The First-Best Sharing Rule in the Continuous-Time Principal-Agent Model with Exponential Utility

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Working Paper Series in Economics and Finance, No 145
December 1996

Abstract

The continuous-time principal-agent model with exponential utility developed by Holmström and Milgrom (1987) and generalized by Schättler and Sung (1993, 1996) and Sung (1995) admits a simple closed-form solution: The second-best sharing rule is linear in output. Unfortunately, the first-best sharing rule has never been derived. In this note, we show that the first-best sharing rule is also linear in output, which fits in nicely with an analogous result from static risk-sharing theory. In addition, we show that the slope is equal to the principal's share of total absolute risk-aversion. This result is consistent with Borch's (1962) fundamental theorem of Pareto-optimal risk-sharing.

JEL Classification Number: D82.
Keywords: Moral Hazard, Continuous-Time Principal-Agent Problem.

[†] This is a revised version of a chapter of an earlier paper entitled "Randomization in Dynamic Principal-Agent Problems," Working Papers in Economics E-96-5, Hoover Institution, Stanford. Financial support from Schweizerischer Nationalfonds and the Stockholm School of Economics is gratefully acknowledged. I would like to thank Heinz Müller for helpful comments and suggestions.

1. Introduction

The continuous-time principal-agent model with exponential utility was developed by Holmström and Milgrom (1987) and later generalized by Schättler and Sung (1993, 1996) and Sung (1995). In this model, an agent continuously controls the drift of a Brownian motion during a given time period. Unlike the static principal-agent problem, the continuous-time version admits a simple closed-form solution: The second-best sharing rule is a linear function of aggregated output.[1] Unfortunately, the first-best sharing rule has never been derived. As is shown here, the first-best sharing rule is also a linear function of aggregated output. This fits in nicely with a well-known result from static risk-sharing theory which says that exponential utility (or, more generally, HARA utility) yields linear sharing rules. Moreover, it is shown that the slope of the first-best sharing rule is equal to the principal's share of total absolute risk-aversion. This finding is consistent with the fundamental theorem of Pareto-optimal risk-sharing derived by Borch (1962).

The remainder of this note is organized as follows: Section 2 presents the continuous-time principal-agent model with exponential utility and discusses the second-best solution derived by Holmström and Milgrom (1987) and Schättler and Sung (1993). Section 3 then examines the first-best problem and states the optimal sharing rule. A detailed proof is subsequently given in the Appendix.

2. The Model and Second-Best Sharing Rule

For ease of exposition, we confine ourselves to the case of one-dimensional Brownian motion. The notation is primarily adopted from Schättler and Sung (1993). At time *0*, the principal and the agent agree on a sharing rule $S: C \to R$ which specifies a payment from the principal to the agent at time *1*. The sharing rule may depend on a stochastic output process X defined on the time interval $[0,1]$ which satisfies $X(0) = 0$ and which is publicly observable. Formally, the process X is governed by a stochastic differential equation of the form

$$dX(t) = f(u)dt + \sigma dB(t), \tag{2.1}$$

where $f(u)$ is the instantaneous mean, $u = u(t, X)$ is the agent's control at time t, σ is the diffusion rate, and B is a standard Brownian motion. Let $(\Omega, \mathcal{F}, P)$ be the underlying probability space. The control $u: [0,1] \times C \to U$ is an $\mathcal{F}_t$-predictable

[1] See, however, Schättler and Sung (1996) and Müller (1996) for a critique of the linearity result.

process with values in some open bounded control set $U \subset R_+$. That is, the agent's control can be revised continuously during the time interval $[0,1]$ and may depend on the history of the process process X in $[0,t]$, but not on the future $(t,1]$. Denote the set of all such control laws by Ξ. The "production function" $f:U \to R$ is bounded with derivatives $f'(u) > 0$ and $f''(u) < 0$, and the diffusion rate σ lies in some bounded subset of R_{++}. The agent incurs a monetary cost at a rate of $c(u)$, where $c:U \to R_+$ is bounded with derivatives $c'(u) > 0$ and $c''(u) \geq 0$. Principal and agent both have negative exponential von Neumann-Morgenstern utility with constant coefficient of absolute risk aversion R and r, respectively.

