A Monte Carlo Analysis of Technical Inefficiency Predictors

Subal C. Kumbhakar
Department of Economics
University of Texas at Austin
Austin, Texas 78712-1173, USA
E-mail: kkar@mundo.eco.utexas.edu

Mickael Löthgren
Department of Economic Statistics
Stockholm School of Economics
P.O. Box 6501
S-113 83 Stockholm, SWEDEN
E-mail: stml@hhs.se

Working Paper Series in Economics and Finance, No. 229
March 1998
Stockholm School of Economics

Abstract

This paper studies performance of both point and interval predictors of technical inefficiency in the stochastic production frontier model using a Monte Carlo experiment. In point prediction we use the Jondrow et al. (1980) results, while for interval prediction the Horrace and Schmidt (1996) and Hjalmarsson et al. (1996) results are used. When ML estimators are used we find negative bias in point predictions. MSEs are found to decline as the sample size increases. The mean empirical coverage accuracy of the confidence intervals are found to be significantly below the corresponding theoretical confidence levels for all values of the variance ratio.

Key words: Bias, MSE, point and interval estimators, production frontier

JEL Classification Number: C15.

1. Introduction

This paper deals with estimation of technical inefficiency in the stochastic production frontier models developed by Aigner et al. (1977) and Meeusen and van den Broeck (1977). We examine the nature of uncertainty associated with inefficiency estimates used in the empirical literature using a Monte Carlo approach. The issue of alternative estimators of parameters in the stochastic production frontier model has been previously addressed (Olsen et al. (1980) (henceforth OSW), Coelli (1995)). Here we focus primarily on technical inefficiency prediction – both point and interval predictors.

Jondrow et al. (1982, henceforth JLMS) suggested a technique to estimate[1] technical inefficiency specific to each observation (producer-specific technical inefficiency in a cross-sectional model). Uncertainty associated with such point estimates has been addressed by Horrace and Schmidt (1996) (henceforth HS) by constructing confidence intervals for technical efficiency estimates. Both the point estimates given by JLMS and the confidence intervals proposed by HS are based on the fact the "true values" of the parameters associated with the production frontier and distributions of the error terms (technical inefficiency and statistical noise) are known. Since estimates of these parameters can only be known, another uncertainty (arising from replacing the true parameter values by their estimates) is added to the problem of predicting technical inefficiency. The effect to this uncertainty on technical inefficiency estimates cannot be assessed analytically. Here we do a Monte Carlo analysis to assess the effect of parameter uncertainty on the estimates of technical inefficiency.

Previous Monte Carlo studies in the stochastic production frontier models by OSW (1980) and Coelli (1995) focused mostly on performance of different estimators such as the corrected OLS, ML, and method of moments. Coelli (1995) reported some results on the estimates of technical efficiency. Our focus here is exclusively on the performance of point and interval estimators of technical inefficiency.

The rest of the paper is organized as follows. In Section 2 we introduce the point and interval estimators of technical inefficiency. Design of the Monte Carlo study is discussed in Section 3. Results are reported in Section 4. Finally, Section 5 summarizes the main

[1] Here we use estimate and predict interchangeably. Similar is the case with estimator and predictor.

findings of the paper.

2. The Model

The stochastic production frontier model proposed by Aigner et al. (1977) and Meeusen and van den Broeck (1977) can be written as

$$y_i = x_i\beta + v_i - u_i, \quad i = 1, \ldots, N \tag{1}$$

where y and x_i are logarithms of output and inputs, and β is the unknown parameter vector. Technical inefficiency is represented by the one-sided term $u_i \geq 0$, technical efficiency by $\exp(-u_i) < 1$, and the stochastic noise component by v_i which is two-sided. In a cross-sectional data model the parameters are mostly estimated by the ML method based on $u_i \sim i.i.d.\, N(0, \sigma_u^2)$ truncated at zero from below, and $v_i \sim i.i.d.\, N(0, \sigma_v^2)$. Furthermore, u_i and v_i are assumed to be independent of each other, and also independent of the input vector x.

