Hierarchical Assignments : Stability and Fairness

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Abstract

We study a simple model of the job market, where workers are assigned to employers. We specify conditions under which the market is hierarchical in a natural sense. For such hierarchies, we can state explicit values for the earnings in the worker-optimal and employer-optimal solutions. This is a discrete analogue to the Ricardian differential rent model of Sattinger (1979). We discuss the compatibility problems between fairness and stability of earnings and assignments and argue for a certain solution.

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1 Introduction

Assignment games, such as the one of Shapley and Shubik (1972), are natural game-theoretic models of many real markets, including auctions and the job market. There are mathematical methods for analyzing the assignment games, but in general they are deterrently complex and give little intuitive feeling for what is going on. For instance, the problem of finding a stable outcome that can be perceived as "fair" is hard to attack.

We present here a special form of assignment games called "hierarchies" that perhaps captures the essence of many real situations. In this special case, we get explicit solutions that provide intuitively tractable information. In particular, we suggest a solution to the problem of finding a "fair outcome".

1.1 Theoretical framework

Most of economic science is about relations between aggregates, and the individual is portrayed as the typical consumer, producer, worker, employer, etc. However, one area where ordering of individuals is crucial is welfare economics, where a tool of analysis is obtained from ordering individuals according to their income into an income distribution.

Another field of research where ordering of individuals is common is in the economics of the organisation, where little can be explained without the concept of hierarchy. Furthermore, the old problem of assigning plants to locations (Koopmans and Beckmann,1957) has found new adherants in game theorists, studying matching problems between men and women, or workers and employers (Roth and Sotomayor, 1990). We will focus on the notion of *stability* of assignments, which is central in Roth and Sotomayor's book. For an excellent survey article on assignment models, cf. Sattinger (1993).

The "O-Ring Model" of Kremer (1993) has parallels to our study. Kremer studies the tendency of individuals of the same level of skills to cluster into cooperative units. For empirical evidence, cf. Kremer and Maskin (1996). For clustering in marriage markets, see Bloch and Ryder (1994), Burdett and Coles (1996) and Edlund (1996), to mention a few of the latest contributions.

1.2 Hierarchies

In the present paper we will present and analyze a special case of a very simple model (Shapley and Shubik, 1972) where workers are assigned to employers. Every pair of a worker and an employer has a potential productivity, should they be assigned to each other. If workers have very diverse abilities and employers similarly diverse demands, the productivity of a worker A might be considerably greater at employer E than at employer F while the situation is the opposite for some other worker B. In our special case, which we term a *hierarchy*, we assume that there is an ordering of workers and an ordering of employers so that all workers have the same preferences on employers and vice versa. We also assume a *local* "mating of likes" optimum property, see Section 2.2.

1.2.1 Why hierarchies?

We shall see that when an assignment situation is hierarchical in our sense, several explicit mathematical facts can be stated. To begin with (Section 2.3), there is a simple algorithm for obtaining the optimal assignment (in terms of maximum total productivity): matching the top worker to the top employer, etc., i.e. a global "mating of likes" scheme, as described in Becker (Ch. 4, 1991).

Second (Section 2.4), we show that in a hierarchy, the payoff to a higher ranked worker (or employer) cannot be less than the payoff to a lower ranked worker (employer), in the sense that other solutions are necessarily unstable, i.e. they will break down when agents see they can do better elsewhere.

Third (Section 2.5), and deepest, is a result that states explicit expressions for the optimal stable outcomes of workers and employers respectively. It is well-known that there exists in the assignment model a unique outcome that is best possible for all workers (among all stable solutions), and dually there exists a unique outcome that is best possible for all employers. In general, these solutions are difficult to identify, but for hierarchies we can readily describe them.

1.2.2 When hierarchies?

There are three different answers to the question of when hierarchies arise, depending on context. In a mathematical context, the answer is that a given productivity matrix is hierarchical if it satisfies a certain characterizing criterion (Section 2.6), which also allows us to generate them all.

Let us now answer the question of what models for obtaining the productivity measures (Section 2.7) would yield hierarchies. E.g. if the productivity is the product of some individual ability measures of the worker and the employer, where the measures are independent of the assignment, the productivity matrix is always hierarchical. More generally, given such invariant measures of ability there are several functions other than direct multiplication, such as the Cobb-Douglas formula, that yield hierarchical matrices. But we must stress that the hierarchy concept is in fact more general; there need be no invariant measures of ability in a hierarchy — the crucial point is just the possibility of establishing ranking orders on both employers and workers.

