Should central banks be more aggressive?*

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Abstract

Simple models of monetary policy often imply optimal policy behavior that is considerably more aggressive than what is commonly observed. This paper argues that such counterfactual implications are due to model restrictions and a failure to account for multiplicative parameter uncertainty, rather than to policymakers being too cautious in their implementation of policy. Comparing a restricted and an unrestricted version of the same empirical model, the unrestricted version leads to less volatility in optimal policy, and, taking parameter uncertainty into account, to policy paths very close to actual Federal Reserve policy.

Keywords: Optimal monetary policy, parameter uncertainty, interest rate smoothing.

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1 Introduction

It is a common observation that central banks implement monetary policy in a gradual manner. As documented by Rudebusch (1995) and the Bank for International Settlements (1998), central banks tend to adjust their interest rate instrument in small, persistent steps, moving the interest rate several times in the same direction before reversing policy. To understand such behavior of policymakers, we need to develop theoretical models that are consistent with the empirical evidence.

Many simple models designed for monetary policy analysis, such as those used by Ball (1997), Cecchetti (1998), Svensson (1997a,b), and Wieland (1998), have the attractive property that the optimal monetary policy rule is a simple linear function of the state of the economy, similar to a Taylor (1993) rule. In a dynamic setting, the central bank acts to minimize the variation over time of the goal variables from their targets, so when facing a shock, the policy instrument is moved away from the initial position, and then gradually returned towards a neutral stance (see, e.g., Ellingsen and Söderström, 1998).

It has been noted that these models often imply considerably more aggressive policy than what is empirically observed. For example, Rudebusch (1998a) and Rudebusch and Svensson (1998) show that the restricted reaction function from an empirical version of the Svensson (1997a,b) model has considerably larger coefficients than those shown by Taylor (1993) to match the behavior of the Federal Reserve.[1] Also, Ellingsen and Söderström (1998) show that the simple Svensson model implies excessive volatility and 'whip-sawing' behavior of the short interest rate for reasonable parameter values. Therefore, to match the observed behavior, it is common to introduce an explicit interest rate smoothing motive into the objective function of the central bank (see, e.g., Rudebusch and Svensson, 1998).

However, although such a smoothing objective might be motivated by central banks' concern about financial market stability (see Goodfriend, 1989, or Cukierman, 1991) or uncertainty about the economic environment (Blinder, 1998; Bank for International Settlements, 1998), if the basic model is misspecified, we should be wary about its policy predictions. Also, as shown by Sack (1998a), an interest rate smoothing objective is not necessary to match the policy path of the Federal Reserve using a standard vector autoregression (VAR) model. Instead, multiplicative parameter uncertainty acts to dampen optimal policy, leading to paths for the

[1]The restricted reaction function allows policy to respond only to current output and inflation, following a simple Taylor rule.

federal funds rate which are very close to those actually observed for the period from 1983 to 1996.

This paper further analyzes the properties of optimal monetary policy in the model developed by Svensson (1997a,b) by estimating a version of the model on U.S. data, and comparing the obtained estimates with results from an unrestricted VAR model of the same variables. The analysis shows that the optimal policy in both the restricted and the unrestricted model is more aggressive than observed policy, implying more volatility in the short interest rate than is observed in reality. However, policy in the restricted model is more aggressive than in the unrestricted model, pointing to the importance of the model's restrictions. Introducing multiplicative parameter uncertainty makes policy less aggressive in both models, following the result of Brainard (1967), but the restricted model still implies far too volatile interest rates to match the data. The unrestricted model, on the other hand, leads to policy that is very close to observed policy, in parallel with the results of Sack (1998a). These results indicate that the general setup with an optimizing central bank is a good approximation of actual policy behavior, whereas the restrictions imposed in the Svensson model are at odds with the data.

Rudebusch and Svensson (1998) estimate a similar version of the model on similar data, without examining the dynamic policy response to shocks, and conclude, on the basis of statistical information criteria, that the model restrictions are *not* at odds with the data. As is shown below, however, formal hypothesis tests of the restrictions leads one to reject the restricted model in favor of the unrestricted model. Rudebusch (1998a) introduces several types of uncertainty into the Svensson model in an attempt to make the coefficients of the optimal restricted Taylor rule match the empirical rule for the U.S. Taking the model setup as given, he finds that combinations of data and parameter uncertainty lead to more reasonable reaction functions.

The present paper is organized as follows. Section 2 presents the dynamic model introduced by Svensson (1997a,b), relates that model to an unrestricted VAR model, and estimates the two models on quarterly U.S. data. In Section 3, optimal policy rules for the models are derived, and the resulting reaction functions and policy responses over time are compared with actual Federal Reserve behavior. Section 4 introduces parameter uncertainty into the models, and discusses the consequences for optimal policy, and Section 5 compares the implied path of the federal funds rate from the models with the actual funds rate path. Finally, Section 6 offers some concluding remarks.