In 1982 JLMS proposed a method to estimate technical inefficiency for each observation. This result opened up the use of stochastic frontier models to empirical applications widely. They showed that $u_i|\varepsilon_i \sim N(\mu_i^*, \sigma_*^2)$ truncated at zero from below, where $\mu_i^* = -\gamma\varepsilon_i$, $\gamma = \sigma_u^2/(\sigma_u^2 + \sigma_v^2)$, $\sigma_*^2 = \gamma\,(1-\gamma)\,\sigma^2$, $\sigma^2 = \sigma_u^2 + \sigma_v^2$, and $\varepsilon_i = u_i - v_i$. Based on this result they suggested two point predictors of technical inefficiency (u_i). These are the mean and the mode of the conditional distribution of $u_i|\varepsilon_i$, viz.,

$$\hat{u}_i \equiv E(u_i|\varepsilon_i) = \mu_i^* + \sigma_*\{\frac{\phi(-\mu_i^*/\sigma_*)}{\Phi(\mu_i^*/\sigma_*)}\}, \tag{2a}$$

$$\tilde{u}_i \equiv Mode(u_i|\varepsilon_i) = \begin{cases} \mu_i^* & \text{if } \mu_i^* \leq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{2b}$$

where $\phi(.)$ and $\Phi(.)$ are the probability density and distribution functions, respectively, of a standard normal random variable.

Horrace and Schmidt (1996) derive expressions for $(1-\alpha)\cdot 100\%$ confidence intervals for technical efficiency $TE_i = \exp(-u_i)$. These confidence intervals are based on monotonic transformations of the $\alpha/2$ and $(1-\alpha/2)$ quantiles of the distribution of $u_i\,|\varepsilon_i$. A $(1-\alpha)\,100\%$ confidence interval (L_i, U_i) of $u_i\,|\varepsilon_i$ is given in Hjalmarsson et al. (1996)

(henceforth HKH)

$$\begin{aligned} L_i &= \mu_i^* + z_L\,\sigma_*, \\ U_i &= \mu_i^* + z_U\,\sigma_*, \end{aligned} \tag{3}$$

where $z_L = \Phi^{-1}\left\{1 - (1-\alpha/2)\,\Phi\left(\mu_i^*/\sigma_*\right)\right\}$, $z_U = \Phi^{-1}\left\{1 - (\alpha/2)\,\Phi\left(\mu_i^*/\sigma_*\right)\right\}$.

The point estimators in (2) and confidence interval estimators in (3) are derived under the assumption that the "true" parameter values are known (which implies that $\varepsilon_i = y_i - x_i\beta$ are also known). So the question is whether estimates of inefficiency (point as well as interval estimates) are biased in one way or the other, when the true parameter values are replaced by their estimates, and ε is replaced by the residuals from (1). JLMS noted that the sampling variability due to replacing the true values by their estimates will disappear asymptotically. We examine this from a Monte Carlo experiment. In addition to the sample size we also examine whether the parameter values (mainly the γ parameter) has any role in getting somewhat precise estimator of technical inefficiency.

3. Design of the Monte Carlo Study

We follow the same design of the Monte Carlo study as used by Coelli (1995). The sample space of the experiment is β, σ^2, γ, N and $X = (x_1, ..., x_N)$. Due to the invariance results noted by OSW (1980) only one value of the variance σ^2 is considered ($\sigma^2 = \sigma_u^2 + \sigma_v^2 = 1.0$). The sample space is further reduced (following Coelli and OSW) by including only an intercept as a regressor which is set at $\beta = 1.0$.

As in Coelli (1995) we control for the variance ratio γ^* that reflects the percentage contribution of the variance of u to the total variance of the error term $\varepsilon = v - u$ in the data generating process. This variance ratio is defined as $\gamma^* = (1-2/\pi)\,\sigma_u^2/\left\{(1-2/\pi)\sigma_u^2 + \sigma_v^2\right\}$.

Eleven variance ratios $\gamma^* = 0.0, 0.10, \ldots, 1.0$ and six sample sizes $N = 25$, 50, 100, 200, 400, and 800 are considered in the study. In each of the 66 combinations of the variance ratio and sample size the simulations involve 1000 Monte Carlo replications, giving a total of 66000 generated data sets.

