A Dynamic Conditionally Heteroscedastic Stochastic Frontier Model

Mickael Löthgren
Department of Economic Statistics
Stockholm School of Economics
P.O. Box 6501
S-113 83 Stockholm
e-mail: stml@hhs.se
fax: +46-8-34 81 61

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Abstract

This paper presents a new dynamic $ARCH$-related conditionally heteroscedastic stochastic frontier model specification where firm and time-specific technical inefficiency is represented by an autoregressive stochastic process in the error components. Monte Carlo results reveal that a one-sided likelihood ratio test of the proposed model has correct small-sample size and has high power for small to medium sized panels. An empirical application is included using a panel of 23 OECD countries over the 26 year period $1965 - 1990$. The estimation results indicate a clear rejection of the standard frontier model and existence of first order dynamic conditionally heteroscedastic technical inefficiency.
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1. Introduction

The stochastic frontier error component model, introduced by Aigner, Lovell and Schmidt (1977) and Meeusen and van den Broeck (1977), has found extensive applications in several areas of production analysis as discussed in, e.g., the survey by Greene (1993). In the original frontier model it is assumed that the error component term consists of two independent and homoscedastic error components: A symmetric term representing the production frontier randomness and a non-negative technical inefficiency term that represents the amount outputs fall short of the maximum stochastic frontier output. The error components are generally assumed to be independent homoscedastic random variables.

The purpose of this paper is to extend the stochastic frontier models to incorporate a new dynamic Conditionally Heteroscedastic (CH) specification of technical inefficiency not previously utilized for frontier models. The proposed model is similar in structure to the Autoregressive Conditionally Heteroscedastic ($ARCH$) time series model by Engle (1982). In the proposed $ARCH$-related specification, the firm and time-specific technical inefficiency variance is conditioned on previous realizations of the composed error term. That is, lagged realizations of both frontier noise and technical inefficiency affect the variance and mean technical inefficiency in future periods. This specification allows for a persistence in the dynamic development of firm-specific technical inefficiency. Firms that are relatively technically inefficient and/or experience large frontier disturbance realizations tend to remain technically inefficient in subsequent periods. On the other hand, firms that are relatively technically efficient and experience small production frontier fluctuations tend to remain relatively technically efficient.

The loglikelihood function of the proposed model, based on a straightforward modification of the original frontier model loglikelihood function presented by, e.g., Aigner et al. (1977), is derived and allows maximum likelihood estimation of the model parameters.

The proposed model provides a tractable specification of a general dynamic conditionally heteroscedasticity process for the non-negative technical inefficiency error component. The dynamic CH model offers an alternative to the specifications of time-varying technical inefficiency proposed by, e.g., Cornwell, Schmidt and Sickles (1990), Kumbhakar (1990), Battese and Coelli (1992), Battese and Coelli (1995), where technical inefficiency is specified as a deterministic non-dynamic function of time. Specifically, Cornwell et al. (1990) model technical inefficiency as a quadratic function of time, Kumbhakar (1990) allows for an ex-

ponential function of quadratic time, Battese and Coelli (1992) models technical inefficiency as an exponential function of time, and Battese and Coelli (1995) models technical inefficiency by a truncated normal variable with mean given by a linear function of time. Few studies attempt to specify a truly dynamic model for inefficiency. Two exceptions are Lee and Schmidt (1993) and a recent paper by Ahn, Good and Sickles (1997). Lee and Schmidt (1993) model technical inefficiency using a nonlinear model that allows for arbitrary temporal patterns of technical inefficiency change, with the restriction that the temporal change is homogeneous across firms in the panel. Ahn et al. (1997) consider a linear model, where the technical inefficiency is autoregressive.

The proposed model offers furthermore an alternative dynamic heteroscedasticity specification extension of the heteroscedasticity specification proposed by Kumbhakar (1993) (for panel data), and Caudil, Ford and Gropper (1995) and Battese, Rambaldi and Broca (1997) (for cross-sectional data), where the variance of the technical inefficiency is specified as a function of exogenous variables mainly related to firm size.

One further advantage of the proposed model is that the standard stochastic frontier model is nested within the conditionally heteroscedastic model. This allows simple likelihood ratio tests of the proposed model specification based on maximum likelihood estimation of both the restricted standard frontier model and the unrestricted conditionally heteroscedastic model.