The principal's problem is to find a sharing rule $S(X)$ and a control u which maximize her expected utility. Under moral hazard, the principal cannot directly observe the agent's control. In order to ensure that the agent exerts u in equilibrium, the tuple $\{S(X),u\}$ must be incentive compatible. Formally, the incentive compatibility constraint requires that, given the sharing rule $S(X)$, u be a solution to the agent's maximization problem. In addition, the solution $\{S(X),u\}$ must also satisfy the participation constraint, which places a lower bound on the agent's expected utility from participating. Let the principal receive the end-of-period output $X(1)$, and let $W_A \in R_+$ denote the agent's certainty equivalent at time *0*. The principal solves

$$\max_{S(X),u} E\left[-\exp\left\{-R\left(X(1) - S(X)\right)\right\}\right] \tag{2.2}$$

$$\text{s.t.} \quad dX(t) = f(u)dt + \sigma dB(t), \tag{2.3}$$

$$E\left[-\exp\left\{-r\left(S(X) - \int_0^1 c(u)dt\right)\right\}\right] \geq -\exp\{-rW_A\}, \text{ and} \tag{2.4}$$

$$u \in \arg\max_{\hat{u} \in \Xi} E\left[-\exp\left\{-r\left(S(X) - \int_0^1 c(\hat{u})dt\right)\right\}\right]. \tag{2.5}$$

Holmström and Milgrom (1987) and Schättler and Sung (1993) suggest the following solution to the principal's second-best problem (2.2)-(2.5):

$$\begin{aligned} S^*(X) = W_A &+ \int_0^1 c(u^*)dt + \int_0^1 \frac{c'(u^*)}{f'(u^*)}\left(dX(t) - f(u^*)dt\right) \\ &+ \frac{r}{2}\int_0^1 \left(\frac{c'(u^*)}{f'(u^*)}\sigma\right)^2 dt \,. \end{aligned} \tag{2.6}$$

The second-best sharing rule (2.6) can be interpreted as follows: The first two terms in (2.6) compensate the agent for his opportunity cost of participating and for the control exerted during the time period $[0,1]$. The third term can be written as two separate integrals: The first integral provides the agent with proper incentives, and the integral subtracted from it constitutes the expected value of the first integral. Observe that the entire third term has mean *0*. In addition, Schättler and Sung (1993) interpret the third term as a compensation error which arises because the sharing rule is based on output X rather than on the agent's control u. The final term constitutes a risk pemium to be paid to the agent for accepting a lottery with mean *0*. Note that if $r = 0$, i.e. if the agent is risk-neutral, this term vanishes. It turns out that the optimal second-best control u^* is constant over time, which implies that the definite stochastic integral can be integrated straightforwardly. The resulting sharing rule is an affine function of end-of-period output $X(1)$ alone with constant coefficient $c'(u^*)/f'(u^*)$, i.e. it has the shape $S^*(X) = \alpha + \beta X(1)$, where the coefficient $\beta = c'(u^*)/f'(u^*)$ reflects the incentive intensity. This is the well-known linearity result.

3. The First-Best Sharing Rule

The principal's first-best problem consists only of equations (2.2)-(2.4). The incentive compatibility constraint (2.5) is no longer needed, since in a first-best setting, the agent's control is directly observable by the principal. Moreover, it is assumed that the principal can costlessly enforce u^* by inflicting an effective punishment upon the agent in case he deviates. Formally, the principal solves

(3.1) $$\max_{S(X),u} E\left[-exp\left\{-R\left(X(1) - S(X)\right)\right\}\right]$$

(3.2) s.t. $dX(t) = f(u)dt + \sigma dB(t)$, and

(3.3) $$E\left[-exp\left\{-r\left(S(X) - \int_0^1 c(u)dt\right)\right\}\right] \geq -exp\{-rW_A\}.$$

This problem can be solved by means of dynamic programming. The solution is given by the following proposition.

PROPOSITION 3.1 (First-Best Sharing Rule): *The solution to the principal's first-best problem* (*3.1*)-(*3.3*) *is*

$$S^*(X) = W_A + \int_0^1 c(u^*)dt + \int_0^1 \frac{R}{R+r}\left(dX(t) - f(u^*)dt\right) + \frac{r}{2}\int_0^1 \left(\frac{R}{R+r}\sigma\right)^2 dt, \tag{3.4}$$

where the first-best control u^* *is constant, unique, and implicitly defined by the efficiency condition* $c'(u^*)/f'(u^*) = 1$.

Proof: See Appendix.

