On the Damodaran Estimator of Price Adjustment Coefficients

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Abstract

This paper investigates the properties of the Damodaran (1993) estimator of price adjustment. It is concluded that strong bias and low precision of the Damodaran estimator renders it useless for empirical work, even when the available sample size is very large.

As an alternative, a GMM-based estimator is derived. Its properties are significantly better than those of the Damodaran estimator. However, for empirical applications it is still preferable to estimate price adjustment speeds using concurrent information from related time series.

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1 Introduction

Amihud and Mendelson (1987) conjecture an intuitive model of prices adjusting to changes in fundamental values. The difference between a price and its corresponding value process is reduced at the rate g_1 per period. Mathematically, the price process of the Amihud and Mendelson (1987) model can be formulated as follows:

$$r_t = P_t - P_{t-1} = g_1 (V_t - P_{t-1}) + u_t, \tag{1}$$

$$V_t = V_{t-1} + e_t, \tag{2}$$

where P_t and V_t are the logarithms of the price and the underlying value respectively. The noise terms u_t and e_t have variance ς^2 and ν^2:

$$\mathrm{Var}(u_t) = \varsigma^2, \qquad \mathrm{Var}(e_t) = \nu^2, \tag{3}$$

and ensure that prices remain noisy estimates of the value process. The parameter g_1 is usually interpreted as a measure of the informational efficiency. If g_1 is less than unity, prices are said to underreact to new information, if g_1 is larger than unity, prices overreact.[1]

Any process defined by equation 1 and the corresponding parameter values for g_1, ς^2 and ν^2 has well-defined time series properties, and it is possible to calculate expected variances and autocovariances for any chosen return interval.

Estimating the speed of price adjustment is complicated, given that there are three unknowns for each return series. Amihud and Mendelson (1987) argue that ς^2 is negligible for index returns since price errors in individual stocks will cancel in a well diversified index. Under the additional assumption of $g_1 \in (0.5, 1.5)$, g_1 can be estimated directly from the first order autocorrelation of index returns.

Damodaran (1993) uses another simplifying assumption to arrive at an estimator for g_1. Setting $g_k = 1$ at an arbitrarily chosen return interval k, it is possible to estimate g_1 from return variances at frequency 1 and k and the autocovariance of k interval returns. The assumption of setting $g_k = 1$ should not be consequential since, for any value of g_1, g_j quickly converges to 1 as j grows.[2] Damodaran's method of estimation thus produces separate estimates for all three unknowns. from a single time series. It is therefore possible to calculate, for example, the noisiness of prices relative to the underlying value process.

The estimator of Damodaran, denoted $\hat{g}_j$, is calculated as follows:

$$\hat{g}_j = \frac{\frac{2}{j}\hat{\sigma}_j^2 + \frac{2}{j}\hat{\rho}_k^2}{\frac{1}{j}\hat{\sigma}_j^2 + \frac{1}{k}\hat{\sigma}_k^2 + \frac{2}{k}\hat{\rho}_k^2}, \tag{4}$$

where $\hat{\sigma}_j^2$ is the estimated return of j interval returns, $\hat{\sigma}_k^2$ is the estimated return of k interval returns and $\hat{\rho}_k$ is the estimated autocovariance of k interval returns.[3]

[1] Empirically, estimates of g_1 are mostly less than unity, that is, prices do not fully reflect concurrent information. Studies seldom use the Amihud and Mendelson (1987) framework, instead the lagged price response is deduced from autocorrelations or cross-autocorrelations. Articles presenting such results include Amihud and Mendelson (1987), Lo and MacKinlay (1990), Damodaran (1993), and Chan (1993).

[2] See, for example, the numerical example in the second column of table 1.

[3] There was an error in Damodaran's original formulation. Equation 4 is the corrected formula of Brisley and Theobald (1996).

Although the Damodaran estimator is easy to calculate and practical for empirical work, its performance is far from satisfactory. Consider the case of applying the estimator to a random walk. Equation 4 suggests that the value of $\hat{g}_1$ should be equal to unity. However, the bias of the estimator is so strong that using Damodaran's choice of $k = 20$ and $T = 1250$, we obtain an average $\hat{g}_1$ of *0.67*.