The small-sample properties of standard and one-sided likelihood ratio (LR) tests of the proposed model is investigated using Monte Carlo simulations. The results reveal that the standard LR test is under-sized whereas the one-sided LR test based on modified critical values has correct size and relatively high power for small panels with cross-sectional (N) and time-series (T) dimension as small as $N = T = 10$. For larger panels, the size of the one-sided test remains correct and the power increases even further indicating that the test is consistent and that the power approaches unity asymptotically.

An empirical application of the model is included using a panel of 23 OECD countries over the period $1965 - 1990$. The estimation results indicate a clear rejection of the standard stochastic frontier model in favor of a first order dynamic conditionally heteroscedastic stochastic frontier model.

The paper unfolds as follows: Section 2 presents the panel data stochastic frontier model and the dynamic CH specification of the technical inefficiency variance. Section 3 presents likelihood ratio tests and the modifications of the critical values necessary to obtain tests of correct size of the standard stochastic

frontier model against the proposed dynamic conditionally heteroscedastic model. Section 4 derives the log likelihood function and gives expressions for firm and time-specific technical inefficiency predictions under the proposed model. Section 5 presents a Monte Carlo simulation study of the small-sample properties of likelihood ratio tests of the model. Section 6 presents the empirical application and Section 7 concludes with a summary.

2. The Model

Consider the following logarithmic linear panel data version of the stochastic frontier model by Aigner et al. (1977)

$$y_{it} = x'_{it}\beta + \varepsilon_{it}, \tag{2.1}$$

where $i = 1, ..., N$, indexes firms, $t = 1, ..., T$, indexes time periods. $x_{it} \in R^k_+$ denotes a $k-$vector of inputs used by firm i in time period t to produce output $y_{it} \in R_+$. β is a k-vector of homogeneous input elasticities common to the N firms. The composed error is given by $\varepsilon_{it} = v_{it} - u_{it}$, where the error component v_{it} represents the effect of random events on the production frontier. $y^f_{it} = x'_{it}\beta + v_{it}$ represents the maximum stochastic frontier output producible using input x_{it}. The standard assumption is that v_{it} is assumed to be independently normally distributed as

$$v_{it} \sim N\left(0, \sigma^2_v\right). \tag{2.2}$$

The error component u_{it} is non-negative and represents the technical inefficiency in terms of the amount the observed output y_{it} falls short of the frontier output y^f_{it}. In the original model u_{it} is specified to be truncated normal with homoscedastic variance $u_{it} \sim |N(0, \sigma^2_u)|$.

In this paper we propose an extension of the *iid* stochastic frontier model specification to allow the technical inefficiency term to follow a general dynamic conditionally heteroscedastic stochastic process.

Let $\mathcal{I}_t$ denote the information set available at time t. The generalization of the model is accomplished by letting u_{it} be conditionally heteroscedastic truncated normal as $u_{it}|\mathcal{I}_{t-1} \sim |U_{it}|$, where $U_{it} \sim N(0, h_{it})$. This can equivalently be specified as

$$u_{it}|\mathcal{I}_{t-1} \sim |\eta_{it}|\, h^{1/2}_{it}, \tag{2.3}$$

where $\eta_{it} \sim N(0,1)$ *iid*. The firm and time-specific conditional variance h_{it} is given by a dynamic Conditional Heteroscedastic ($CH(q)$) process in the composed

error

$$h_{it} = \alpha_0 + \sum_{j=1}^{q} \alpha_j \varepsilon_{it-j}^2. \tag{2.4}$$

This specification is similar in structure to the Autoregressive Conditionally Heteroscedastic ($ARCH$) time series model by Engle (1982). Analogously as in the $ARCH$ model the restrictions $\alpha_0 = 0$ and $\alpha_j \geq 0$ $j = 1, ..., q$, are imposed to ensure non-negativity of the conditional variance h_{it} and the restriction $\sum_{j=1}^{q} \alpha_j < 1$ are needed to ensure covariance-stationarity and existence of the unconditional variance of the technical inefficiency term.

One difference between the $ARCH$-related panel data specification in (2.4) and the $ARCH$ model is that the variance h_{it} of the technical inefficiency error component u_{it} is specified as a function of previous realizations of the error component term ε_{it}, and not of lags of u_{it} itself as in the $ARCH$ model. We argue that the $ARCH$-related specification considered here is a sensible modification of the standard $ARCH$ specification since it is reasonable that technical inefficiency is affected not only by previous realizations of technical inefficiency but also that previous random effects on the production environment facing the firm, as represented by the error component term v_{it}, are likely to affect technical inefficiency.

