, we know that when the denominator, the Jacobian determinant, is positive, we have the case where the substitution effect is so strong that it makes first period consumption decrease with increased risk. In that case, we know that the derivative between the two tax parameters has to be negative, since the tax base has then become larger. In other words, to obtain a positive derivative between the two tax parameters, the substitution effect has to be strong, but it cannot be too strong.

5.6 DETERMINISTIC TAXES

We begin our analysis by investigating the effect that a purely deterministic tax rate has on consumption, without involving the government's budget constraint. This is basically the same as conducting a comparative statics analysis of the interest rate's effect on consumption paths. As a starting point for the main analysis, it will be useful to know what the effect on first period consumption is from an increase in a *known* tax rate. The experiment will be conducted as simply as possible, by only performing a comparative statics exercise on the individual's problem. Let the individual solve the problem

$$\begin{aligned} &\max_{c_1, c_2} U(c_1, c_2) \\ &\text{s.t.} \quad c_1 = y_1 - \tau_1 - a_1 \\ &\qquad\;\; c_2 = y_2 + (1 + r - \theta r)(y_1 - c_1 - \tau_1) \,, \end{aligned} \tag{45}$$

where θ is the deterministic tax rate on capital income. The first and second order conditions are then

FOC: $$F \equiv U_1 - (1 + r - \theta r)U_2 = 0 \tag{46}$$

SOC: $$S \equiv U_{11} - 2(1 + r - \theta r)U_{12} + (1 + r - \theta r)^2 U_{22} < 0 \,. \tag{47}$$

Define $R \equiv 1 + r - \theta r$, and differentiate the FOC with respect to the deterministic tax parameter to obtain

$$\frac{\partial F}{\partial \theta} = \quad U_{11}\frac{\partial c_1}{\partial \theta} - U_{12}\left(R\frac{\partial c_1}{\partial \theta} + ra_1\right) + rU_2 - RU_{21}\frac{\partial c_1}{\partial \theta} + RU_{22}\left(R\frac{\partial c_1}{\partial \theta} ra_1\right) = 0 \ . \tag{48}$$

Rewrite to get the partial derivative of interest

$$\frac{\partial c_1}{\partial \theta} = -\frac{r}{S}\left[\underbrace{U_2}_{+} \underbrace{-a_1\left(U_{12} - RU_{22}\right)}_{-}\right] \ . \tag{49}$$

If the first term within brackets dominates the second, the sign of the derivative is positive, while if the second term dominates, the expression is negative.

5.7 SANDMO'S GENERAL CAPITAL INCOME RISK

This section presents the analysis in Sandmo [1970] with the general approach and notation used in this paper. The two equations now represent a mean preserving spread in capital income and the first order condition of the households respectively.

Mean preserving spread

The idea is that the expected value of the return on capital should be kept constant (at κ), so that

$$F^1 \equiv E\left[px^i + \theta\right] = p\bar{x} + \theta - \kappa = 0 \ , \tag{50}$$

where $\bar{x} \equiv E\left[x^i\right]$.

Households

The households are expected utility maximizers and solve

$$\max_{c_1,c_2} E\left[U\left(c_1,c_2\right)\right] = \int_X U\left(c_1,c_2\right) f\left(x^i\right) dx^i$$
$$\text{s.t.} \quad c_1 = y_1 - a_1 \tag{51}$$
$$c_2 = y_2 + (1 + \left(px^i + \theta\right)) a_1 \quad ,$$

which gives rise to the first and second order conditions

FOC: $$F^2 \equiv E\left[U_1 - R^i U_2\right] = 0 \tag{52}$$

SOC: $$S \equiv E\left[U_{11} - 2R^i U_{12} + {R^i}^2 U_{22}\right] < 0 \ , \tag{53}$$

where $R^i \equiv 1 + \left(px^i + \theta\right)$.

The first order condition is the second equation in our system, so now we have the system of equations

$$\begin{cases} F^1 = 0 \\ F^2 = 0 \ . \end{cases} \tag{54}$$

If we totally differentiate the system, the Jacobian matrix is

$$\begin{bmatrix} \dfrac{\partial F^1}{\partial \theta} & \dfrac{\partial F^1}{\partial c_1} & \dfrac{\partial F^1}{\partial p} \\ \dfrac{\partial F^2}{\partial \theta} & \dfrac{\partial F^2}{\partial c_1} & \dfrac{\partial F^2}{\partial p} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \bar{x} \\ E[\Lambda] & S & E\left[x^i \Lambda\right] \end{bmatrix}, \tag{55}$$

where $\Lambda \equiv -\left[U_2 - a_1\left(U_{12} - R^i U_{22}\right)\right]$.

In order to use the implicit function theorem we have to check that the Jacobian determinant of the endogenous variables is non-zero. Or

$$|J| = \begin{vmatrix} \dfrac{\partial F^1}{\partial \theta} & \dfrac{\partial F^1}{\partial c_1} \\ \dfrac{\partial F^2}{\partial \theta} & \dfrac{\partial F^2}{\partial c_1} \end{vmatrix} = S \neq 0 \ , \tag{56}$$

which is satisfied if we have a well defined optimization, since then $S < 0$.

Proceed by dividing both equations by ∂p to obtain the system of equations.

$$\begin{bmatrix} \frac{\partial F^1}{\partial \theta} & \frac{\partial F^1}{\partial c_1} \\ \frac{\partial F^2}{\partial \theta} & \frac{\partial F^2}{\partial c_1} \end{bmatrix} \begin{bmatrix} \frac{\partial \theta}{\partial p} \\ \frac{\partial c_1}{\partial p} \end{bmatrix} = \begin{bmatrix} -\frac{\partial F^1}{\partial p} \\ -\frac{\partial F^2}{\partial p} \end{bmatrix} . \tag{57}$$

From this we can solve for the derivatives of interest by using Cramer's rule. We start by investigating how the additive return parameter, θ, has to change when p changes, in order to satisfy the mean preserving spread condition.

$$\frac{\partial \theta}{\partial p} = \frac{\begin{vmatrix} -\frac{\partial F^1}{\partial p} & \frac{\partial F^1}{\partial c_1} \\ -\frac{\partial F^2}{\partial p} & \frac{\partial F^2}{\partial c_1} \end{vmatrix}}{|J|} = -\bar{x} . \tag{58}$$

This condition could of course equally well be obtained by total differentiation of the first equation, which is the way it is done in Sandmo. Our unnecessarily tedious approach is used to conform to the steps needed in the following sections.

The next step is to investigate how consumption changes in response to a mean preserving spread. We have that

$$\frac{\partial c_1}{\partial p} = \frac{\begin{vmatrix} \frac{\partial F^1}{\partial \theta} & -\frac{\partial F^1}{\partial p} \\ \frac{\partial F^2}{\partial \theta} & -\frac{\partial F^2}{\partial p} \end{vmatrix}}{|J|} , \tag{59}$$

or,

$$\frac{\partial c_1}{\partial p} = -\frac{1}{S} E\left[\Lambda\left(x^i - \bar{x}\right)\right] = \underbrace{\frac{1}{S} E\left[U_2\left(x^i - \bar{x}\right)\right]}_{+} \underbrace{- \frac{1}{S} a_1 E\left[\left(U_{12} - R^i U_{22}\right)\left(x^i - \bar{x}\right)\right]}_{-} . \tag{60}$$

The first term, the substitution effect, can be shown to be negative if risk aversion is assumed. The second term, the income effect, will be negative if decreasing temporal risk aversion is assumed. Combining these assumptions renders the total effect an ambiguous sign. Proofs of these statements could be found in Sandmo’s [1970] original paper.