The random terms $v_i, i = 1, \ldots, N$, are generated from $v_i \sim i.i.d.N(0, (1-\gamma))$ and the technical inefficiency terms $u_i, i = 1, \ldots, N$, are generated from $u_i \sim i.i.d.N(0, \gamma)$ truncated at zero from below. Note that $\sigma^2 = 1 \Rightarrow \sigma_u^2 = \gamma$ and $\sigma_v^2 = (1-\gamma)$. Once u and v are generated, the y values are calculated from $y_i = 1.0 + v_i - u_i,\ i = 1, \ldots, N$.

The numerical maximizations are done using the DFP-algorithm with numerical gradient. The expressions for the likelihood function and the gradients can be found in Battese and Coelli (1993).

In each Monte Carlo replicate, we keep track of whether the firm-specific confidence intervals for technical inefficiency cover the true inefficiencies u_i or not by defining an indicator variable. The simulated estimate of the firm-specific empirical coverage accuracy is obtained from the average of the indicator variable over the 1000 Monte Carlo replicates. The total of N firm-specific empirical coverage accuracies are finally summarized by using the average of the N firm-specific coverage accuracies (as well as the minimum, maximum, and standard deviation). Furthermore, we calculate the bias, variance, and MSE of the point-predictors of technical inefficiency.

In addition to the above-mentioned features of the confidence interval coverage accuracy and the point predictors, we also calculate the bias, variance, and MSE of the ML parameter estimates which were examined both in OSW (1980) and Coelli (1995).

4. Results

Properties of the ML and some other estimators have been investigated quite extensively in studies by OSW and Coelli (1995). The reader is referred to these studies for a detailed discussion on this issue. Our results on the bias, variance, and MSE of the ML parameter estimates are similar to those in Coelli (1995) and OSW (1980). Because of this similarity we are not reporting these results.[2] Instead, here we focus our attention to the results regarding the confidence intervals and point predictors of technical inefficiency.

Monte Carlo estimates of bias and MSE for the point predictions of technical inefficiency are based on the JLMS predictor given in (2a). These predictors are calculated based on (i) the true parameter values and (ii) the ML parameter estimates. In calculating bias, $E(u_i|\varepsilon_i)$ in (i) is obtained from (2a), while in (ii) we use the same formula for $E(u_i|\varepsilon_i)$ but γ is replaced by the ML estimate of γ and ε_i are replaced by the residuals from (1). Bias is calculated for each Monte Carlo replicate. The reported bias is the average of the 1000 Monte Carlo replicates. Similarly, $\mathrm{Var}(\hat{u})$ is calculated based on known and MLE of

[2] These results can be obtained from the authors upon request.

the parameters. Finally, MSE of $\hat{u}$ is obtained by adding the variance and squared bias of $\hat{u}$, in both (i) and (ii).

Table/Figure 1a and 1b present bias and MSE results for $\hat{u}$ based on the true parameter values and Table/Figure 2a and 2b present bias and MSE results for $\hat{u}$ based on the ML estimates of the parameters. From Table/Figure 1a we see that the mean technical inefficiency predictions based on true parameters are almost zero. This result follows from the law of iterative expectations $(E(E(u_i|\varepsilon_i)) = E(u_i))$.

Although mean bias is almost zero for all values of γ^* between 0.1 and 0.9,[3] the variance of the point predictor based on the true parameters is not necessarily zero or close to zero. This is because $E(u_i|\varepsilon_i)$ cannot extract all the information about u_i given $\varepsilon_i = v_i - u_i$. The size of the variance (which is not reported here because the variance is the same as MSE, $E\{E(u_i|\varepsilon_i)\}$ being equal to $E(u_i)$) is reflected in the MSE reported in Table/Figure 1b.

It can be seen from Table/Figure 1b that MSE first increased for $\gamma^* = 0.1$ to 0.4 and then decreased monotonically from about 0.13 associated with $\gamma^* = 0.4$ to about 0.03 for $\gamma^* = 0.9$. The pattern is almost identical for all sample sizes.