The proposed specification introduces both a conditional heteroscedasticity and a general dynamic process for the conditional technical inefficiency. A large realization of the technical inefficiency and/or the frontier disturbance term increases the variance of the technical inefficiency term in the next period. Hence a persistence is introduced for the dynamic development of technical inefficiency. The probability of large realizations of technical inefficiency in a given period is increased if the previous period has been subject to large frontier noise and/or inefficiency realizations. On the other hand, high technical efficiency and small frontier disturbances tend to be followed by high realizations of technical efficiency.

The specification also implies that the firm and time-specific expected inefficiency is conditionally given by

$$E\left(u_{it} \left| \mathcal{I}_{t-1}\right.\right) = \sqrt{2h_{it}/\pi} \tag{2.5}$$

Furthermore, the conditional second moment of the technical inefficiency term u_{it} corresponds to the conditional variance of the non-truncated normal random

variable U_{it} used to obtain u_{it}.[1] That is

$$E\left(u_{it}^2 \left|\mathcal{I}_{t-1}\right.\right) = h_{it}. \tag{2.6}$$

The unconditional second moment of the technical inefficiency random term corresponds to the unconditional variance, denoted h, of the non-truncated normal random variable U_{it} which can be expressed in terms of the variance parameters in the model (given that the restrictions on α_j, $j = 0, 1, ..., q$, are satisfied and that the error components v_{it} and u_{it} are independent for all i and t) as[2]

$$h = E\left(u_{it}^2\right) = \frac{\alpha_0 + \sigma_v^2 \sum_{j=1}^{q} \alpha_j}{1 - \sum_{j=1}^{q} \alpha_j}. \tag{2.7}$$

Furthermore, the conditional variance of the technical inefficiency term is obtained (similarly as in the standard model) as

$$Var\left(u_{it} \left|\mathcal{I}_{t-1}\right.\right) = (1 - 2/\pi)\, h_{it} = (1 - 2/\pi)\left(\alpha_0 + \sum_{j=1}^{q} \alpha_j \varepsilon_{it-j}^2\right). \tag{2.8}$$

And the unconditional technical inefficiency variance is given by

$$Var\left(u_{it}\right) = (1 - 2/\pi)\, h = (1 - 2/\pi)\left[\frac{\alpha_0 + \sigma_v^2 \sum_{j=1}^{q} \alpha_j}{1 - \sum_{j=1}^{q} \alpha_j}\right]. \tag{2.9}$$

3. Hypothesis testing

The original stochastic frontier model is nested within the dynamic CH model. This allows model specification testing of the original homoscedastic frontier model against the proposed dynamic CH specification. Under the restriction $\alpha_j = 0$, $j = 1, ...q$, the conditional variance $Var\left(u_{it} \left|\mathcal{I}_{t-1}\right.\right)$ equals the restricted unconditional homoscedastic variance $Var\left(u_{it}\right) = (1 - 2/\pi)\,\alpha_0$, for all i and t.

[1] This follows since the second moment of a normal random variable, truncated at zero, is given by the variance of the non-truncated variable. I.e., if $u \sim |U|$, where $U \sim N\left(0, \sigma^2\right)$, then $E\left(u^2\right) = \sigma^2$.

[2] This corresponds to the expression for the unconditional variance in the $ARCH\,(q)$ model with the added tem $\sigma_v^2 \sum_{j=1}^{q} \alpha_j / \left(1 - \sum_{j=1}^{q} \alpha_j\right)$ that corresponds to the variance component of the symmetric error component v_{it}.

Furthermore, the conditional expected technical inefficiency $E(u_{it}|\mathcal{I}_{t-1})$ equals the restricted unconditional average technical inefficiency $E(u_{it}) = \sqrt{2\alpha_0/\pi}$, for all i and t. These restricted quantities correspond to the technical inefficiency variance and expectation in the original homoscedastic model by Aigner et al. (1977), with $\alpha_0 = \sigma_u^2$.

The null hypothesis $H_0 : \alpha_j = 0$, $j = 1, ...q$, that there are no dynamic conditional heteroscedasticity is easily tested against the one-sided alternative dynamic CH hypothesis $H_1 : \alpha_j > 0$, for some j, using a likelihood ratio test. The likelihood ratio test statistic is given by $\lambda_{NT} = 2(\ln L_{1,NT} - \ln L_{0,NT})$, where $L_{1,NT}$ and $L_{0,NT}$ denote the loglikelihood under the alternative and the null hypothesis, respectively.