2 A dynamic framework

2.1 The Svensson model

The monetary policy model to be analyzed is the dynamic framework developed by Lars Svensson (1997a,b). This framework, which has been primarily used to study issues of inflation targeting, contains the important aspects that the policymaker has imperfect control over the inflation rate, and that policy, implemented through an interest rate instrument, affects the economy with a lag. Most importantly, the policymaker cannot affect the inflation rate directly, but only via the output gap, and with an extra control lag. Thus, monetary policy affects the output gap with a one-period lag and the inflation rate with a lag of two periods. As shown below, this feature, designed to be consistent with the stylized facts of the monetary transmission mechanism, has important implications for the behavior of monetary policy when responding to innovations to inflation and output.

The model consists of two relationships between inflation, output (or the output gap), and a short (one-period) interest rate, controlled by the central bank. In a general formulation, with an unspecified number of lags, the output gap in period $t+1$ is determined by the IS-relationship

$$y_{t+1} = \alpha(L)y_t + \beta(L)\left(i_t - \pi_t\right) + \varepsilon^y_{t+1}, \tag{1}$$

where y_t is the percentage deviation of output from its trend (or 'potential') level; i_t is the central bank's interest rate instrument (or its deviation from the long-run mean) at an annualized rate; π_t is the annualized inflation rate, in percentage points (also its deviation from its long-run mean, or target); and ε^y_{t+1} is an i.i.d. demand shock, with zero mean and constant variance. The output gap is thus assumed to depend on past values of itself and past realizations of the ex-post short real interest rate, or the 'pseudo-real' interest rate (Svensson, 1997a). The inflation rate is assumed to follow an accelerationist-type Phillips curve;

$$\pi_{t+1} = \delta(L)\pi_t + \gamma(L)y_t + \varepsilon^\pi_{t+1}, \tag{2}$$

thus being determined by past inflation, past values of the output gap, and an i.i.d. supply shock ε^π_{t+1}, also with zero mean and constant variance. To close the model, a quadratic loss function is assigned to the central bank, and then the bank's optimal control problem is solved to obtain a decision rule for the short interest rate, contingent on the development of output and inflation.[2]

[2]Note that the model is formulated in deviations from targets or long-run means, so negative values of all variables are allowed.

This setup is clearly a severe simplification of the true economy, but it could be interpreted as reduced-form relationships from a more complete model including sticky prices and some kind of transmission mechanism of monetary policy, such as the standard interest rate channel or a credit channel. Under this reduced-form interpretation, any policy experiments in this model are clearly at odds with the Lucas (1976) critique. However, Fuhrer (1995) argues that Phillips curve specifications like equation (2) are very close to being 'structural' relationships, since they do not seem to change much over time.

The model is also subject to criticism for not incorporating forward-looking behavior of agents. In particular, the Phillips curve (2) does not include inflation expectations, except in the adaptive form of a distributed lag of past inflation rates. The IS-specification (1) includes an ex-post real interest rate instead of an ex-ante real rate, which arguably is more important for investment behavior, or credit market considerations.[3] Again, however, Fuhrer (1997) shows that expectations of future prices are not very important in determining price and inflation behavior: backward-looking price specifications are actually favored by the data. On the other hand, backward-looking models exhibit long-run dynamics which are less consistent with existing evidence. Accepting equation (2) as a reasonable specification for the inflation rate, Svensson (1997b) shows how an IS-equation with an ex-ante real interest rate is easily transformed into an IS-equation like equation (1).[4]

Finally, Estrella and Fuhrer (1998) argue that many dynamic models incorporating rational expectations and optimizing behavior have the counterfactual implication that the inflation rate (or real spending) jumps in response to shocks, making them unsuitable for short-run monetary policy analysis. A version of the Svensson setup, with partially forward-looking behavior, is shown to be more consistent with the data.

2.2 A VAR interpretation

As pointed out by Rudebusch and Svensson (1998) and Rudebusch (1998a), the model (1)–(2) can be interpreted as restrictions on the first two equations of a trivari-

[3]Eijffinger et al. (1998) include an ex-ante long real interest rate in the specification of the IS-curve, and find that optimal policy becomes more aggressive than in the original formulation with the short real rate.

[4]A third criticism of the model is that it does not strictly obey the natural-rate hypothesis, since the central bank could increase output indefinitely by accepting accelerating inflation. Given the loss function assigned to the central bank (see below), such behavior will never be optimal (Svensson, 1997b).

ate vector autoregression (VAR) model containing the output gap, the inflation rate, and the short interest rate.[5] Writing out the three equations, and assuming that the central bank responds to current output and inflation when setting the interest rate, but that policy has no contemporary effects on the economy, such an unrestricted VAR system is given by

$$y_t = \sum_{s=1}^{L} A_s^y y_{t-s} + \sum_{s=1}^{L} B_s^y \pi_{t-s} + \sum_{s=1}^{L} C_s^y i_{t-s} + \xi_t^y, \tag{3}$$

$$\pi_t = \sum_{s=1}^{L} A_s^\pi y_{t-s} + \sum_{s=1}^{L} B_s^\pi \pi_{t-s} + \sum_{s=1}^{L} C_s^\pi i_{t-s} + \xi_t^\pi, \tag{4}$$

$$i_t = \sum_{s=0}^{L} A_s^i y_{t-s} + \sum_{s=0}^{L} B_s^i \pi_{t-s} + \sum_{s=1}^{L} C_s^i i_{t-s} + \xi_t^i. \tag{5}$$

The Svensson model then puts restrictions on the parameters in the first two equations, and assumes that the parameters of the third equation are obtained from the central bank's optimization problem. The parameter restrictions imply that $B_s^y = -C_s^y$ and $C_s^\pi = 0$ for all s.