This leads us over to the question hardest to answer: in what practical situations are hierachies plausible? Obviously, it is necessary to restrict the setting to a particular line of trade, since otherwise worker abilities and employer demands are too diverse. Furthermore, it is necessary that employers in this line of trade for some reasons have a ranking order upon which workers agree. A plausible example may be management consultants applying for jobs at the local branches of consultant firms. Allowing for moderate individual variation, it will in general be so that the top-notch firms will, thanks to good reputation, get the biggest and highest-paying commissions. Our paper is, however, a theoretical one and we have not even tried to meet the difficulties of devising or carrying out empirical studies in the matter. This would be the subject of future research.

1.2.3 Fairness of outcome

In Section 3 we look at economic implications of our model, regarding the compatibility between fairness and stability of earnings and assignments. Notions of fairness that may seem sensible often fail to be stable, and are hence irrelevant. For hierarchies, we can present a notion of fairness that does turn out to be stable: an average between the worker-optimal and employer-optimal solutions. This is a fair outcome in the "maxmin" sense of Rawls, and it is also a solution the agents might arrive at in decentralized bargaining (Section 3.1).

The fourth section contains a few concluding remarks on possible uses for our model. In an appendix, we further show that our hierarchy concept is a discrete analogue to the Ricardian differential rent model of Sattinger (1979), and that the latter one can easily be derived from our model.

1.3 Acknowledgment

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2 Hierarchies in the assignment model

We start by recalling the assignment model of Shapley and Shubik (1972). Let $\{p_1, \ldots, p_n\}$ be a set of workers and $\{q_1, \ldots, q_n\}$ a set of employers. Assigning worker p_i to employer q_j is denoted by $p_i \leftrightarrow q_j$ or, equivalently, by $\mu(i) = j$ where μ is a permutation of $\{1, 2, \ldots, n\}$. Every pair (p_i, q_j) has a potential productivity α_{ij}. Let us immediately make a remark on how this differs from standard macroeconomic models, where usually heterogeneity is considered only among workers, not among the employers. The assignment model allows not only for differing productivities among employers, but also for different workers to interact differently with different employers; productivities are not measured independently but only for the pair of interacting agents.

The problem is to find an optimal assignment μ, i.e. an assignment $p_i \leftrightarrow q_{\mu(i)}$ for $i = 1, \ldots, n$ so that the total productivity

$$S_\mu = \sum_{i=1}^{n} \alpha_{i\mu(i)}$$

is maximal.

The total productivity is distributed as *payoffs* u_i to worker p_i and v_i to employer q_i for $i = 1, \ldots, n$. An *outcome* is an assignment combined with a payoff.

Clearly, in an optimal assignment every outcome is Pareto optimal, in that no agent can do better without someone else being worse off.

The outcome is *stable* if it is individually rational (no payoffs are negative) and if it contains no "blocking pairs" of agents that are not matched but who both have an incentive to disrupt their current matches in order to join each other instead. It is well-known that stable outcomes must be Pareto optimal (i.e. the underlying assignment must be optimal) and that no side payments will occur (i.e. in every matched pair $p_i \leftrightarrow q_j$ the payoffs satisfy $u_i + v_j = \alpha_{ij}$, so that the employer and worker share their joint productivity). Therefore, if the agents are numbered such that $p_1 \leftrightarrow q_1, \ldots, p_n \leftrightarrow q_n$ is an optimal assignment, the stable outcomes are given by the payoff vectors $\bar{u}$ and $\bar{v}$ that satisfy:

$$u_i \geq 0, \quad v_i \geq 0, \quad u_i + v_i = \alpha_{ii}, \quad u_i + v_j \geq \alpha_{ij} \text{ for all } i, j = 1, \ldots, n.$$

2.1 Hierarchies

The problem of finding optimal assignments and stable outcomes is relatively complicated. Hence it might be interesting to consider special situations where the solution is easier to come by. We will now discuss a situation that we will call a "hierarchy", in which there is a natural order (rank) of the workers and employers, respectively.

A hierarchy is an instance of the assignment game where the matrix of joint productivities satisfy some simple conditions. The restriction imposed by these conditions enable us to prove some special nice properties of hierarchies, which will be presented below: an easy algorithm for optimal assignment; a payoff hierarchy; and explicit expressions for the payoffs in the solution optimal for workers, as well as for the solution optimal for employers.

The interpretation of the restrictions imposed in the hierarchy concept is that employer and worker productivities are approximately independent in this particular market, as we briefly discussed in the introduction (Section 1.2.2) and return to in the conclusion (Section 4).

2.2 Definition of a hierarchy

The matrix (α_{ij}) of potential productivities is a **hierarchy** if the following two conditions are satisfied for all indices between 1 and n: [1]

1. If $j < k$ then $\alpha_{ij} \leq \alpha_{ik}$ and $\alpha_{ji} \leq \alpha_{ki}$ (i.e. columns and rows are increasing)

2. If $i < k$ and $j < \ell$ then $\alpha_{il} + \alpha_{kj} \leq \alpha_{ij} + \alpha_{k\ell}$ (i.e. in every 2 by 2 submatrix the main diagonal is heavier than the opposite diagonal).