Due to the fact that the α_j coefficients under the null hypothesis $H_0 : \alpha_j = 0$ is on the boundary of the parameter space, the asymptotic distribution of the one-sided λ_{NT} statistic under the null hypothesis is not a standard χ^2-distribution, with q degree of freedom. Instead the asymptotic distribution of the one-sided λ_{NT} is a mixture of χ^2 with weights depending upon q, the number of restrictions of boundary values in the null hypothesis. For instance, in testing the $CH(1)$ model, with $q = 1$, the distribution of the one-sided λ_{NT} under the null is given by $(1/2)\chi_0^2 + (1/2)\chi_1^2$ (where χ_0^2 has unit mass at zero). The critical value for the one-sided LR test of level θ corresponds to the critical value of the standard LR test with level 2θ. The implication of this is that the one-sided test rejects the null hypothesis more often than the standard LR test.[3]

When testing the general $CH(q)$ model, with $q > 1$, the situation is more complicated. See, e.g., Gouriéroux, Holly and Montfort (1982) and Kodde and Palm (1986) for a complete treatment of the theory for bounded hypothesis testing.

4. The Likelihood Function

The pdf of the composed error ε_{it} given in Aigner et al. (1977) for the homoscedastic original stochastic frontier model is easily extended to the dynamic CH specification considered here. Specifically, the conditional pdf for ε_{it} under the distri-

[3]See, e.g., Coelli (1995) and Lee (1993) for a related discussion of one-sided tests of the null hypothesis $H_0 : \gamma = \sigma_u^2/(\sigma_u^2 + \sigma_v^2) = 0$ of a zero variance ratio against the alternative $H_1 : \gamma > 0$ in the standard stochastic frontier model that also has a mixed chi-square distribution under the null.

butional assumptions given in (2.2), (2.3) and (2.4) is

$$f\left(\varepsilon_{it}\left|\mathcal{I}_{t-1}\right.\right)=\frac{2}{\sqrt{2\pi\left(\sigma_v^2+h_{it}\right)}}\exp\left\{-\frac{\varepsilon_{it}^2}{2\left(\sigma_v^2+h_{it}\right)}\right\}\Phi\left(-\frac{h_{it}^{1/2}\varepsilon_{it}}{\sqrt{\sigma_v^2\left(\sigma_v^2+h_{it}\right)}}\right), \quad (4.1)$$

where $\Phi(\cdot)$ denotes the distribution function of a standard normal random variable and h_{it} is given by (2.4).

Collect the parameters to be estimated in the $(k+q+2)$-vector $\theta=(\beta',\ \alpha',\ \sigma_v^2)'$, $\alpha=(\alpha_0,\alpha_1,...,\alpha_q)'$. Assuming that there are a total of T_i+q-1 observations available for each firm $i=1,...,N$, conditional on the first q observations $(t=-q+1,-q+2,...,0)$, the pooled loglikelihood function is then

$$\mathcal{L}(\theta)=\sum_{i=1}^{N}\sum_{t=1}^{T_i}\ln L_{it}(\theta), \quad (4.2)$$

where $\ln L_{it}(\theta)$ denotes the contribution of the period t observation of firm i to the loglikelihood function. By taking the logarithm of (4.1) and substituting $\varepsilon_{it}=y_{it}-x_{it}'\beta$, $\ln L_{it}(\theta)$ is given by

$$\begin{aligned}\ln L_{it}(\theta) &= \frac{1}{2}\ln(2/\pi)-\frac{1}{2}\ln\left(\sigma_v^2+h_{it}\right)-\frac{1}{2\left(\sigma_v^2+h_{it}\right)}\left(y_{it}-x_{it}'\beta\right)^2 \quad (4.3)\\ &\quad +\ln\Phi\left[-\frac{\left(y_{it}-x_{it}'\beta\right)h_{it}^{1/2}}{\sqrt{\sigma_v^2\left(\sigma_v^2+h_{it}\right)}}\right].\end{aligned}$$

Maximum likelihood estimation of this model is similar to estimation of $ARCH$ models. For a given value of the parameter vector θ, the conditional variances is given by $h_{it}=\alpha_0+\sum_{j=1}^{q}\alpha_j\left(y_{it-j}-x_{it-j}'\beta\right)^2$, which is used to evaluate the loglikelihood function (4.2). The loglikelihood function can then be maximized using various numerical optimization procedures to obtain the maximum likelihood estimates of the parameter vector θ.