Although these restrictions may seem plausible, it is conceivable that they are not consistent with the true transmission mechanism of monetary policy. If, for example, output were affected by the ex-ante real interest rate (or even the long real rate), and inflation expectations were not directly related to past inflation, the restriction on the output equation would be rejected. Also, one could argue that the restricted inflation equation is likely to be at odds with the data: although Phillips curve relationships like (2) seem to hold empirically (see, e.g., Fuhrer, 1997, or Blanchard and Katz, 1997), monetary policy could possibly affect inflation without affecting the level of output first, for example, if there were bottlenecks in the economy. In that case, a monetary easing would create excess demand, that could not be satisfied directly with increased output. Then inflation would increase *before* output, leading to a direct link from monetary policy to inflation.

Following Rudebusch and Svensson (1998), a first test of the Svensson model is to estimate the restricted equations (1)–(2) on quarterly U.S. data, and compare the results with those obtained from estimating the unrestricted VAR model (3)–(5).[6]

[5]That the methodology behind VAR models is not entirely uncontroversial can be seen from the debate between Rudebusch (1998b,c) and Sims (1998).

[6]When estimating the Svensson model, Rudebusch and Svensson (1998) use four lags of inflation and one lag of output in the inflation equation, and two lags of output and one lag of the average real interest rate for the last four quarters in the output equation. Also, the sum of the B_s^π coefficients is not significantly different from unity, so the authors impose that restriction in the estimation. Thus, a third restriction of $B_j^y = B_h^y$ and $C_j^y = C_h^y$ for $j, h = 1, \ldots, 4$, and a fourth restriction

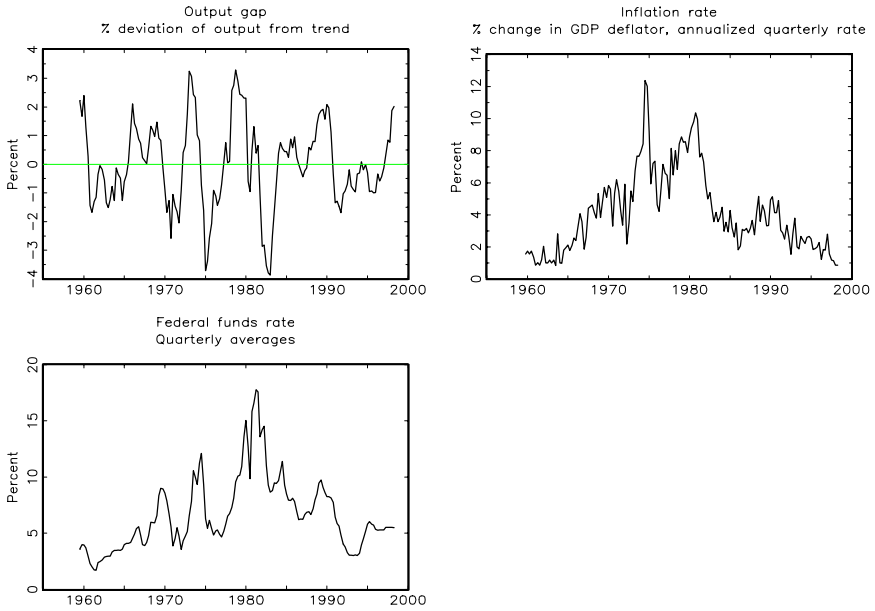

Figure 1: Data series 1959:3–1998:2

2.3 Data

The two models are estimated on quarterly U.S. data from 1959:3 to 1998:2; graphs of the data series are shown in Figure 1. The output series used is real GDP, measured in billions of fixed 1992 dollars and seasonally adjusted. The output gap is defined as the percentage deviation of output from trend, where the trend is calculated using a Hodrick-Prescott filter. The price series is the implicit GDP deflator, seasonally adjusted, and the inflation rate is the quarterly percentage change in the price index, at an annual rate. Both of these series are from the Bureau of Economic Analysis at the U.S. Department of Commerce. The interest rate used is the quarterly average of the effective federal funds rate, taken from the Board of Governors of the Federal Reserve.[7] All data have been downloaded from the FRED

of $\sum_s B_s^\pi = 1$ are imposed. Since the Rudebusch-Svensson setup leads to extreme volatility (and sometimes exploding paths) in the optimal interest rate, I choose not to impose these additional restrictions here, and instead concentrate on the restrictions from the original Svensson model.

[7]During this sample period, the Federal Reserve has occasionally changed its policy instrument, most notably during the experiment of non-borrowed reserves targeting from 1979 to 1982. Although the preferred choice of policy indicator varies across researchers, Bernanke and Mihov (1998), while concluding that no simple measure of policy is appropriate for the entire period from 1965 to 1996, show that a federal funds rate targeting model marginally outperforms models of borrowed reserves and non-borrowed reserves targeting for the whole sample period.

database of the Federal Reserve Bank of St. Louis at *http://www.stls.frb.org/fred/*.[8] Since the Svensson model is formulated in deviations from long-run means or targets, all variables are de-meaned before estimation, so no constants will appear in the regressions.