Example. Among the following two matrices,

$$\begin{pmatrix} 2 & 4 & 5 \\ 3 & 6 & 8 \\ 4 & 7 & 10 \end{pmatrix} \text{ is a hierarchy, while } \begin{pmatrix} 2 & 4 & 5 \\ 3 & 7 & 6 \\ 4 & 8 & 10 \end{pmatrix} \text{ is not.}$$

The latter matrix fails in two ways: the second row is not increasing, and in the upper right 2 by 2 submatrix the main diagonal $4 + 6$ is less heavy than

[1] It is an easy exercise to verify that the hierarchy conditions can equivalently be formulated using only local comparisons: 1. $\alpha_{ij} \leq \alpha_{i,j+1}$ and $\alpha_{ij} \leq \alpha_{i+1,j}$. 2. $\alpha_{i,j+1} + \alpha_{i+1,j} \leq \alpha_{ij} + \alpha_{i+1,j+1}$.

the opposite diagonal $5 + 7$. Furthermore, there is no permutation of rows or columns such that it becomes a hierarchy.

In a hierarchy there is an ordering $p_1, \ldots, p_n$ of the workers, and an ordering $q_1, \ldots, q_n$ of the employers Condition 1 says that every worker will be better off cooperating with a high-ranked employer than with a low-ranked one, and vice versa. Condition 2 says that given a local situation of two workers and two employers, the optimal assignment of these is for the high-ranked agents to match with each other, and the low-ranked to match with each other. With reference to the assignment matrix, this could be called the NW-SE max rule.

2.3 The optimal matching algorithm

Let us now prove the first claimed property of hierachies, that the optimal assignment respects the hierarchic order in that we are mating like with like. As it turns out, for this property the second hierarchy condition is sufficient.

Proposition 2.1 *Given a hierarchy, the optimal assignment is $p_i \leftrightarrow q_i$ for all i between 1 and n. Hence the optimal total productivity is $\alpha_{11} + \alpha_{22} + \ldots + \alpha_{nn}$, the trace of the assignment matrix.*

Proof. To prove this, we must show that no other assignment can give higher total productivity. Suppose there is one. Then there is some such assignment with maximal i, where i is defined as the least index such that p_i is matched to q_ℓ for some $\ell > i$, and hence q_i is matched to p_j for some $j > i$. But by the hierarchy assumption, Condition 2, we have

$$\alpha_{i\ell} + \alpha_{ji} \leq \alpha_{ii} + \alpha_{j\ell}$$

and hence by matching $p_i \leftrightarrow q_i$ and $p_j \leftrightarrow q_l$, we can change the assignment to one with at least as great total productivity, but with larger i, contradicting the maximality assumption.

■

Example. For the example of a hierarchy that we saw earlier,

$$(\alpha_{ij}) = \begin{pmatrix} 2 & 4 & 5 \\ 3 & 6 & 8 \\ 4 & 7 & 10 \end{pmatrix},$$

the optimal assignment yields total productivity $2+6+10 = 18$. The other five possible assignment give total productivities $2 + 8 + 7 = 17$, $4 + 3 + 10 = 17$, $4 + 8 + 4 = 16$, $5 + 3 + 7 = 15$ and $5 + 6 + 4 = 15$ respectively.

2.4 The hierarchy is also a payoff hierarchy

Moreover, we claim that independently of which stable outcome we have chosen, the payoffs will mirror the hierarchy. In other words, if worker p_i is higher in the hierarchy than worker p_j, then he must always get at least as high income; similarly for the employers. Here the first hierarchy condition is crucial.

Proposition 2.2 *In a hierarchy, the payoff vectors $\bar{u}$ and $\bar{v}$ will in every stable outcome satisfy $u_1 \leq u_2 \leq \ldots \leq u_n$ and $v_1 \leq v_2 \leq \ldots \leq v_n$.*

Proof. Suppose that the payoff vectors $\bar{u}$ and $\bar{v}$ yield a stable outcome (together with the assignment of p_i to q_i for every i between 1 and n). It is known that in stable outcomes there are no side payments, so $u_i + v_i = \alpha_{ii}$ for all i. Also, stability implies $u_i + v_j \geq \alpha_{ij}$ for all i, j. Now suppose that $i > j$. We have

$$u_i + v_j \geq \alpha_{ij} \Rightarrow u_i + \alpha_{jj} - u_j \geq \alpha_{ij} \Rightarrow u_i - u_j \geq \alpha_{ij} - \alpha_{jj},$$

and $\alpha_{ij} - \alpha_{jj}$ is nonnegative by the first hierarchy condition. Hence $u_i \geq u_j$. In a similar way we get $v_i \geq v_j$.