The Monte Carlo simulation study and the empirical application presented in this paper are based on the DFP-algorithm with numerical approximations of the gradient. The start value θ^0 of the parameter vector is found using a two-step procedure based on ML estimates from a standard frontier model specification. First, the gridsearch procedure suggested by Coelli (1995) is used. In this procedure OLS estimates of the slope coefficients and a grid of variance ratios $\gamma=\sigma_u^2/\left(\sigma_u^2+\sigma_v^2\right)$

and a corresponding grid of corrected intercept and variance estimates are used to obtain the initial start values of the parameters (that is, those corresponding to the largest loglikelihood value). These initial parameter start values are then used in the second step to obtain standard stochastic frontier ML coefficient estimates. These standard model estimates provide the basis for the starting values in the ML estimation of the dynamic conditionally heteroscedastic model. Specifically, the start values of the regressor coefficients and the frontier variance component are set to the corresponding standard model estimates. The start value for the intercept in the conditional variance function are set to the technical inefficiency variance estimate in the standard model and the remaining conditional variance coefficients α_i, $i = 1, ..., q$, are set to zero.

4.1. Technical inefficiency prediction

Jondrow, Lovell, Materov and Schmidt (1982) proposed predicting/estimating the technical inefficiency error component term for cross-sectional data using either the conditional expectation or the mode of the conditional distribution of $u_i \,|\varepsilon_i$. The expressions for technical inefficiency predictions given in Jondrow et al. (1982) are easily extended to the dynamic models considered in this paper. The predictor of the firm and time-specific technical inefficiency error component term $\widehat{u}_{it}$, based on the conditional expectation, is obtained as

$$E\left(u_{it}\,|\varepsilon_{it},\ \varepsilon_{it-1}, ...,\ \varepsilon_{it-q}\right) = \sigma_{*it}\left[\frac{\phi\left(\varepsilon_{it}\lambda_{it}/\sigma_{it}\right)}{\Phi\left(\varepsilon_{it}\lambda_{it}/\sigma_{it}\right)} - \frac{\varepsilon_{it}\lambda_{it}}{\sigma_{it}}\right], \tag{4.4}$$

where $\phi\left(\cdot\right)$ and $\Phi\left(\cdot\right)$ denotes the probability density and distribution functions, respectively, of a standard normal random variable. $\sigma_{it}^2 = h_{it} + \sigma_v^2$, $\sigma_{*it} = h_{it}^{1/2}\sigma_v/\sigma_{it}$, $\lambda_{it} = h_{it}^{1/2}/\sigma_v$, (using a direct analogy with the notation in Jondrow et al. (1982)). The conditioning on both ε_{it} and the q lags ε_{it-1}, ..., ε_{it-q} follows from the specification of the conditional variance h_{it}. The expression for the mode of the conditional distribution follows analogously.

Battese and Coelli (1988) proposed predicting technical inefficiency, which under the multiplicative model in (2.1) is given by $TE_{it} = \exp\left(u_{it}\right)$, using the conditional expectation $E\left(\exp\left(-u_{it}\right)|\varepsilon_{it}\right)$. Under the dynamic model specification considered in this paper, the modified expression for this predictor is given by

$$E\left(\exp\left(-u_{it}\right)|\varepsilon_{it},\ \varepsilon_{it-1},...,\ \varepsilon_{it-q}\right)=\exp\left\{-\mu_{it}^{*}+\sigma_{*it}^{2}/2\right\}\frac{\Phi\left(\mu_{it}^{*}/\sigma_{*it}-\sigma_{*it}\right)}{\Phi\left(\mu_{it}^{*}/\sigma_{*it}\right)}, \tag{4.5}$$

where $\mu_{it}^{*}=\varepsilon_{it}h_{it}/\sigma_{it}^{2}$, $\sigma_{it}^{2}=h_{it}+\sigma_{v}^{2}$ and $\sigma_{*it}=h_{it}^{1/2}\sigma_{v}/\sigma_{it}$ as above.