In addition to the strong bias, parameter estimates also have low precision. For the same parameter choice as above, 76% of the g_1 estimates are outside the range of economically meaningful estimates, that is $\hat{g} \notin (0, 1.5)$.[4] The standard error of individual estimates is so high (*2.62*) that individual estimates are virtually useless. Even an average of 1 000 individual and uncorrelated estimates will have a standard error of *0.08*.[5] Since observed differences between securities groups tend to be small, this implies the estimator can rarely produce significant results.

The following four sections of this paper provides a brief analysis of the problems of the Damodaran estimator. The next section, section 2, discusses how Jensen's inequality and division-by-zero bias the estimated parameter values. Section 3 reports simulations results for the Damodaran estimator. Section 4 uses GMM to derive an alternative estimator and reports simulation results comparing the two estimators.. Section 5 summarises and concludes.

2 Properties of the Damodaran estimator

2.1 Variance of point estimates

It is straightforward to derive the asymptotic distribution of variance and autocovariance estimates. As an illustration, consider the null of prices following a random walk. Using GMM we obtain:

$$\hat{\sigma}_1^2 \xrightarrow{\mathrm{d}} \mathrm{N}\left(\sigma_1^2, 2/T\right), \tag{5}$$

$$\hat{\sigma}_k^2 \xrightarrow{\mathrm{d}} \mathrm{N}\left(k\sigma_1^2, 2k^3/T\right), \tag{6}$$

$$\hat{\rho}_k^2 \xrightarrow{\mathrm{d}} \mathrm{N}\left(0, k^3/T\right), \tag{7}$$

where σ_1^2 is the true variance of unit interval returns. Note how the variance and covariance estimate of the k interval returns has k^3 and $\frac{1}{2}k^3$ times higher variance than the estimates of the variance of unit interval returns. This is due to the higher variance of k interval returns as well as the effectively shorter sample period using non-overlapping k interval returns. Damodaran (1993) uses $k = 20$. This implies that the variance of $\hat{\sigma}_k^2$ and $\hat{\rho}_k^2$ are 8 000 and 4 000 times higher than the variance of $\hat{\sigma}_1^2$.

[4] In this paper values of $g_1 \in (0, 1.5]$ are taken to be economically meaningful. The upper limit can be set even lower. A g_1 of 1.5 implies a very strong overreaction of the market for unit interval returns. In addition it implies an unintuitive *underreaction* to two interval returns, with $g_2 = 0.75$.

[5] The numbers in this and the preceding paragraph are based on the simulation results reported in table 2.

Table 1: Bias resulting from Jensen's inequality, approximated using a econd order Taylor expansion of the $\hat{g}_1$ function.

Expected bias	$k=5$	$k=10$	$k=20$
From $\hat{\sigma}_j^2$	0.000	0.000	0.000
From $\hat{\sigma}_k^2$	0.002	0.004	0.008
From $\hat{\rho}_k^2$	−0.016	−0.072	−0.304
Sum	−0.014	−0.068	−0.296

The bias is calculated using analogues to equation 8 under the parameter values: $g_1 = 1$, $\varsigma^2 = 0$, $\nu^2 = 1$, $T = 1500$.

2.2 The direct effect of Jensen's inequality

Recall equation 4 and note that the estimator is a ratio of sums of stochastic variables. Depending on parameter choices, $\hat{g}_1\left(\hat{\sigma}_j^2, \hat{\sigma}_k^2, \hat{\rho}_k^2\right)$ may be both convex and concave in the estimated variances. As the variances and covariances are estimated with error, Jensen's inequality introduces a potentially large bias in the estimator.