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2.5 Explicit expressions for optimal outcomes

A further advantage with hierarchies is that they let us state explicit expressions for certain stable outcomes. Shapley and Shubik (1972) showed that the set of stable outcomes has a unique employer-optimal outcome. This means that no other stable outcome is better for any employer q_j. Dually, there is also a unique worker-optimal outcome, which is for every worker p_i the best stable outcome. Since the employers and the workers are sharing the total payoff, it is necessarily so that the employer-optimal outcome is the worst possible outcome for the workers, and vice versa.

Readers familiar with the theory of stable marriages (originally by Gale and Shapley, but see the book of Roth and Sotomayor, 1990) will recognize the concept of a matching that is optimal for one side and worst possible for the other side. However, while for the marriage problem there is a well-known simple algorithm for finding the optimal matchings, in the assignment problem where there is money involved no such simple scheme exists in general. For a hierarchy, though, we can explicitly compute these optimal outcomes.

The key observation is the one made in the proof above, that the payoffs must satisfy

$$u_{i+1} - u_i \geq \alpha_{i+1,i} - \alpha_{i,i}. \tag{1}$$

From this it follows that the the worst possible payoff for the workers cannot be worse for them than the payoff $\bar{u}$ obtained by taking $u_1 = 0$ and then choosing the smallest payoff differences between workers according to the inequality above, i.e. taking $u_{i+1} - u_i = \alpha_{i+1,i} - \alpha_{i,i}$ for $i = 1, \ldots, n-1$. We shall see that this outcome is stable; therefore it is the employer-optimal outcome.

Proposition 2.3 *Suppose that (α_{ij}) is a hierarchy. Then the employer-optimal stable outcome has worker payoff vector $\bar{u}^Q$ given by $u_1^Q = 0$ and*

$$u_i^Q = (\alpha_{i,i-1} - \alpha_{i-1,i-1}) + (\alpha_{i-1,i-2} - \alpha_{i-2,i-2}) + \ldots + (\alpha_{2,1} - \alpha_{1,1})$$

for all $i = 2, \ldots n$. The employer payoff is then $v_i^Q = \alpha_{i,i} - u_i$ for all i between 1 and n.

PROOF. By the foregoing argument, it suffices to show that the payoffs $\bar{u}^Q$ and $\bar{v}^Q$, described in the proposition, give a stable outcome. In other words, we must show that for all i, j we have $u_i^Q + v_j^Q = u_i^Q + \alpha_{jj} - u_j^Q \geq \alpha_{ij}$. We distinguish between two cases: $i > j$ and $i < j$, the case of $i = j$ being obvious.

To begin with, note that for $i > j > k$ we have the following rule:

$$\alpha_{ij} - \alpha_{jj} + \alpha_{jk} = \alpha_{ij} - (\alpha_{jj} - \alpha_{jk}) \geq \alpha_{ij} - (\alpha_{ij} - \alpha_{ik}) = \alpha_{ik}. \tag{2}$$

Now let us treat the case when $i > j$. Then

$$u_i^Q + \alpha_{jj} - u_j^Q = (\alpha_{i,i-1} - \alpha_{i-1,i-1}) + \ldots + (\alpha_{j+2,j+1} - \alpha_{j+1,j+1}) + \alpha_{j+1,j}.$$

By using (2) repeatedly (we show the last repetition below) we get

$$u_i^Q + v_j^Q \geq \ldots \geq \alpha_{i,j+1} - \alpha_{j+1,j+1} + \alpha_{j+1,j} \geq \alpha_{ij}.$$

The case when $i < j$ works in an analogous way.

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For the worker-optimal outcome we can reason analogously; the worst possible payoff for the employers cannot be worse than the payoff v obtained by taking $v_1 = 0$ and the smallest possible payoff differences between employers respecting the inequalities $v_{i+1} - v_i \geq \alpha_{i,i+1} - \alpha_{i,i}$ for $i = 1, \ldots, n-1$. By a completely analogous argument to the proof above we get the dual result.

Proposition 2.4 *Suppose that (α_{ij}) is a hierarchy. Then the worker-optimal stable outcome has payoff vector $\bar{v}^P$ for employers given by $v_1^P = 0$ and*

$$v_i^P = (\alpha_{i-1,i} - \alpha_{i-1,i-1}) + (\alpha_{i-2,i-1} - \alpha_{i-2,i-2}) + \ldots + (\alpha_{1,2} - \alpha_{1,1})$$

for $i = 2, \ldots, n$. The payoff for workers is then $u_i^P = \alpha_{i,i} - v_i$ for all i between 1 and n.