The first two terms in (3.4) and (2.6) are identical. As in (2.6), the third term is stochastic and varies with output. In the absence of moral hazard, this output-dependence is solely due to risk-sharing. This qualifies a claim made in Schättler and Sung (1993) that the third term in the second-best sharing rule (2.6) represents a compensation error resulting from the non-observability of effort. In the light of (3.4), it is clear that output-dependence in (2.6) is driven by both the provision of incentives *and* risk-sharing. The fourth term in (3.4) constitutes a risk premium which is to be paid to the agent for accepting a lottery with mean *0*. The proof of Proposition 3.1 shows that the first-best control u^* is constant, which implies that the stochastic integral in (3.4) can be solved straightforwardly. Integration of (3.4) then gives $S^*(X) = \alpha + \beta X(1)$, i.e. the first-best sharing rule is a linear function of aggregated output alone.

While conceptually similar, the sharing rules (2.6) and (3.4) differ in one important aspect: In (2.6), the constancy of the second-best control u^* drives $X(1)$-dependence since it allows direct integration of the stochastic integral. In contrast to this, $X(1)$-dependence of the sharing rule (3.4) does not arise from the constancy of the first-best control u^*. This is simply because the first-best control is no longer part of the stochastic integral. Intuitively, letting $S^*(X)$ depend on the entire process X does not improve the first-best solution since per definition, shirking during the time period $[0,1]$ is not a problem.

The Appendix shows that the slope of the first-best sharing rule is equal to $\left(R/(R+r)\right)\left(c'(u^*)/f'(u^*)\right)$. From Pareto-efficiency, it follows that marginal cost $c'(u^*)$ equals marginal productivity $f'(u^*)$, which is why this term eventually reduces to $R/(R+r)$. This is summarized in the following corollary.

COROLLARY 3.1 (Efficient Risk-Sharing): *The first-best sharing rule* (*3.4*) *implies efficient risk-sharing, i.e.* $\partial S(X)/\partial X = R/(R+r)$, *where* $R/(R+r)$ *is the*

principal's share of total absolute risk aversion. In particular, if $R = 0$*, all risk is borne by the principal.*

Proof: Integrate (3.4) and take derivatives with respect to X.

Q.E.D.

REMARK: Corollary 3.1 is consistent with Borch's (1962) fundamental theorem of Pareto-optimal risk-sharing.

Appendix

Proof of Proposition 3.1: We solve the principal's first-best problem (3.1)-(3.3) by means of dynamic programming. The principal's value function is

$$V(t,X) \equiv \max_{S(X),u} E\left[\left\{exp\left(-R\left(-dS(X)\right)\right)\right\}V(t+dt,X)\right], \tag{A.1}$$

where $V(\cdot)$ is of the form $-exp\{\cdot\}$. By Ito's Lemma, $dS(X)$ can be expressed as

$$dS(X) = \frac{\partial S(X)}{\partial X}dX + \frac{1}{2}\frac{\partial^2 S(X)}{\partial X^2}\sigma^2 dt = \alpha dt + \beta dX \ . \tag{A.2}$$

Throughout the proof, terms of order higher than dt are dropped. In addition, we follow the usual convention that $E[dB] = 0$ and $(dB)^2 = dt$. Inserting (A.2) into (A.1) and using Taylor series expansions yields

$$\begin{aligned} V(t,X) \equiv \max_{\alpha,\beta,u} E\Bigg[&\left(1 + R(\alpha dt + \beta dX) + \frac{1}{2}R^2(\alpha dt + \beta dX)^2\right) \\ &\left(V(t,X) + \frac{\partial V(t,X)}{\partial t}dt + \frac{\partial V(t,X)}{dX}dX + \frac{1}{2}\frac{\partial^2 V(t,X)}{\partial X^2}\sigma^2 dt\right)\Bigg]. \end{aligned} \tag{A.3}$$

Multiplying out (A.3), taking expectations and rearranging terms, we obtain the following nonlinear second-order partial differential equation:

$$\begin{aligned} 0 \equiv &\frac{\partial V(t,X)}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V(t,X)}{\partial X^2}\sigma^2 dt \\ &+ \max_{\alpha,\beta,u}\left[\frac{\partial V(t,X)}{\partial X}\left(f(u) + R\beta\sigma^2\right)dt + RV(t,X)\left(\beta f(u) + \alpha + \frac{R}{2}\beta^2\sigma^2\right)dt\right] \end{aligned} \tag{A.4}$$

with terminal condition $V(1,X) \equiv -exp\{-RX(1)\}$.