Table/Figure 2a shows that technical inefficiency predictions based on ML estimates are negatively biased except for the variance ratio $\gamma^* = 0.0$ (for all sample sizes) and $\gamma^* = 0.1$ (for all sample sizes except for $N = 800$). When $\gamma^* = 0$ there are no technical inefficiency in the true model, and thus a non-negative predictor of u will naturally be positively biased. The magnitude of biases (in absolute value) is much higher compared to the case when the parameters are assumed to be known. Although the MLEs are consistent, the point predictor of u is not. Thus, the bias in estimating technical inefficiency is not likely to vanish with large sample. However, we see a few patterns which are worth mentioning. Barring the case when $\gamma^* = 0$ the biases (in absolute value) tend to get smaller when N gets larger. This is true for all values of γ^*. In particular, when $N = 800$ and γ^* takes values between 0.5 and 0.9 the magnitude of bias (absolute value) varies within the range 0.003 to 0.008 (not in ascending or descending order of γ^*). The evidence of smaller bias with large N might be due to the fact that MLEs approach to the true

[3] Note that the point predictions are not defined for variance ratio $\gamma^* = \gamma = 0$ or 1.0 as can be seen from equation (2a). For this reason no bias and MSE are reported for $\gamma^* = 0$ and 1.

parameters as N gets larger. Thus, the source of uncertainty associated with unknown parameters is reduced when N is increased. This phenomenon is reflected in the MSE of technical inefficiency estimates as well.

It can be seen from Table/Figure 2b that MSEs for the technical inefficiency predictors based on MLEs follow a hump-shaped pattern. For each sample size, the MSE reaches a maximum at the variance ratio (γ^*) that approximately maximizes (absolute) bias of the technical inefficiency predictor. Furthermore, similar to the bias, MSEs decrease with an increase in the sample size for each variance ratio. The MSE of technical inefficiency is higher for each sample size when compared to the case of known parameters. We also find that the bias squared component in MSE is less than the variance part. Furthermore, the bias squared part declines with an increase in N faster than the variance part.

We now report results associated with confidence intervals of technical inefficiency predictors. We calculate empirical coverage accuracies based on whether the calculated interval includes the true value of u for confidence levels 0.80, 0.90, 0.95 and 0.975, respectively. These interval calculations are based on both true and estimated parameters. When evaluated at the true parameter values the mean coverage accuracies are always equal to the confidence level for all sample sizes and irrespective of variance ratio γ^*. Because of this we are not reporting empirical coverage accuracies based on the true parameter values. The standard deviations of the coverage accuracies can easily be obtained from $\sqrt{(}p(1-p)/N \cdot 1000)$ for various confidence levels (0.80, 0.90, 0.95 and 0.975, etc.,) and sample sizes ($N = 25$, 50, 100, 200, 400, and 800). The standard deviations of the simulated coverage accuracies are found to be quite small. For example, when the sample size is 25 the standard deviation of the 95% confidence interval is slightly below 0.0014. (The corresponding standard deviation for $N = 50$ can be obtained similarly by dividing these numbers by $\sqrt{2}$, and so on for other N.) From this result, we conclude that a 95% confidence interval for the simulated coverage accuracy has a width of only 0.54%. This indicates the high precision in the coverage accuracies obtained from the simulation study.

The intervals based on the MLEs show a clear tendency to undercover, i.e., the (average) empirical coverage accuracy of the confidence intervals are clearly below the theoretical confidence level. The undercoverage is maximum for $N = 25$ and the confidence interval is 0.80. When N increases coverage accuracy tends to increase for each value of γ^*. This result is expected from the consistency of the ML estimators. Figure/Table 3a presents the results for the 95%[4] confidence interval. It can be seen from the table that for $N = 400$ or more mean coverage accuracy is very close to the theoretical confidence levels for γ^* between 0.6 and 0.9.