Given that variance and covariance estimates are stochastic, it is straightforward to calculate the effect of Jensen's inequality on $\hat{g}$. For expositional simplicity, assume that the parameter estimates $\hat{\sigma}_1^2$, $\hat{\sigma}_k^2$ and $\hat{\rho}_k^2$ are uncorrelated. We may then calculate the bias introduced by the variance of, for example, $\hat{\rho}_k^2$ using a second order Taylor series approximation.

$$\begin{aligned}\text{Bias} &= \mathrm{E}\left[\hat{g}_1 - E\left[\hat{g}_1 | \hat{\rho}_k^2\right] \middle| \hat{\sigma}_j^2, \hat{\sigma}_k^2\right] \approx \frac{1}{2} \left.\frac{\partial^2 \hat{g}}{\partial\left(\hat{\rho}_k^2\right)^2}\right|_{g_1^*} \mathrm{Var}\left(\hat{\rho}_k^2\right), &(8)\\ g_1^* &= \frac{\frac{2}{j}E\left[\hat{\sigma}_j^2\right] + \frac{2}{j}E\left[\hat{\rho}_k^2\right]}{\frac{1}{j}E\left[\hat{\sigma}_j^2\right] + \frac{1}{k}E\left[\hat{\sigma}_k^2\right] + \frac{2}{k}E\left[\hat{\rho}_k^2\right]}. &(9)\end{aligned}$$

Table 1 reports the bias under the null of prices following a random walk. The bias introduced by the noisiness of $\hat{\rho}_k^2$ dominates the other sources of bias and increases rapidly as k is increased. Damodaran's choice of $k = 20$ implies a bias of -0.30 for a time series of length 1 250. Since the bias is proportional to the variance of estimates, doubling the sample size will reduce the bias by half. In a time series of, say 10 000 or more, this bias will be negligible at least for low value of k.

2.3 A close-to-zero effect

A further problem resulting from the fact that the Damodaran estimator is calculated as a ration of stochastic variables is a close-to-zero effect. Although the denominator cannot be equal to zero it can get relatively close to zero. In this case estimates will be disproportionately large. This problem is more severe insofar as it cannot be mitigated using longer time series. The effect will be strong when σ_j^2 is small relatively to σ_k^2, that is, when g_1 is low. This bias can go either way. If ς^2 is large and g_1 small there will be strong negative autocovariance in k interval returns. This makes the nominator negative when the denominator is close to zero. For low ς^2 the bias is instead positive. Since we have no way of estimating ς^2 separately within the model, this problem can only be solved by increasing the set of independent variables.

2.4 High variance

The low precision of variance and covariance estimates also make g_1 estimates very noisy. Using $k = 20$, the standard error of individual g_1 estimates is above 0.5 even when using time series with 10 000 observations for the estimation. Given the range of permissible values for g_1, this implies that individual estimates cannot be used for inference.

Reducing k reduces the problems due to noise in variance estimates but also makes the estimator less theoretically convincing. Setting k too low introduces biases from the fact the g_k may not, in fact, be equal to unity.[6]

2.5 Non-uniqueness of parameters

In the Amihud and Mendelson (1987) model parameters cannot always be uniquely determined from return series properties. For example, all return series with parameters satisfying, $g_1 = \sqrt{1 - \varsigma^2/\nu^2}$, will follow a random walk. This is also the standard prediction of g_1 from noisy rational expectations equilibrium models such as, Hellwig (1980) or Kyle (1985). Another case when it is hard to determine parameters from data is the case of weak overreaction of prices. For such parameters, time series properties are close to those of a series with higher ς^2 with g_1 equal to unity.

Depending on parameters, return properties are quite different "close" to the random walk. Therefore it is only meaningful to estimate g_1 once the random walk hypothesis has been rejected. Using the single series estimation of price adjustment coefficients thus always implies an implicit assumption of market imperfections.