Any solution pair $\{S(X),u\}$ must satisfy the agent's participation constraint (3.3). The intertemporal version of (3.3) is

(A.5) $$E\left[-exp\left\{-r\left(dS(X)-c(u)dt\right)\right\}\right]\geq -1$$

for all dt in $[0,1]$. It can be shown that (A.5) implies (3.3) by integrating (A.5) over $[0,1]$ and defining $S(0,X)\equiv W_A$. Expanding (A.5) and taking expectations, we have

(A.6) $$\alpha^* = c(u)-\beta f(u)+\frac{r}{2}(\beta\sigma)^2 .$$

The agent's participation constraint (3.3) is satisfied if (A.6) holds. Hence, by choosing α^* according to (A.6), the sole constraint in the principal's first-best problem is satisfied. What remains is an unconstrained maximization problem with respect to both β and u. The solution value u^* is called the first-best control, and the solution value β^* determines by how much the agent's compensation $S(X)$ varies with output X. In the absence of further constraints, the solution value β^* implies first-best risk-sharing. This is in contrast to the moral hazard problem (2.2)-(2.5), in which both parameters α^* and β^* are used up to satisfy the participation and incentive compatibility constraints, respectively (cf. Schättler and Sung (1996)).

Inserting (A.6) into (A.4) and collecting terms gives

(A.7) $$\begin{aligned} 0 \equiv\ & \frac{\partial V(t,X)}{\partial t}dt+\frac{1}{2}\frac{\partial^2 V(t,X)}{\partial X^2}\sigma^2 dt \\ & + \max_{\beta,u}\left[\frac{\partial V(t,X)}{\partial X}\left(f(u)+R\beta\sigma^2\right)dt + RV(t,X)\left(c(u)+\frac{(R+r)}{2}\beta^2\sigma^2\right)dt\right] \end{aligned}$$

with terminal condition $V(1,X)\equiv -exp\{-RX(1)\}$. The partial differential equation (A.7) can be expressed as a partial differential equation for the principal's certainty equivalent by defining $V(t,X)\equiv -exp\{-RW(t,X)\}$. Inserting $V(t,X)\equiv -exp\{-RW(t,X)\}$ into (A.7), we get

(A.8) $$\begin{aligned} 0 \equiv\ & \frac{\partial W(t,X)}{\partial t}dt+\frac{1}{2}\frac{\partial^2 W(t,X)}{\partial X^2}\sigma^2 dt-\frac{R}{2}\left(\frac{\partial W(t,X)}{\partial X}\right)^2\sigma^2 dt \\ & + \max_{\beta,u}\left[\frac{\partial W(t,X)}{\partial X}\left(f(u)+R\beta\sigma^2\right)dt - c(u)dt-\frac{(R+r)}{2}\beta^2\sigma^2 dt\right] \end{aligned}$$

with terminal condition $W(1,X)\equiv X(1)$.

The first-order conditions for the principal's maximization problem in (A.8) are

(A.9) $$\frac{\partial W(t,X)}{\partial X}=\frac{c'(u^*)}{f'(u^*)}, \qquad \text{and}$$

(A.10) $$\beta^* = \frac{\partial W(t,X)}{\partial X}\frac{R}{R+r}.$$

As is shown in (A.17), $\partial W(t,X)/\partial X$ is constant. Moreover, R, r and σ^2 are constant, $c(u)$ is convex, and $f(u)$ is strictly concave. From this it follows that the principal's maximization problem (A.8) is strictly concave in both β and u, which implies that the first-order conditions (A.9) and (A.10) are sufficient conditions for a unique global maximum. Inserting (A.9) and (A.10) back in (A.8), dividing through by dt and collecting terms, we obtain

$$\begin{aligned} 0 \equiv\ & \frac{\partial W(t,X)}{\partial t} + \frac{1}{2}\frac{\partial^2 W(t,X)}{\partial X^2}\sigma^2 - \frac{R}{2}\left(\frac{\partial W(t,X)}{\partial X}\right)^2\sigma^2 \\ & + \frac{1}{2}\frac{c'(u^*)}{f'(u^*)}\left(\frac{c'(u^*)}{f'(u^*)}\frac{R^2}{R+r}\sigma^2 + 2f(u^*)\right) - c(u^*) \end{aligned} \tag{A.11}$$

with terminal condition $W(1,X) \equiv X(1)$.