2.4 Estimation and hypothesis tests

Table 1 shows the results from estimating the unrestricted VAR model and the restricted model on quarterly data, using four lags.[9] Since the independent variables are likely to be highly multicollinear, it does not make much sense to discuss the significance of individual coefficients. Note, however, that the coefficients on inflation in the unrestricted model's output equation are not very close to the negative of the interest rate coefficients.

Table 2 shows some simple criteria for model selection. Depending on how strongly the different criteria penalize extra explanatory variables, one or the other model is selected. Using the most common criterion, adjusted R^2, leads to a preference for the unrestricted model, for both the output and the inflation equation. As criteria are chosen to punish extra right-hand variables more heavily, there is a gradual shift towards the restricted model. The Akaike information criterion chooses the unrestricted output equation but the restricted inflation equation, whereas using the Schwarz information criterion, the restricted model is preferred (recall that a smaller value of the information criteria is preferred to a larger). Consequently, as noted by Rudebusch and Svensson (1998), the simple criteria give a split decision as to which model to choose.

For formal hypothesis tests, Table 3 shows the results from univariate F-tests of each restriction separately (in the upper panel) and bivariate likelihood ratio tests for the two-equation system (in the lower panel), both on each restriction separately and jointly on both restrictions. The univariate and the bivariate tests of the separate hypotheses give very similar results: at the 5% confidence level we reject the hypothesis of $B_s^y = -C_s^y$ in the output equation, but we cannot reject the

[8]Many authors, for example, McCallum (1993), Orphanides (1998), Rudebusch (1998a), and Ghysels et al. (1998), stress the importance of data uncertainty for economic modeling; using final (revised) data in econometric estimation, and in particular in policy rules, is highly inappropriate, since these data are typically not available at the time of the policy decisions. On the other hand, although data on GDP and prices are only available with a delay, central banks do have access to a number of indicators of output and prices, which they use when formulating policy.

[9]Likelihood ratio tests on the VAR model reject the hypotheses of two and three lags in favor of four lags, but do not reject the hypothesis of four lags against five lags. See Hamilton (1994) for details.

Table 1: Estimated coefficients in restricted and unrestricted models

	Restricted		Unrestricted		
	y_t	π_t	y_t	π_t	i_t
y_t					0.465**
					(0.114)
y_{t-1}	1.070**	0.213	1.050**	0.077	−0.005
	(0.085)	(0.127)	(0.089)	(0.139)	(0.168)
y_{t-2}	−0.023	−0.002	0.005	0.074	−0.146
	(0.123)	(0.185)	(0.124)	(0.195)	(0.167)
y_{t-3}	−0.175	0.128	−0.177	0.206	0.046
	(0.121)	(0.183)	(0.120)	(0.188)	(0.162)
y_{t-4}	−0.061	−0.050	−0.056	−0.081	−0.069
	(0.085)	(0.127)	(0.085)	(0.134)	(0.114)
π_t					0.086
					(0.072)
π_{t-1}	0.045	0.579**	0.084	0.564**	−0.010
	(0.042)	(0.083)	(0.053)	(0.084)	(0.083)
π_{t-2}	0.063	0.006	−0.051	0.042	0.122
	(0.051)	(0.095)	(0.061)	(0.096)	(0.082)
π_{t-3}	−0.093	0.201*	−0.058	0.185	0.003
	(0.050)	(0.095)	(0.061)	(0.096)	(0.083)
π_{t-4}	0.027	0.142	0.053	0.180*	−0.102
	(0.043)	(0.082)	(0.055)	(0.086)	(0.075)
i_{t-1}	−0.045		0.051	0.162	0.929**
	(0.042)		(0.063)	(0.099)	(0.086)
i_{t-2}	−0.063		−0.277**	−0.215	−0.291*
	(0.051)		(0.085)	(0.133)	(0.119)
i_{t-3}	0.093		0.260**	−0.034	0.290*
	(0.050)		(0.087)	(0.136)	(0.120)
i_{t-4}	−0.027		−0.079	0.036	−0.007
	(0.043)		(0.064)	(0.100)	(0.086)
$\bar{R}^2$	0.799	0.835	0.807	0.838	0.930

Coefficient estimates from quarterly restricted Svensson model and unrestricted VAR model, 151 observations 1960:4 to 1998:2. Standard errors in parentheses, **/* denote significance at the 1%-/5%-level. In the y_t regression of the restricted model, the coefficients on i_{t-s} are restricted to be the negative of those on π_{t-s}.

Table 2: Simple criteria for model selection

	Output equation		Inflation equation	
	Restricted	Unrestricted	Restricted	Unrestricted
$\bar{R}^2$	0.799	0.807	0.835	0.838
Akaike	641.150	637.997	773.356	774.309
Schwarz	665.289	674.204	797.494	810.516

Adjusted R^2, Akaike, and Schwarz information criteria for the restricted and the unrestricted model.