3 Simulation results

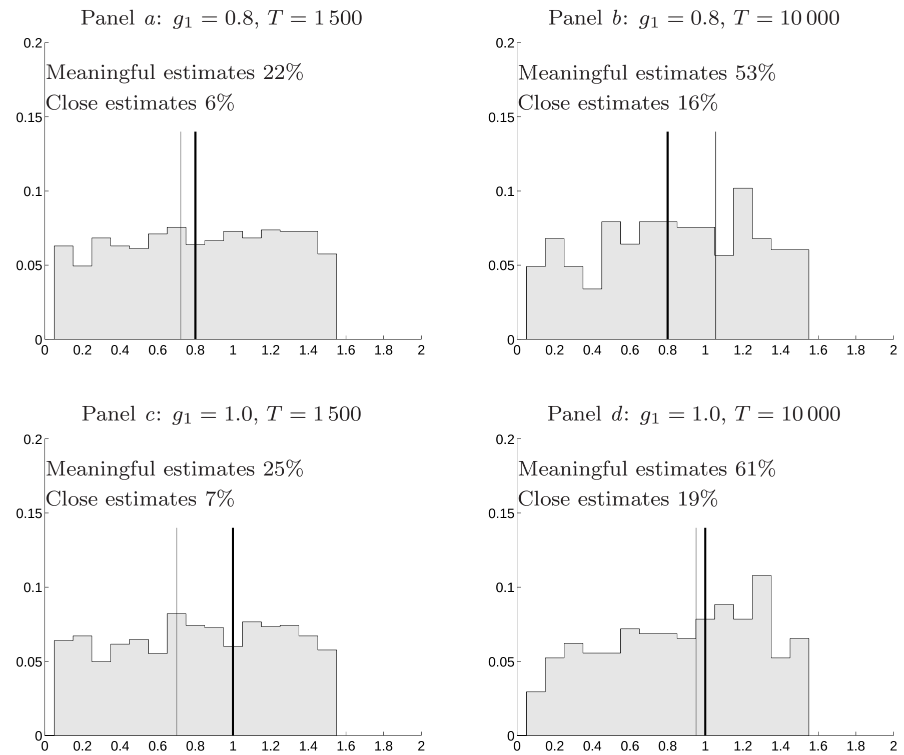
This section investigates the properties of the Damodaran estimator using simulations.[7] Figure 1 presents the distribution of $\hat{g}_1$ in the form of histograms. It is evident that the proportion of meaningful estimates is too low. In addition the estimates seem to be "all over the map". Increasing the sample size does not improve the properties of the estimator.

Figures 2 and 3 present the bias of the Damodaran estimator for varying choices of parameter choices and sample lengths. An unbiased estimator would follow the dashed line closely, but for the Damodaran estimator, this is clearly not the case. For many parameter combinations, the function $E[\hat{g}_1|g_1]$ is flat

[6] Compare with the results presented in figure 4. The results from setting $k = 5$ are more well-behaved than those obtained for $k = 20$, but also more strongly biased.

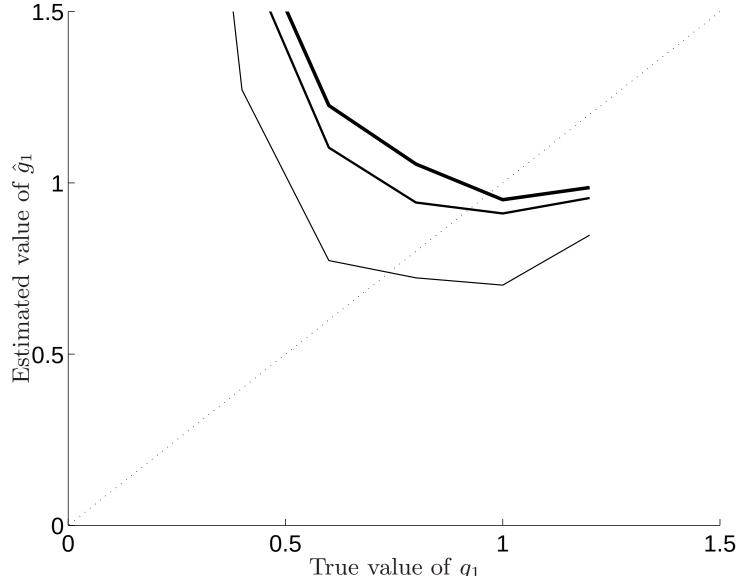
[7] Simulation details: $\hat{g}_j$ is calculated using equation 4. Variances and covariances are estimated using overlapping returns. This reduces the variance estimates by approximately one third, see Richardson and Smith (1991). Value innovations and errors are normally distributed. Tests using fat-tailed and skewed distributions produce similar results (not reported). The Matlab programs used for the simulations are included in the working paper version of this paper (available from the author).

Figure 1: Simulation of the Damodaran estimator: Histograms of simulation results.