The partial differential equation (A.11) can be solved by separation of variables. A standard guess is a solution of the form $W(t,X) \equiv \omega(t)X^2 + \xi(t)X + \psi(t)$ with terminal conditions $\omega(1) = 0$, $\xi(1) = 1$ and $\psi(1) = 0$. Inserting this into (A.11) and rearranging terms, we have

$$\begin{aligned} 0 \equiv\ & X^2\{\omega'(t) - 2R\sigma^2\omega^2(t)\} + X\{\xi'(t) - 2R\omega(t)\xi(t)\} + \psi'(t) + \omega(t)\sigma^2 - \frac{R}{2}\xi^2(t)\sigma^2 \\ & + \frac{1}{2}\frac{c'(u^*)}{f'(u^*)}\left(\frac{c'(u^*)}{f'(u^*)}\frac{R^2}{R+r}\sigma^2 + 2f(u^*)\right) - c(u^*). \end{aligned} \tag{A.12}$$

Comparing terms of equal powers gives rise to the following system of ordinary differential equations:

$$0 \equiv \omega'(t) - 2R\omega^2(t)\sigma^2, \tag{A.13}$$

$$0 \equiv \xi'(t) - 2R\omega(t)\xi(t)\sigma^2, \text{ and} \tag{A.14}$$

$$0 \equiv \psi'(t) + \omega(t)\sigma^2 - \frac{R}{2}\xi^2(t)\sigma^2 + \frac{1}{2}\frac{c'(u^*)}{f'(u^*)}\left(\frac{c'(u^*)}{f'(u^*)}\frac{R^2}{R+r}\sigma^2 + 2f(u^*)\right) - c(u^*) \tag{A.15}$$

with terminal conditions $\omega(1) = 0$, $\xi(1) = 1$ and $\psi(1) = 0$, respectively.

In (A.13), $\omega(1) = 0$ implies $\omega(t) \equiv 0$. Inserting $\omega(t) \equiv 0$ and $\xi(1) = 1$ in (A.14) gives $\xi(t) \equiv 1$. Similarly, inserting both $\omega(t) \equiv 0$ and $\xi(t) \equiv 1$ in (A.15), we obtain the definite solution $\psi(t) = \left\{(R/2)\sigma^2 - (1/2)\left(c'(u^*)/f'(u^*)\right)\left[\left(c'(u^*)/f'(u^*)\right)\left(R^2/R+r\right)\sigma^2 + 2f(u^*)\right] + c(u^*)\right\}(t-1)$. And finally, inserting all three solutions in $W(t,X) = \omega(t)X^2 + \xi(t)X + \psi(t)$, we have

$$W(t,X) = X + \left\{\frac{R}{2}\sigma^2 - \frac{1}{2}\frac{c'(u^*)}{f'(u^*)}\left(\frac{c'(u^*)}{f'(u^*)}\frac{R^2}{R+r}\sigma^2 + 2f(u^*)\right) + c(u^*)\right\}(t-1). \tag{A.16}$$

Differentiating (A.16) with respect to X gives

(A.17) $$\frac{\partial W(t,X)}{\partial X} = 1 .$$

Inserting (A.17) into (A.9) yields the efficiency condition $c'(u^*) / f'(u^*) = 1$, which implicitly defines the first-best control u^*. Furthermore, this condition shows i) that u^* is constant, for neither $c(u)$ nor $f(u)$ depend on t, and ii) that u^* is unique, because $c(u)$ is convex and $f(u)$ is strictly concave. Inserting (A.17) into (A.10) yields

(A.18) $$\beta^* = \frac{R}{R+r},$$

and using this result and the first-best control u^* in (A.6), we obtain

(A.19) $$\alpha^* = c(u^*) - \frac{R}{R+r} f(u^*) + \frac{r}{2}\left(\frac{R}{R+r}\sigma\right)^2 .$$

Finally, inserting both (A.18) and (A.19) in (A.2), integrating over $[0,1]$ and setting $S(0,X) \equiv W_A$, we have

(A.20) $$S^*(X) = W_A + \int_0^1 c(u^*)dt + \int_0^1 \frac{R}{R+r}\left(dX(t) - f(u^*)dt\right) + \frac{r}{2}\int_0^1 \left(\frac{R}{R+r}\sigma\right)^2 dt,$$

which completes the proof.

Q.E.D.

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