Table 3: Hypothesis tests

Null	Test statistic	Distribution	Significance level
Univariate F-tests			
$B_s^y = -C_s^y$	2.664	$F(4, 139)$	0.035
$C_s^\pi = 0$	1.660	$F(4, 139)$	0.163
Bivariate LR-tests			
$B_s^y = -C_s^y$	10.268	$\chi^2(4)$	0.036
$C_s^\pi = 0$	6.488	$\chi^2(4)$	0.166
Joint hypothesis	16.732	$\chi^2(8)$	0.033

Hypothesis tests of restrictions in the model (3)–(4).

hypothesis of $C_s^\pi = 0$ in the inflation equation. The joint hypothesis is nevertheless rejected at the 5%-level. Thus, using formal hypothesis tests, we lean towards a rejection of the restricted Svensson model in favor of the unrestricted VAR model, and both hypothesis tests and the information criteria hint that the restriction on the output equation is more severe than that on the inflation equation.

3 Optimal policy

In the previous section we have seen that the restrictions of the simple Svensson model do not find very strong support in the data. In the remainder of this paper, we shall see how important these restrictions are for the optimal path of monetary policy. Assigning a loss function to the central bank, it is straightforward to calculate the bank's optimal decision rule for both the restricted and the unrestricted model. Since the Svensson model is a special case of the unrestricted VAR model, let us derive the optimal policy rule for the unrestricted model, and then apply the rule to both models.

The central bank is assumed to minimize the expected discounted sum of future values of a loss function, which is quadratic in output and inflation deviations from target (here normalized to zero). Thus, the central bank solves the optimization

problem

$$\min_{\{i_{t+\tau}\}_{\tau=0}^{\infty}} E_t \sum_{\tau=0}^{\infty} \phi^{\tau} L(y_{t+\tau}, \pi_{t+\tau}), \tag{6}$$

subject to (3)–(4), where in each period the loss function $L(\cdot)$ is given by

$$L(y_t, \pi_t) = \pi_t^2 + \lambda y_t^2, \tag{7}$$

and where $\lambda \geq 0$ is the weight of output stabilization relative to inflation fighting.[10] The parameter ϕ is the central bank's discount factor, set to 0.987 per quarter, implying an annual discount rate of around 5%.

To calculate the optimal policy rule, it is convenient to rewrite the general model (3)–(4) in state-space form as

$$x_{t+1} = Ax_t + Bi_t + \varepsilon_{t+1}. \tag{8}$$

Here x_t is an (11×1) state vector, given by current and lagged values of y_t and π_t, and lags of i_t,

$$x_t = \{y_t, \ldots, y_{t-3}, \pi_t, \ldots, \pi_{t-3}, i_{t-1}, \ldots, i_{t-3}\}; \tag{9}$$

the (11×11) matrix A has its first and fifth rows filled with the parameters from the VAR according to

$$A_1 = \left[\begin{array}{cccccccccccc} A_1^y & A_2^y & A_3^y & A_4^y & B_1^y & B_2^y & B_3^y & B_4^y & C_2^y & C_2^y & C_3^y \end{array} \right] \tag{10}$$

$$A_5 = \left[\begin{array}{cccccccccccc} A_1^\pi & A_2^\pi & A_3^\pi & A_4^\pi & B_1^\pi & B_2^\pi & B_3^\pi & B_4^\pi & C_2^\pi & C_2^\pi & C_3^\pi \end{array} \right], \tag{11}$$

and occasional ones on the other rows, to complete the identities; and the (11×1) vector B has zeros everywhere except for the first and fifth elements, which correspond to C_1^y and C_1^π, and the ninth element, which is 1.[11]

[10] This formulation of the central bank objective function brings to mind at least three comments. First, it is widely accepted that all modern central banks put some weight on output stabilization, even when their ascribed goal only includes inflation or price stability (see Svensson, 1998). Fischer (1996) criticizes the tendency of central banks to only acknowledge price stability as their objective. Second, note that the loss function is formulated in terms of the quarterly inflation rate, and not the yearly rate, which would be the average rate of the last four quarters. Targeting the yearly inflation rate often makes it optimal (if λ is small enough) to move the instrument in four-period cycles in response to shocks. Therefore the quarterly inflation rate is chosen in the loss function. Third, as mentioned in the Introduction, Rudebusch and Svensson (1998) choose to include an interest rate smoothing motive in the loss function. Since such a motive seems warranted only to make the model fit the data, and since Sack (1998a) finds that a dynamic model which takes parameter uncertainty into account leads to optimal policy that fits the actual data very well, I choose to not include such an objective.

[11] As above, the Svensson restrictions imply that $B_s^y = -C_s^y$ and $C_s^\pi = 0$.