Example. Taking once again the old example,

$$(\alpha_{ij}) = \begin{pmatrix} 2 & 4 & 5 \\ 3 & 6 & 8 \\ 4 & 7 & 10 \end{pmatrix},$$

the worker-optimal outcome has worker payoff $\bar{u}^P = (2, 4, 6)$ and employer payoff $\bar{v}^P = (0, 2, 4)$. In the employer-optimal outcome we get instead payoffs $\bar{u}^Q = (0, 1, 2)$ and $\bar{v}^Q = (2, 5, 8)$.

2.6 Characterization and generation of hierarchical matrices

We will describe a potentially useful characterization of hierarchical matrices, as sums of certain fundamental matrices. Assuming a fixed matrix size n, let M_{ij} denote the n by n matrix that has ones in all positions (i', j') satisfying $i' \geq i, j' \geq j$; all other entries are zero. For example, with $n = 3$ we have

$$M_{12} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, M_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, M_{32} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \text{ etc.}$$

Proposition 2.5 *The set of hierarchical matrices of a given size n is the positive hull of the fundamental matrices M_{ij}, i.e.*

$$\{\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}M_{ij} : m_{ij} \geq 0\}.$$

It is quite obvious that the M_{ij} are themselves hierarchies, and that positive linear combinations of hierarchies can only result in hierarchies. This proves that the above set of matrices contains hierarchies only. For the other direction, that every hierarchy is obtained in this way, we refer to the following algorithm.

Algorithm 2.6 *Any hierarchy (α_{ij}) can be (uniquely) decomposed as a sum $\sum_{i,j} m_{ij}M_{ij}$ in n^2 steps, indexed by (i,j) for $i = 1, \ldots, n$ and $j = 1, \ldots, n$. The algorithm starts by setting $H = (h_{ij}) := (\alpha_{ij})$. In step (i,j) it sets m_{ij} to h_{ij} and resets $H := H - m_{ij}M_{ij}$.*

Example. Our usual example hierarchy is decomposed as follows:

$$\begin{pmatrix} 2 & 4 & 5 \\ 3 & 6 & 8 \\ 4 & 7 & 10 \end{pmatrix} = 2M_{11}+2M_{12}+1M_{13}+1M_{21}+1M_{22}+1M_{23}+1M_{31}+0M_{32}+1M_{33}.$$

The correctness of the algorithm depends on the fact that in each step the remaining matrix H is still a hierarchy, easily verified by the reader.

2.7 Models of hierarchies

As discussed in the introduction (Section 1.2.2), hierarchies are job markets where productivities are approximately independent of the interactions between employers and employees. Let us end this section by assuming an idealized situation where there are indeed assignment-independent competences of workers, $g_1 \leq g_2 \leq \ldots \leq g_n$, and of employers, $k_1 \leq k_2 \leq \ldots \leq k_n$, and discuss a few models for computing the productivity from the measures that will always result in hierarchies.

The simplest model of productitivity is a product rule: $\alpha_{ij} = g_i k_j$. For this case, it is easy to see that the optimal assignment is to match the most competent worker p_n to the most competent employer q_n, worker p_{n-1} to employer q_{n-1} and so on, as stated in Becker (1991).

A generic model has the productivity given by $\alpha_{ij} = f(g_i, k_j)$ where f is some function. This will yield hierarchies under the following liberal conditions: $f(g,k)$ is any twice differentiable function satisfying $\frac{\partial f}{\partial g} \geq 0$, $\frac{\partial f}{\partial k} \geq 0$, $\frac{\partial^2 f}{\partial g \partial k} \geq 0$. [2] We will consider this type of structure in the appendix. This model is generic in the following sense. If a job market has an observed productivity matrix (α_{ij}) which turns out to satisfy the hierarchy conditions, then one can find artificial measures of competence and a twice differentiable function f of the above kind such that this model gives the observed productivity matrix.

[2] Differentiability of $f(g,k)$ is not a necessary condition for a hierarchy; e.g. the famous Leontief function $\alpha_{ij} = \min(g_i, k_j)$ produces a hierarchy.

Another model is a that of competence being measured as an ability vector of m values measuring different aspects, and productivity being given by the scalar product of the ability vectors of worker and employer. If the ability vectors are such that a higher-ranked agent have higher ability than a lower-ranked agent in every component, then we get a hierarchy.

3 Is there a fair stable outcome?

The two most famous guiding principles for choosing outcomes to a game like the assignment game are the *utilitaristic* principle, which tells you to maximimize the overall productivity, and the *egalitaristic* principle which tells you to minimize the differences between the payoffs, i.e. to be as fair as possible (cf. Moulin, 1988). These principles are often in conflict with each other. Our approach, as well as most standard methods in game theory, is utilitaristic since our outcomes maximize the productivity. However, we will now try to analyze how far such outcomes can satisfy the demands of egalitarians.