Each histogram represents the results across 5 000 simulations. The thick vertical line indicates the true value of g_1 in the simulations, while the narrow vertical line indicates the sample average of $\hat{g}_1$. Common parameter values: $\varsigma^2 = 0.1$, $\nu^2 = 1.0$, $k = 20$. Observations outside the interval $(0, 1.5)$ are truncated from the histograms, but included in the calculated mean. In addition to showing the distribution of estimates, each panel also shows the proportion of economically meaningful ($\hat{g}_1 \in (0, 1.5]$) and "close" estimates (defined as $|\hat{g}_1 - g_1| \leq 0.1$).

Figure 2: Bias of Damodaran's estimator for different sample lengths



Parameter values: $\nu^2 = 1$, $\varsigma^2 = 0.1$, $T \in \{1\,500, 5\,000, 10\,000\}$ (thin, medium, thick solid line), $k = 20$. The dotted diagonal line indicates the true value of g_1. Deviations from the dotted line implies positive (above) or negative bias (below).

or downward sloping. A *smaller* true coefficient will thus result in a *larger* estimated coefficient. For low values of g_1, the close-to-zero effect is so severe that $\hat{g}_1$ seems to be wholly unrelated to the true value. As shown in figure 2, an increased sample size reduces the bias due to Jensen's inequality, thereby increasing the average value of $\hat{g}_1$, however the close-to-zero effect is not affected.[8,9]

Figure 3 reports similar results, this time for different choices of ς^2/ν^2. The same pattern similar to that of figure 2. However, the figure shows how the "close-to-zero bias" is highly sensitive to the values of ς^2/ν^2. For high ς^2/ν^2, the nominator of equation 4 is more likely to be negative when the denominator is close to zero. This makes cross-sectional inference of estimated g_1 parameters close to impossible, unless separate estimates or assumptions on the noisiness of prices are made.

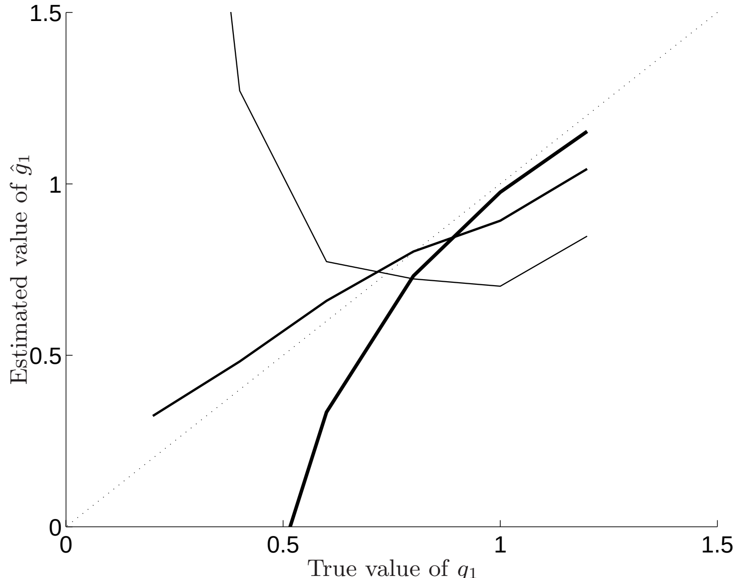
In addition to the bias discussed above, the estimates of g_j are not consistent across choices of j. Given an estimate of g_1, estimates for all higher g_j follow easily from standard algebra:

$$g_{j+1} = g_j + g_1 - g_j g_1. \tag{10}$$

[8]It should be noted that an increase in the sample size to 100 000 corresponds to using, for example, 3 years worth of 5 minute returns. Using such short return interval implies a lower value of the true g_1 (a daily g_1 of 0.99 corresponds to a 5 minute g_1 of 0.04) and probably a higher value for k (Assuming $g_{1\ \text{day}} = 1$, implies $k = 120$). Given the properties of $\hat{g}_1$ for low g_1 and high k this implies that estimates will deteriorate with increased data frequency.

[9]In this context it should be mentioned that Damodaran (1993) presents simulation results in support of the derived estimator. Unfortunately, the simulation uses the parameter specification, $g = 0.7$, $\sigma^2/\nu^2 = 0.4$, T =1 250. For this parameter combination the estimator is close to unbiased.