The loss function (7) can then be written as

$$L_t = x_t' Q x_t, \tag{12}$$

where the preference matrix Q has λ as element $(1,1)$, 1 as element $(5,5)$, and zeros elsewhere. The central bank solves the control problem

$$J(x_t) = \min_{i_t} \left\{ x_t' Q x_t + \phi E_t J(x_{t+1}) \right\}, \tag{13}$$

subject to (8), and Appendix A shows that the optimal interest rate is given by

$$i_t = f x_t, \tag{14}$$

where the decision vector f is given by

$$f = -(B'VB)^{-1} B'VA, \tag{15}$$

and the matrix V is determined by the Ricatti equation

$$V = Q + \phi(A + Bf)'V(A + Bf). \tag{16}$$

(See also Chow, 1975 or Sargent, 1987, ch. 1.) Consequently, it is optimal for the central bank to set the interest rate instrument in each period as a function of current and lagged values of the output gap and the inflation rate, and lagged values of the instrument itself.

3.1 Reaction functions

Using the parameter values obtained from the unrestricted VAR model and the restricted model in Table 1 in the A-matrix and the B-vector, we can calculate the optimal policy rule from (14)–(16) numerically for the two models, for different values of the preference parameter λ. Table 4 shows the policy rules, or reaction functions, obtained for the two models with $\lambda = 0$ and $\lambda = 1$, along with the empirical estimates of the reaction function from the VAR model from equation (5) and Table 1.

As has been noted elsewhere, the coefficients in the optimal reaction functions are typically larger (in absolute value) than the empirical estimates. This is true for both models, although the restricted model has even larger coefficients than the unrestricted model, leading to more aggressive policy in the restricted model. This is especially striking for the response to current output and inflation, where the coefficients in the restricted model are extremely large. When $\lambda = 1$, that is, with equal weights on inflation and output stabilization, the coefficients are typically smaller than when $\lambda = 0$, but not for all variables. The optimal rules thus imply more aggressive policy than the empirical rule in the last column, which also seems more persistent, with a larger coefficient on the lagged interest rate.

Table 4: Optimal reaction functions

	Restricted		Unrestricted		Empirical
	$\lambda = 0$	$\lambda = 1$	$\lambda = 0$	$\lambda = 1$	
y_t	20.071	11.848	3.110	3.926	0.465
y_{t-1}	2.131	−0.719	0.392	−0.194	−0.005
y_{t-2}	−1.529	−1.843	−0.760	−0.838	−0.146
y_{t-3}	−1.623	−0.765	−0.178	−0.240	0.046
y_{t-4}					−0.069
π_t	16.405	4.018	1.487	1.173	0.086
π_{t-1}	11.559	1.426	1.224	0.598	−0.010
π_{t-2}	8.052	0.388	0.849	0.562	0.122
π_{t-3}	3.017	0.717	0.223	0.295	0.003
π_{t-4}					−0.102
i_{t-1}	−0.189	−0.091	−0.489	−0.314	0.929
i_{t-2}	0.874	0.779	0.638	0.684	−0.291
i_{t-3}	−0.298	−0.273	−0.168	−0.238	0.290
i_{t-4}					−0.007

Optimal reaction function (the vector f in equation (14)) from restricted and unrestricted models, and estimated empirical reaction function from Table 1.

3.2 The policy response over time

Using the calculated reaction functions and the transition dynamics of the models, we can calculate how policy responds over time to shocks to output and inflation by conducting the following experiment. In the first period, the economy is hit by a shock, either to output (ξ_t^y) or to inflation (ξ_t^π). This shock is then transmitted through the economy by equation (8), and the central bank responds optimally in each period according to its reaction function (14). Proceeding for a number of periods, we can trace the dynamic effects of a shock on policy by calculating how the central bank reacts in each period. This policy response is then similar to the impulse response function obtained from the VAR model, and thus the optimal response from the two models can be compared with the empirical response to shocks.

To calculate the impulse response functions of the VAR, however, we need to make some identifying assumptions concerning the structural relationships between the variables. A convenient method to identify the dynamic effects on one variable of a shock to another variable in the VAR is to assume that there is a causal ordering between the variables. A reasonable assumption in this particular model is that monetary policy is affected by current values of the output gap and the inflation