The above presented predictors only give point estimates of technical inefficiency. Confidence intervals for the predictor (4.4) and (4.5) can be derived by straightforward modification of the confidence interval expressions given by Horrace and Schmidt (1996) for the original homoscedastic stochastic frontier model. As a final remark, note that the expressions for point predictions of technical inefficiency are derived under the assumption that the true parameters are known and in using any of the above point predictors the parameters have to be replaced by the corresponding ML estimates pointed out by Jondrow et al. (1982).

5. A Monte Carlo Study

A Monte Carlo simulation study is conducted to assess the small-sample properties of likelihood ratio tests of the null hypothesis $H_0:\alpha_1=0$, that the data generating process (DGP) is characterized as a standard non-dynamic homoscedastic stochastic frontier model, against the alternative dynamic conditionally heteroscedastic hypothesis $H_1:\alpha_1>0$.

The simulations are based the following DGP: A first order $CH\ (1)$ model with an intercept as the only regressor[4] is used, where the observed output y_{it} is given by

$$y_{it}=\beta_0+\varepsilon_{it}, \tag{5.1}$$

where $\varepsilon_{it}=v_{it}-u_{it}$. $v_{it}\sim N\left(0,1\right)$ *iid* and $u_{it}\sim\left|\eta_{it}\right|h_{it}^{1/2}$, with $\eta_{it}\sim N\left(0,1\right)$ *iid*. The firm and time-specific variance h_{it} given by

$$h_{it}=\alpha_0+\alpha_1\varepsilon_{it-1}^{2}. \tag{5.2}$$

The true values of the parameter β_0, α_0 and α_1 are set as follows: $\beta_0^{*}=\alpha_0^{*}=1.0$; three values for α_1 are considered in the study $\alpha_1^{*}\in\{0,\ 0.1,\ 0.25\}$. All combinations of cross-sectional dimension $N=10$, 25, 50 and 100 and time-series dimension $T=10$, 25 and 50 are considered in the study. $T+1$ observations are

[4] We also perform simulations including a time trend as an additional regressor. Since the results for the two models are similar, we only include results for the intercept-only case.

generated according to the given DGP for each cross-sectional unit $i = 1, ..., N$ where the dynamic process for the conditional variance is initialized by using $u_{i0} = 0$ for all i, i.e., all firms are technically efficient in the first period upon which the ML estimation is conditioned.

The reported results on size and power are based on 1000 Monte Carlo replicates for each different combination of N and T.

Table 1 presents the size results for standard and one-sided likelihood ratio tests for nominal 5% level tests standard and mixed χ_1^2 distribution critical values, respectively.[5]

Table 1: Size of LR test of $H_0 : \alpha_1 = 0$.

	$T = 10$		$T = 25$		$T = 50$	
	standard χ_1^2	mixed χ_1^2	standard χ_1^2	mixed χ_1^2	standard χ_1^2	mixed χ_1^2
$N = 10$	0.025	0.053	0.032	0.052	0.028	0.051
$N = 25$	0.024	0.049	0.031	0.053	0.033	0.058
$N = 50$	0.025	0.046	0.031	0.053	0.032	0.047
$N = 100$	0.033	0.060	0.023	0.056	0.015	0.049

The size results reveal that tests based on standard asymptotic χ^2 critical values are significantly[6] under-sized with size about 2.5% for the $T = 10$ case (for $N = 10$, 25 and 50) and size about 3.3% for the $T = 25$ case. The results for the one-sided LR test based on the mixed χ^2 critical values that take the inequality restriction into account indicate that this modification of the critical values gives the test the correct small-sample size. These results are in favor of using modified mixed χ^2 critical values for the LR test. The standard LR test is conservative and will naturally have lower power than the one-sided LR test. *Table 2* presents power results for nominal 5% level one-sided LR test based on the mixed χ^2 critical value.

[5] The critical values for the 5% tests are 2.71 for the one-sided and 3.84 for the standard test.

[6] Note that a 95% confidence interval for the simulated size of a 5% level test based on 1000 Monte Carlo replicates equals 0.05 ± 0.014.

Table 2: Power of one-sided LR test of $H_0 : \alpha_1 = 0$.