Figure 3: Bias of the Damodaran estimator for three values of σ^2



Parameter values: $\nu^2 = 1$, $\varsigma^2 \in \{0.1, 1.0, 5\}$ (thin, medium, thick line), $T = 1500$, $k = 20$. The dotted diagonal line indicates the true value of g_1. Deviations from the dotted line implies positive (above) or negative bias (below).

However, the Damodaran estimator does not have this property. Instead the speed of convergence to $g_j = 1$ is too low. This illustrated by the relatively large difference between the first two columns of table 2.

3.1 Damodaran's empirical results

In the light of the simulation results it is possible to reject most of the empirical findings reported in Damodaran (1993). Table 2 compares the results reported by Damodaran to results from two matching simulations. Results are consistent with figure 2. The estimated value of $\hat{g}_1$ is close to 0.65 both for a random walk and for a process with a true g_1 of 0.7. We may therefore conclude that the underreaction results presented by Damodaran are predominantly generated from the choice of k and T, not the properties of actual price adjustment. The same conclusion must be made about the reported medium term overreaction.

Damodaran also report evidence of cross-sectional differences in price adjustment coefficients between AMEX and size-sorted NYSE stocks. According to Damodaran small stocks seem to react slower than large stocks, however differences between categories are small. Given the small sample length and the high variance of $\hat{g}_1$, these differences are cannot be interpreted as statistically significant difference in the speed of price adjustment.

4 A GMM estimator

Using the General Method of Moments technique (GMM) of Hansen (1982) it is straightforward to derive a consistent estimator of g_1. The model of Amihud and

Table 2: Estimates of g_j as reported by Brisley and Theobald (1996) compared to simulation results

	Original estimates[†]	Theoretical value[‡]	Matched simulation[§]	Random walk prices[¶]
$\hat{g}_1$	0.6408	...	0.6476	0.6673
$\hat{g}_2$	0.7336	0.8710	0.8361	0.8455
$\hat{g}_5$	0.9503	0.9940	0.9525	0.9505
$\hat{g}_{10}$	1.0050	1.0000	0.9886	0.9853
$\hat{g}_{15}$	1.0080	1.0000	0.9957	0.9935
$\hat{g}_{20}$	1.0000	1.0000	1.0000	1.0000

[†]Values taken from Brisley and Theobald (1996), table I. [‡]Calculated using equation 10 and $g_1 = 0.6408$. [§]Average estimated g_1 using equation 4 under the true parameters $g = 0.7$, $\varsigma^2 = 0.4$, $\nu^2 = 1.0$, $T = 1250$, $k = 50$, 5 000 simulations. [¶]Average g_1 under the true parameters $g = 1.0$, $\varsigma^2 = 0.0$, $\nu^2 = 1.0$ and $T = 1250$, $k = 50$, 5 000 simulations.

Mendelson (1987) provides a large number of possible moment conditions based on predicted variances and covariances. Predicted variances give us moment conditions of the form:

$$\mathrm{E}\left[r_{k,t}^2 - \frac{g_k}{2-g_k}k\nu^2 + \frac{2}{2-g_k}\varsigma^2\right] = 0, \tag{11}$$

for k interval returns. Similarly we have:

$$\mathrm{E}\left[r_{k,t}r_{k,t-i} - g_k\left(1-g_k\right)i\nu^2 - g_k\varsigma^2\right] = 0 \tag{12}$$

from the predicted autocovariance of k interval returns. Combined with the equation for g_i, we can construct a GMM estimator using an arbitrary number of moment conditions. The simulation use an estimator based on $2k$ moment conditions, k variance moments and k covariance moments.