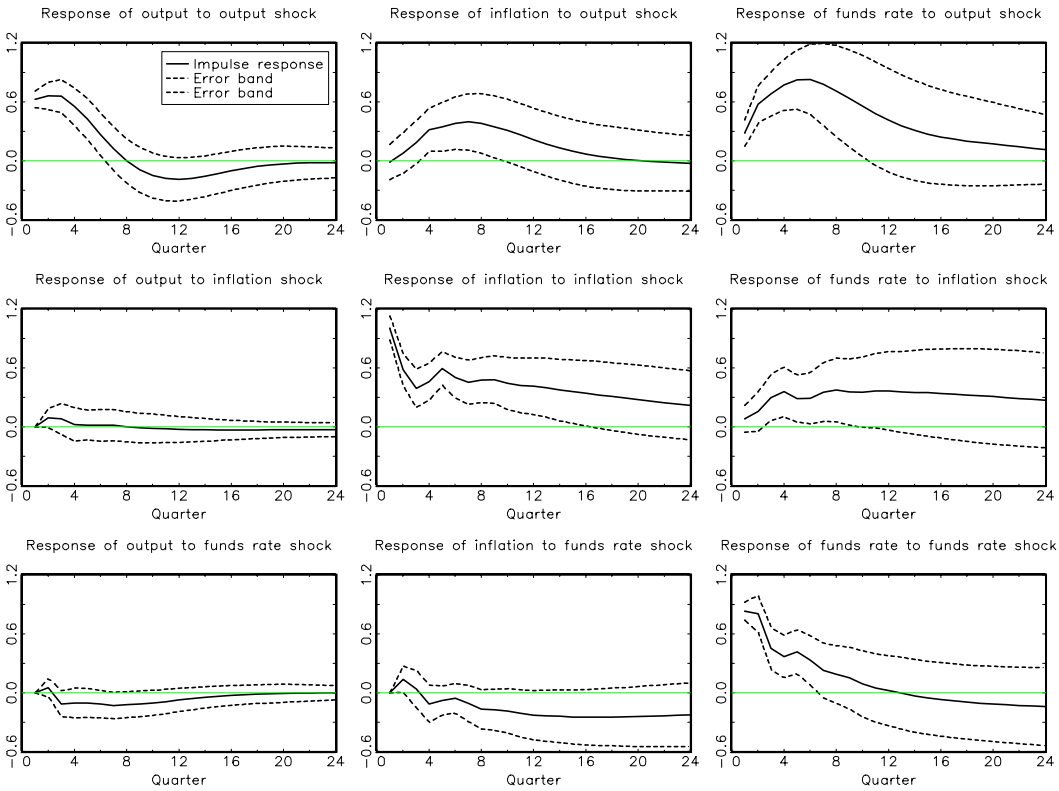

Figure 2: Impulse responses from VAR

rate, but that these do not respond to contemporaneous policy.[12] To identify the response of monetary policy to shocks to output and inflation, we need a further assumption, and following Rudebusch and Svensson (1998), Sack (1998a), Bagliano and Favero (1998), and many others, I shall assume that the inflation rate is affected by contemporaneous output, but not vice versa. Consequently, we end up with the ordering (y_t, π_t, i_t), and identification can be achieved through a Choleski decomposition.[13]

The resulting impulse responses are graphed in Figure 2 as the response of each variable to a unit shock to an orthogonalized innovation in another variable. The solid line is the estimated impulse response, and the dotted lines are confidence intervals of two standard deviations, calculated with Monte Carlo simulations. The impulse responses are consistent with the conventional view of the monetary trans-

[12]This recursive assumption is very common in the VAR literature, see Christiano et al. (1998) and references therein.

[13]The alternative ordering, with inflation before output, leads to very similar impulse responses.

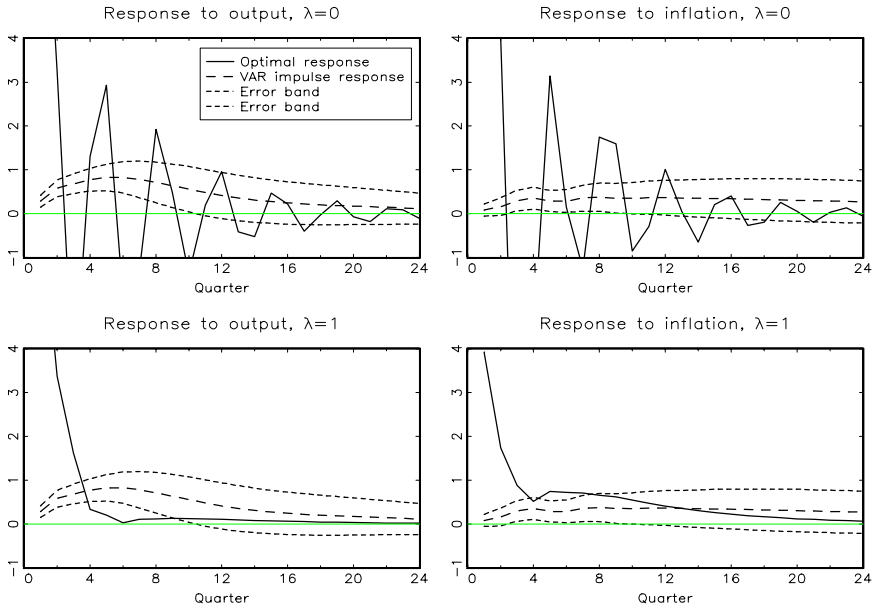

Figure 3: Estimated and optimal policy response to shocks, restricted model

mission mechanism from a number of VAR studies: after a funds rate shock, there is a sustained decline in output and inflation, and output reaches its minimum after four to eight quarters (these responses are not significant, however).[14] The funds rate response to output and inflation shocks is positive and persistent, and significant for the first ten quarters.

Letting the dynamic systems of the restricted and unrestricted models be hit by a shock of comparable size to that in the impulse responses,[15] we can trace the optimal policy response over time and compare it with the empirical impulse responses.[16] Figure 3 shows the optimal response of monetary policy in the restricted model along with the empirical impulse responses from the VAR including the two-standard deviation confidence intervals. The two left-hand graphs show the response

[14]Note that there is a tendency to a 'price puzzle' and an 'output puzzle,' i.e., that inflation and output increase slightly before falling after a monetary contraction (see the third row of columns 1 and 2), indicating that the VAR model is misspecified. The standard method of solving the price puzzle is to include commodity prices in the VAR as a leading indicator of inflation. See Christiano et al. (1998) for a discussion.