	$T = 10$		$T = 25$		$T = 50$	
	$\alpha_1^* = 0.10$	$\alpha_1^* = 0.25$	$\alpha_1^* = 0.10$	$\alpha_1^* = 0.25$	$\alpha_1^* = 0.10$	$\alpha_1^* = 0.25$
$N = 10$	0.228	0.497	0.386	0.892	0.640	0.987
$N = 25$	0.366	0.865	0.667	0.995	0.930	1.000
$N = 50$	0.595	0.983	0.910	1.000	0.997	1.000
$N = 100$	0.841	1.000	0.997	1.000	1.000	1.000

The power results reveal that the one-sided LR test of the proposed model has fairly good power properties. For the rather small panel with $N = T = 10$ for $\alpha_1 = 0.10$ the power of 0.23 are, as expected, not that high, but for an increase in the true α_1^* to $\alpha_1^* = 0.25$ the power is more than doubled to around 0.50. As the number of cross-sections and/or the time-series dimension increase the power increases rather rapidly. For panels with size around the size of the panel in the included application in this paper (with N and T around 25) the results reveal that the power is as high as 0.67 for $\alpha_1 = 0.10$, and almost 1.0 for $\alpha_1^* = 0.25$. The results for larger panel combinations (with $N \geq 50$ and $T \geq 25$) reveal that the one-sided LR test for this model is consistent and its power approaches unity asymptotically.

6. An Empirical Example

We present an empirical macroeconomic application of the dynamic conditionally heteroscedastic stochastic frontier model using a panel of 23 OECD countries[7] observed over the 26 years $1965 - 1990$. This paper makes use of an updated version of the Penn World Tables (PWT), Mark 5.6. Summers and Heston (1991) give a detailed description of the construction of the benchmark studies and national accounts data used in the construction of PWT.

The purpose of this application is mainly to provide an illustration of a typical application of the model. For that purpose, we specify and estimate a simple Cobb-Douglas stochastic frontier model under constant returns to scale with a first order dynamic conditional heteroscedastic specification for the technical inefficiency.

[7] The following countries are included in the study: Canada, U.S.A., Japan, Austria, Belgium, Denmark, Finland, France, Germany (West), Greece, Iceland, Ireland, Italy, Netherlands, Norway, Portugal, Spain, Sweden, Switzerland, Turkey, U.K., Australia, and New Zealand.

The stochastic frontier model is given by

$$y_{it} = x'_{it}\beta + v_{it} - u_{it}, \tag{6.1}$$

where $i = 1, ..., 23$ indexes countries and $t = 0, 1, ..., 25$ indexes time periods ($t = 0$ correspond to the year 1965 on which the ML estimation of the $DCH(1)$ model is conditioned). The frontier random term v_{it} is assumed *iid* normally distributed according to (2.2) as $v_{it} \sim N(0, \sigma_v^2)$ and the first order dynamic conditional heteroscedastic specification is given by (2.3) and (2.4) as $u_{it} \sim |\eta_{it}| h_{it}^{1/2}$, with $\eta_{it} \sim N(0, 1)$ *iid* and $h_{it} = \alpha_0 + \alpha_1 \varepsilon_{it-1}^2$.

The output $y_{it} = \ln RGDPW$ is given by the logarithm of real gross domestic product per worker, the input vector is three dimensional and contains (1) an intercept, $x_{1,it} = 1$ (2) the logarithm of capital stock per worker, $x_{2,it} = \ln KAPW$, and (3) a time trend, $x_{3,it} = YEAR = 1, 2, ..., 26$, for the years $1965 - 1990$ to capture the technical change. The $RGDPW$ and $KAPW$ variables are measured in 1985 years international prices.

Table 3 gives some descriptive statistics of the data.

Table 3: Summary statistics

Variable	Mean	Std. Dev.	Min.	Max.
$\ln RGDPW$	9.920	0.403	8.233	10.514
$\ln KAPW$	9.942	0.602	7.768	11.204

6.1. Results

The ML estimation results are given below (absolute t-values in parentheses)[8]:

The frontier model:

$$\ln RGDPW_{it} = \underset{(30.6)}{5.66} + \underset{(23.3)}{0.446} \ln KAPW_{it} + \underset{(2.46)}{0.0023} YEAR, \tag{6.2}$$

The conditional variance:

$$h_{it} = \underset{(2.4E-6)}{0.00} + \underset{(12.9)}{0.799} \varepsilon_{it-1}^2, \tag{6.3}$$

[8]The parameter start values are obtained as described in Section 4. A sensitivity check revealed stable convergence to the reported estimates with start values $\alpha_1^0 > 0$.