In addition to the mean values of coverage accuracies, we also present the corresponding standard deviations of the simulated coverage accuracies in Table 3b. The standard deviations of the confidence interval coverage accuracy for various combinations of sample sizes and variance ratios γ^* are calculated from $\sqrt{(\hat{p}(1-\hat{p})/(N \cdot 1000)}$, where $\hat{p}$ denotes the estimate of coverage accuracy. From the results in Table 3b we see that the standard deviations are basically constant for variance ratios $\gamma^* > 0$, and for all sample sizes. The standard deviations reach a maximum of 0.003, for the smallest sample size $N = 25$, and tend to decrease for larger samples. For the largest sample size $N = 800$, the smallest standard deviation is 0.0002 for $\gamma^* > 0$. These standard deviations can be used to construct confidence intervals of coverage accuracies, and/or to test hypotheses regarding coverage accuracies, e.g., testing whether empirical coverage accuracies are equal to the theoretical confidence level. For instance, a 95% confidence interval for the coverage accuracy estimate is given by $\hat{p} \pm 1.96 \cdot \sqrt{(\hat{p}(1-\hat{p})/(N \cdot 1000))}$, based upon which one can conduct hypothesis tests (at the 5% level of significance) whether the simulated coverage accuracy equals the theoretical confidence level or not. From the standard deviation results in Table 3b, we can reject the null hypothesis that the simulated empirical coverage accuracy equals the theoretical confidence level for all variance ratios, and all sample sizes.

These results reveal that there is a potential risk in drawing inference on technical inefficiency using the confidence interval procedure outlined in HS and HKH especially when the sample size is small (below 200), and the estimated value of γ^* is small (below 0.5).

5. Conclusions

This paper studies performance of both point and interval predictors of technical inefficiency in the stochastic production frontier model. In point prediction of technical

[4] We only include the results for the 95% confidence intervals to economize space. The results for the other confidence levels are similar and can obtained from the authors upon request.

inefficiency using the JLMS result we find that, on average, there is no bias if the parameters are *known*. So the MSEs are entirely from the variance part which tends to decline with an increase in the sample size. When the MLEs are used instead of the true parameters, we find negative biases for values of the variance ratio $\gamma^* > 0.2$, and in all sample categories. These biases tend towards zero as sample size increases. In general, there is a declining trend in MSE with higher values of the variance ratio. This is true for all sample categories.

In interval estimation, mean coverage accuracies are always equal to the corresponding confidence level when the *true* parameters are used. The standard deviations of the coverage accuracies are quite low. The mean coverage accuracy of the confidence intervals are below the theoretical confidence level when the MLEs are used. However, the standard deviations are small and quite stable for all values of the variance ratios.

Based on these results we conclude that there is a high risk in both point and confidence interval prediction of technical inefficiency when the parameters are not known. We find that biases play smaller role in the MSE of the point predictors.

References

Aigner, D., C. A. K. Lovell, and P. Schmidt (1977), Formulation and Estimation of Stochastic Frontier Production Function Models, *Journal of Econometrics 6*, 21-37.

Battese, G. E. and T. J. Coelli (1993), A Stochastic Frontier Production Function Incorporating a Model for Technical Inefficiency Effects, Working Papers in Econometrics and Applied Statistics No. 69, University of New England, Armidale.

Coelli, T. J. (1995), Estimators and Hypothesis Tests for a Stochastic Frontier Function: A Monte Carlo Analysis, *Journal of Productivity Analysis 6*, 247-268.

Hjalmarsson, L., S. C. Kumbhakar, and A. Heshmati (1996), DEA, DFA and SFA: A Comparison, *Journal of Productivity Analysis 7*, 303-327.

Horrace, W. C. and P. Schmidt (1996), Confidence Statements for Efficiency Estimates from Stochastic Frontier Models, *Journal of Productivity Analysis 7*, 257-282.

Jondrow, J., C. A. K. Lovell, I. S. Materov, and P. Schmidt (1982), On the Estimation of Technical Inefficiency in the Stochastic Frontier Production Function Model, *Journal of Econometrics 19*, 233-238.

Meeusen, W., and J. van den Broeck (1977), Efficiency Estimation from Cobb-Douglas Production Function with Composed Errors, *International Economic Review 18*, 435-444.

Olson, J. A., P. Schmidt, and D. W. Waldman (1980), A Monte Carlo Study of Estimators of Stochastic Frontier Production Functions, *Journal of Econometrics 13*, 67-82.