One property of hierarchies that we have discussed earlier is that every stable outcome $(\bar{u}; \bar{v})$ satisfies

$$u_i - u_j \geq \alpha_{ij} - \alpha_{jj} \geq 0 \quad \text{when } i > j.$$

If $\alpha_{ij} - \alpha_{jj} > 0$ holds, then necessarily the payoff u_i to worker p_i must be greater than the payoff u_j to worker p_j. It follows that stable outcomes cannot be *fair in the sense that all payoffs be equal.* What one can do is to look within the set of feasible solutions, which in our case is the set of stable outcomes, for an outcome that is in some other sense fair. A classical approach is to follow Rawls's "veil of ignorance" argument and look for a stable outcome that will maximize the minimal individual payoff. (cf. Rawls, 1971) There are many such "maxmin" solutions. An advanced method of choosing among them is to find the stable outcome that is optimal in the so called "lexmax" order. (Moulin, 1988) A disadvantage with the latter approach is that the computation of the explicit "lexmax" solution will usually be very complicated.

We will take another approach and present an outcome that can be given explicitly in a simple way and still be regarded as fair in certain senses, e.g. in the "maxmin" sense.

3.1 Finding a decentralized, stable and fair outcome

We know that all stable outcomes have some common characteristics: p_i must be assigned to q_i for all i; the sum of the payoffs u_i and v_i must equal α_{ii}; u_i must be in the interval $[u_i^Q, u_i^P]$ and v_i must be in the interval $[v_i^P, v_i^Q]$, where $(\bar{u}^P; \bar{v}^P)$ is the P-optimal (worker optimal) outcome and $(\bar{u}^Q; \bar{v}^Q)$ is the Q-optimal (employer optimal) outcome.

We shall present our solution as the result of a bargaining between p_i and q_i. We postulate that the bargaining is subject to the following three criteria:

- **Decentralization.** The agents p_i and q_i shall determine p_i's payoff u_i as $u_i = f(u_i^P, u_i^Q, v_i^P, v_i^Q)$ where f is the same function for all pairs p_i, q_i (i.e.

independent of i) and depending on the numbers $u_i^P, u_i^Q, v_i^P, v_i^Q$ only (i.e. no other information about the market is known). This is "decentralized" in the sense that no global information is used, just knowledge of the feasible intervals $[u_i^Q, u_i^P]$ and $[v_i^P, v_i^Q]$.

- **Stability.** The function f must be such that if all pairs share their productivity according to f, then the resulting outcome is stable.
- **Fairness.** The bargaining should result in payoffs u_i and v_i such that this distribution is fair in some intuitive sense.

Two interpretations of the fairness condition would seem to be the most natural ones:

Alt. I. Setting $f(u_i^P, u_i^Q, v_i^P, v_i^Q)$ to the value u_i that minimizes $|u_i - v_i| = |2u_i - \alpha_{ii}|$ under the restriction $u_i^Q \leq u_i \leq u_i^P$ corresponds to a natural interpretation of fairness. However, this function f does not in general yield stability, so it fails the second criterion.

Alt. II. Start by setting $u_i' = u_i - u_i^Q$ and $v_i' = v_i - v_i^P$. The numbers u_i' and v_i' can be said to measure the agents' net profit from bargaining. Now setting u_i' to $(u_i^P - u_i^Q)/2$ and v_i' to $(v_i^Q - v_i^P)/2$, i.e. letting each agent get half his maximum net profit, gives

$$f(u_i^P, u_i^Q, v_i^P, v_i^Q) = (u_i^P + u_i^Q)/2,$$

and this function does give a stable outcome for the following reason. It is well-known and easy to verify that the set of stable outcomes is convex. In particular, this means that every weighted mean

$$(\lambda \bar{u}^Q + (1-\lambda)\bar{u}^P; \lambda \bar{v}^Q + (1-\lambda)\bar{v}^P)$$

is also a stable outcome for any value of the parameter λ between zero and one. Our suggested fair outcome is the result of taking $\lambda = 1/2$ in this expression.

As a result of our investigations, we suggest that u_i is set to $(u_i^P + u_i^Q)/2$ and that v_i is set to $(v_i^P + v_i^Q)/2$, which satisfies the three criteria above. In addition, ours is a "maxmin" solution, since the two smallest payoffs will be $u_1 = v_1 = \alpha_{11}/2$, and it is impossibe to raise both u_1 and v_1 and still have $u_1 + v_1 = \alpha_{11}$. Explicitly, this "fair outcome" takes the shape

$$u_i = \frac{\alpha_{i,i} - \sum_{j=i+1}^{n} (\alpha_{j,j-1} - \alpha_{j-1,j})}{2},$$

and

$$v_i = \frac{\alpha_{i,i} + \sum_{j=i+1}^{n} (\alpha_{j,j-1} - \alpha_{j-1,j})}{2},$$

thanks to Propositions 2.3 and 2.4.