Variance parameter: $\hat{\sigma}_v^2 = \underset{(9.53)}{0.0050}$,

Loglikelihood: $\ln L = 422.96$.

All coefficient estimates are significant, except the intercept α_0 in the conditional variance function. The loglikelihood ratio test statistic of the null hypothesis $H_0 : \alpha_1 = 0$, that there are no dynamic conditional heteroscedasticity and that the data could be described by the standard production frontier model has a value of $\lambda = 556.65$. Based on this result the standard stochastic frontier model is clearly rejected[9] in favor of the alternative dynamic $CH(1)$ hypothesis $H_1 : \alpha_1 > 0$.

7. Summary

This paper has presented a new specification of the stochastic frontier models that introduce a general time-varying dynamic and heteroscedastic stochastic process for the technical inefficiency disturbance term. The proposed dynamic Conditionally Heteroscedastic (CH) specification specifies the variance of the firm and time-specific technical inefficiency to be conditionally heteroscedastic, conditioned on past realizations of the composed error term. That is, large frontier shocks or technical inefficiency shocks in previous periods increase the variance of the technical inefficiency in the present period. The proposed model specification offers a two-fold extension of the stochastic frontier models:

- Firm and time-specific technical inefficiency are allowed to follow a general dynamic stochastic process as opposed to other proposed specifications of time-varying technical inefficiency, such as Cornwell et al. (1990), Kumbhakar (1990), Battese and Coelli (1992) and Battese and Coelli (1995), where the time-varying pattern of technical inefficiency is governed by deterministic functions of time, in essence without dynamics. The specification offers an alternative to other dynamic specifications, such as the nonlinear dynamic model by Lee and Schmidt (1993) and the recent linear autoregressive model by Ahn et al. (1997).

[9] The critical values for this test, based on Table 1 in Kodde and Palm (1986) are given by 2.707, 3.841 and 5.412 for tests with significance level 0.05, 0.025 and 0.01, respectively.

- The proposed model incorporates a heteroscedastic specification of the technical inefficiency as opposed to the standard homoscedastic frontier models. The heteroscedasticity specification offers an alternative dynamic extension of the heteroscedasticity panel data model specification by Kumbhakar (1993). Other heteroscedasticity models are mainly specified for cross-sections, such as, e.g., Caudil et al. (1995) and Battese et al. (1997), where for the cross-sectional setup, the variance of the technical inefficiency is specified as a function of exogenous variables related to firm size.

The standard stochastic frontier model is nested within the proposed conditionally heteroscedastic model. This allows a simple likelihood ratio test of the proposed model specification, where the null specifies that the data generating process can be characterized as a standard stochastic frontier model and the alternative represents the proposed conditionally heteroscedastic model. We argue that a one-sided likelihood ratio test of the of the null hypothesis should be based on asymptotic distributions that incorporate the boundary restrictions of the parameters in the CH-model. That is, the asymptotic distribution is mixed χ^2 rather than standard χ^2 as for the standard likelihood ratio test. Monte Carlo simulations are performed to investigate the small-sample size and power properties of the standard and the one-sided LR tests. The results reveal that the standard LR test is under-sized, whereas the one-sided LR tests based on mixed χ^2 critical values have correct size for all considered small to medium sized panels (with cross-sectional dimension $N \leq 100$ and time-series dimension $T \leq 50$). This leads us to motivate the use of one-sided LR tests with mixed χ^2 critical values. The simulation results reveal that the one-sided LR test has relatively high power for small panels with cross-sectional and time-series dimension as small as $N = T = 10$. As the panel-size increase, in either the cross-sectional or the time-series dimension, the power of the LR test increases quite rapidly. The results for larger panel combinations (with $N = 50$ and 100 and $T = 25$ and 50) indicate that the one-sided LR test for this model is consistent and it power approaches unity asymptotically.

An empirical application of the model is included using a panel of 23 OECD countries over the period $1965 - 1990$. The estimation results indicate a clear rejection of the standard stochastic frontier model in favor of a first order dynamic conditionally heteroscedastic stochastic frontier model.

The proposed model is the first approach to model technical inefficiency in stochastic frontier models to follow a dynamic stochastic process based on the time-series $ARCH$ model for conditional heteroscedasticity. Important extensions

of the proposed homogeneous model left to future research are to allow technical inefficiency to follow various heterogeneous dynamic conditional heteroscedastic stochastic processes, possibly under more general $GARCH$ models of conditional heteroscedasticity.

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