The GMM estimate is obtained by setting the corresponding in-sample moments to zero. This is equivalent to minimising:

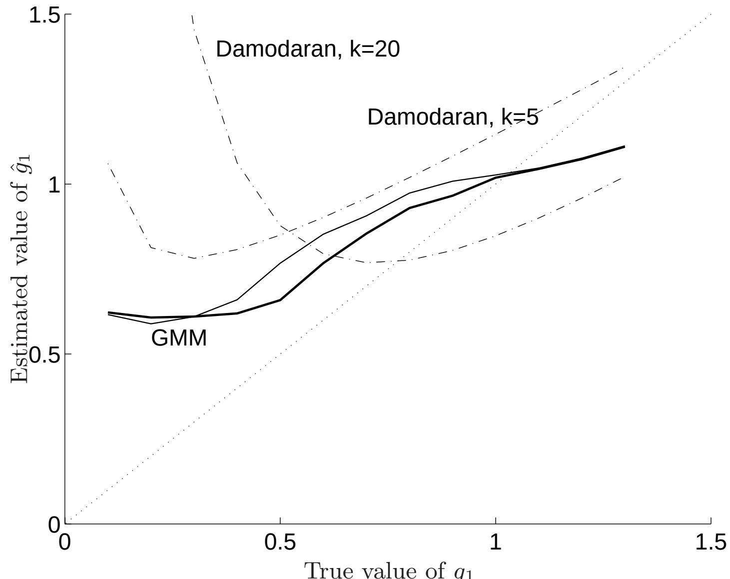
$$\left(\hat{g}_1, \hat{\varsigma}^2, \hat{\nu}^2\right) = \arg \min_{g,\varsigma^2,\nu^2} \sum_{t=1}^{T} \mathbf{g}_t\left(g_1, \varsigma^2, \nu^2\right)' S_w^{-1}\mathbf{g}_t\left(g_1, \varsigma^2, \nu^2\right), \tag{13}$$

where S_w^{-1} is the inverse covariance matrix of moment conditions, estimated at $\left(\hat{g}_1, \hat{\varsigma}^2, \hat{\nu}^2\right)$ using k lags and $\mathbf{g}_t$ is a vector of $2k$ sample moment conditions at time t. For details, see Hansen (1982).

An illustration of the results obtained using the GMM estimator is given in figure 4. The GMM estimator has several advantages over the Damodaran estimator. Firstly, its estimates are by construction always within the range of permissible parameter values. Secondly, individual estimates are significantly less noisy and the estimator is much more "well-behaved". The main drawback of the GMM method is that it is computationally much more demanding.

Although it is clear that GMM provides significantly better results than the Damodaran estimators, results are still quite wobbly, particularly for intermediate values of g_1, since $g_1 = 0.75$ implies that prices follow a random walk. The experienced positive bias is due to a large number of observations with $\hat{g}_1$ close to unity.

Figure 4: Comparision of the Damodaran estimator and a simple GMM estimator



The figure presents the results from four separate estimators: *1* the Damodaran estimator with $k = 5$ (thin dotted line), *2* the Damodaran estimator with $k = 20$ (thick dotted line), *3* the GMM estimator with $k = 5$ (thin solid line), and *4* the GMM estimator with $k = 10$ (thick solid line). The GMM estimator uses $2k$ moment conditions as described in section 4. All estimators use *identical* simulated return data, estimated using the Amihud and Mendelson (1987) model (equation 1) with the parameter values: $\varsigma^2 = 0.5$, $\nu^2 = 1.0$, $T = 1\,500$. The dotted diagonal line indicates the true value of g_1. Deviations from the dotted line implies positive (above) or negative bias (below).

5 Conclusion

This paper demonstrates that the estimator of price adjustment coefficients, derived in Damodaran (1993), cannot be used for empirical work. Results are biased and extremely noisy. In addition, parameter estimates are often not meaningful in the context of the Amihud and Mendelson (1987) model. In some cases the bias is so severe that estimated parameter values are negatively related to the true values.

The cursory exposition made in section 4 shows that, for the purpose of estimating price adjustment coefficients, a simple GMM estimator produces robust and consistent results that are significantly better than those of the Damodaran estimator. However, given the ambiguity of parameter values, even the GMM estimator should only be used where there is a theoretical foundation for prices actually following a Amihud and Mendelson (1987) type process, and not, say a random walk with measurement noise.

Even when this condition is satisfied, the information in a single time series is often not sufficient to estimate price adjustment coefficients efficiently. However, using additional information, such as returns in concurrent time series, it will be feasible to measure the price response to a common factor quite efficiently. Such an estimator is outside the scope of this paper, and is probably best derived with a particular testing environment in mind.

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