We discuss in the appendix when studying Sattinger's continuous model that when the number of agents tends to infinity the worker-optimal and employer-optimal solutions will differ by a constant term only. Hence in that case *all* stable solutions will be on the line

$$\lambda(\bar{u}^Q; \bar{v}^Q) + (1-\lambda)(\bar{u}^P; \bar{v}^P).$$

This may also be a justification for looking for fair outcomes on this line in the discrete case too.

3.2 Minimizing envy

The concept of *envy* is often measured by the differences between payoffs to agents in the market; if $u_i > u_j$ then agent p_j's envy of p_i is $u_i - u_j$. The total envy E in a market is the sum of all these differences, i.e.

$$E = \sum_{i,j} [u_i - u_j]^+,$$

where $[a-b]^+ = a - b$ if $a - b > 0$ and 0 otherwise. (See Feldman and Kirman, 1974; and Varian, 1974.)

We will make the simplifying assumption that all envy is between pairs of P-agents or pairs of Q-agents, so that a P-agent is never envious of a Q-agent or vice versa. It turns out that hierarchies are particularly well-behaved with respect to envy: *every* stable outcome will yield the same envy and the value of this envy is

$$E = \sum_{k=1}^{n} (2k - n - 1)\alpha_{kk}.$$

This invariance implies that the smaller the envy among P-agents, the greater the envy among Q-agents. If we actually want to minimize the envy among the P-agents, we shall minimize the sum $\sum_{i>j}(u_i - u_j)$. But we already know a way to minimize this sum: just choose the P-worst outcome! By symmetry, the P-optimal outcome in fact maximizes the envy among P-agents. This is a good illustration of the conflict between the utilitaristic and egalitaristic principles.

4 Concluding remarks

We have discussed a model for assigning workers to jobs, and in particular we have studied a certain kind of market which might be named 'hierarchical', since it induces a linear ordering of workers and employers according to ability. These hierarchies were mathematically convenient objects, in that they allowed us to state explicitly the worker-optimal and employer-optimal stable distribution of earnings; we then suggested that a fair distribution should be an average between these two extremes. But how applicable is this model? Certainly the idea of a linear ordering of *all* workers in the world is counterfactual, since different kinds of jobs emphasize different skills of workers. But in a particular sector, the idea may not be too farfetched as we speculated in the introduction. A Tayloristic division of labor (Tinbergen, 1959) could be something of the past, especially in the more developed segments of the post-industrial society, cf. Lindbeck and Snower (1996). Under the assumption that the ability to increase one's skill is positively correlated to one's current position, a dynamic model of repeated assignments would eventually lead to a hierarchy-like situation.

We have assumed that agents will play by the rules whatever the utility they get out of the stable assignments. An obvious generalization would be to introduce reservation wages, so that a worker would remain self-employed if his/her proposed salary is below a certain limit.

As suggested by a referee, taxing is a popular tool for achieving a higher degree of fairness. We have found that a progressive tax will in general distort the hierarchical structure while a regressive tax is nondistortionary.

Future research should include an attempt to an empirical study of actual productivity matrices.

5 Appendix: Derivation of Sattinger's model

Sattinger (1979, 1993) considers an assignment model, called the Ricardian differential rents model, where there is a continuous distribution of workers' skills and a continuous distribution of machine sizes. Given certain assumptions on the potential productivity function, he can compute the (unique) stable matching of workers and machines and the wage function. We will show that Sattinger's results can be derived from ours.

5.1 The differential rents model

Each job (in Sattinger's terminology: each machine) is described by a single characteristic, say its size, and each worker is described by a single skill. If g_i is a measure of worker i's skill and k_j is a measure of the size of machine j, we suppose that the potential productivity α_{ij} of matching worker i to machine j is described by $\alpha_{ij} = f(g_i, k_j)$ where $f(g, k)$ is an increasing function of both g and k, and has continuous first and second derivatives, such that the mixed partial derivative is positive:

$$\frac{\partial^2 f}{\partial g \partial k} > 0.$$

Sattinger now lets the number of workers and jobs grow indefinitely so that we get a continuous distribution of skills and machine sizes.

Given this model, Sattinger argues that the assignment in equilibrium of workers to jobs will be strictly top–down, that is, the nth worker, in order of decreasing skill, will be assigned to the nth machine, in order of decreasing size. From the continuous distributions of skills and machine sizes it then follows that the relationship between them can be described by a monotonic function $k(g)$ determining which machine size will be assigned to a worker of skill g.

Then Sattinger proceeds by determining the wage function in equilibrium, i.e. the amount $w(g)$ of the productivity that the employer will pay for a worker of skill g. He shows that the wage function must satisfy

$$w'(g) = \left[\frac{\partial f(g, k)}{\partial g}\right]_{k=k(g)} \tag{3}$$

where the partial derivative is taken treating k as a constant. This equation determines the wage function up to a constant term. The constant term, describing the absolute level of wages, will be determined by the conditions for the last match in order of decreasing skill and machine size, basically by choosing a minimum wage.

5.2 Deriving the differential rents model from hierarchies

We shall now see how the differential rents model can be derived from our theory of hierarchies. Recall that a hierarchy is a matrix (α_{ij}), with $i, j = 1, 2, \ldots, n$, of potential productivities satisfying

$$\alpha_{ij} \leq \alpha_{i,j+1}, \qquad \alpha_{ij} \leq \alpha_{i+1,j} \tag{4}$$

and

$$\alpha_{ij} + \alpha_{i+1,j+1} \geq \alpha_{i,j+1} + \alpha_{i+1,j}. \tag{5}$$

We can then introduce an artificial measure of workers' skills and machine sizes by defining $g_i = i/n$ and $k_j = j/n$ respectively — although for our purposes any bounded (uniformly for all n) monotonic functions $g(i)$ and $k(j)$ would do. Further, define the function $f(g, k)$ on a discrete set by $f(g_i, k_j) = \alpha_{ij}$.

With these definitions, and letting n tend to infinity, we obtain a uniform distribution of skills and machine sizes in the unit interval. Consider the function $f(g, k)$ that is now defined on the unit square. In the limit, Eq. (4) says exactly that $f(g, k)$ is increasing in both g and k, while Eq. (5) says exactly that the mixed partial derivative $\frac{\partial^2 f}{\partial g \partial k}$ is positive. Hence, we have obtained the differential rents model as the limit case of hierarchies.

Let us now derive the differential equation (3) for the wage function $w(g)$ given by $w(g_i) = u_i$. For a discrete hierarchy we have already computed the employer-optimal and worker-optimal stable outcomes in Proposition 2.3 and Proposition 2.4 respectively. In the worker-optimal outcome we had

$$u_i - u_{i-1} = \alpha_{i,i} - \alpha_{i,i-1}$$

while in the employer-optimal outcome we had

$$u_{i+1} - u_i = \alpha_{i+1,i} - \alpha_{i,i}.$$

The function $f(g, k)$ being continuously differentiable means that in the limit the left-hand derivative equals the right-hand derivative:

$$\alpha_{i,i} - \alpha_{i,i-1} = \alpha_{i+1,i} - \alpha_{i,i}.$$

Hence we have that both the employer-optimal and worker-optimal wage functions satisfy the differential equation (3), so in fact they coincide (except for a constant term) and thus we have a unique wage function in equilibrium, up to the absolute level of wages.

References

[1] Becker, G.: *A treatise on the Family*, Harvard University Press, 2nd ed., 1991.

[2] Bloch, F. and H. Ryder: Two-sided search, marriages and matchmakers, CORE discussion paper 9428, 1994.

[3] Burdett, K. and M.G. Coles: Marriage and Class, University of Essex Working Papers, 1996.

[4] Edlund, L.: *The Marriage Market: How Do You Compare?*, thesis, Stockholm School of Economics, 1996.

[5] Feldman, A. and Kirman, A.: Fairness and envy, *The American Economic Review*, December 1974, 995–1005.

[6] Koopmans, T.C. and M. Beckmann: Assignment problems and the location of economic activities, *Econometrica*, January, 1957, 53–76.

[7] Kremer, M.: The O-ring theory of economic development, *Quarterly Journal of Economics*, CVIII August 1993, 551–576.

[8] Lindbeck, A. and D.J. Snower: Reorganization of firms and labor market inequality, Seminar Paper no. 605, 1996, Institute for International Economic Studies.

[9] Moulin, H.: *Axioms of Cooperative Decision Making*, Cambridge University Press, 1988.

[10] Rawls, J.: *A theory of justice*, Harvard University Press, 1971.

[11] Roth, A.E. and M.A.O. Sotomayor: *Two-sided matching — A study in game-theoretic modeling and analysis*, Cambridge University Press, 1990.

[12] Sattinger, M.: Differential rents and the distribution of earnings, *Oxford Economic Papers*, March, 1979, 60–71.

[13] Sattinger, M.: Assignment models and the distribution of earnings, *Journal of Economic Literature*, XXXI, June, 1993, 831–880.

[14] Shapley, L. and Shubik, M.: The assignment game I: the core, *International Journal of Game Theory* **1**, 1972, 111–130.

[15] Tinbergen, J.: On the theory of distribution of income, *Jan Tinbergen Selected Papers*, ed. L.H. Klaassen, L.M. Koyck and H.J. Witteveen, North Holland Publ. Co. 1959.

[16] Varian, H.: Equity, Envy, and Efficiency, *Journal of Economic Theory*, 9:63–91.