[15]A unit shock to the orthogonalized innovations in output and inflation corresponds to a 0.621 shock to output, and a 0.976 shock to inflation, respectively.

[16]Rudebusch and Svensson (1998) calculate the impulse response using the dynamics of their estimated restricted model, but including an estimated reaction function from a VAR model. Since the resulting impulse responses are not far from those of the VAR, they conclude that the model restrictions do not significantly alter the dynamics of the model relative to the unrestricted VAR.

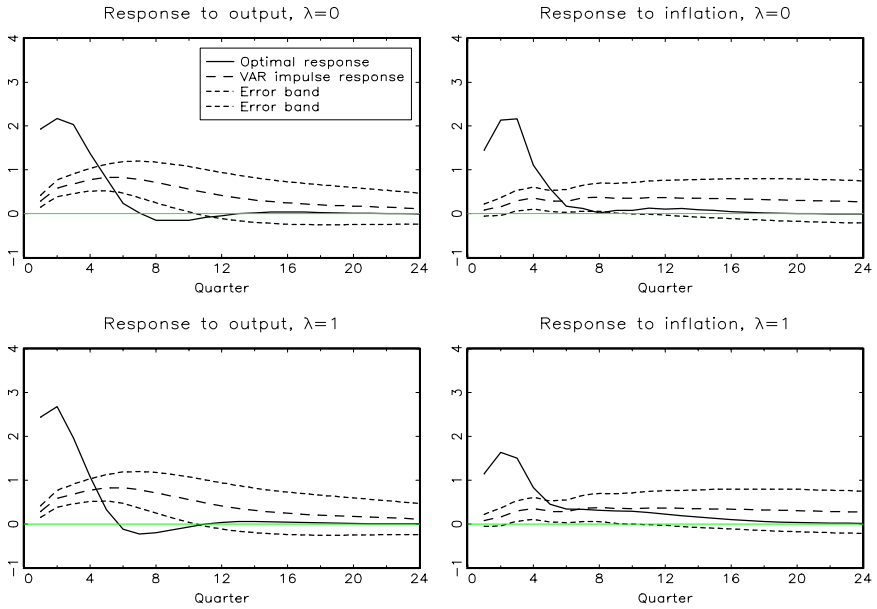

Figure 4: Estimated and optimal policy response to shocks, unrestricted model

to output shocks and the right-hand graphs the response to inflation shocks, with 'strict inflation targeting' ($\lambda = 0$) at the top and 'flexible inflation targeting' ($\lambda = 1$) at the bottom.

In the top row, where $\lambda = 0$, the restricted model implies an extremely volatile response to shocks, with large fluctuations in the central bank instrument.[17] When $\lambda = 1$, the response is less volatile, and more reasonable. Still, the initial response to both output and inflation shocks is very strong, whereas the impulse response is weak at first, and then increases somewhat before reverting back to zero. Consequently, the restricted Svensson model leads to substantially more aggressive policy behavior than what seems to be observed in practice.

Figure 4 shows the optimal response from the unrestricted model. As was clear from the reaction functions, optimal policy is less aggressive in the unrestricted model, and the response is much closer to the empirical impulse responses, even if the initial response is always too aggressive. As compared with the restricted model, the unrestricted policy response is more intuitively attractive, since it implies more interest rate smoothing, in the sense that policy is adjusted in the same direction at least twice before returning towards zero. In the restricted model, at least for

[17]The response in the first periods falls outside the graphs, with the response to output being 12.48 in the first period and -4.04 in the third period, and the response to inflation being 16.01 and -7.43, respectively.

$\lambda = 1$, it is always optimal to make a large initial adjustment and then quickly return towards zero.

From this experiment, we can conclude that the restrictions of the Svensson model have serious counterfactual implications not only for the coefficients of the optimal reaction function, but also for the path of monetary policy over time. Still, however, we are far away from a reasonable model of monetary policy, since the unrestricted model also implies considerably more interest rate volatility than what is empirically observed. In an attempt to add some realistic features to the models, the next section will evaluate the consequences of multiplicative parameter uncertainty for the optimal response of policy.

4 Parameter uncertainty

The assumption of additive uncertainty in macroeconomic modeling is very convenient when deriving optimal policy rules, since, coupled with a quadratic loss function, the optimal policy rule depends only on the first moments of the goal variables (so 'certainty equivalence' holds). It has long been known that multiplicative uncertainty, for example uncertainty about the parameters in a model, has important implications for the optimal behavior of policymakers. The analysis of Brainard (1967) shows that a policymaker who is uncertain about the multiplier of policy should be less aggressive in his policy moves, at least if covariances are small.[18] This result has recently been stressed by Blinder (1997, 1998) and Goodhart (1998) as having a major relevance for practical policymaking within the Federal Reserve and the Bank of England. Also, Sack (1998a) has shown that allowing for parameter uncertainty makes the optimal policy path from a standard unrestricted VAR model very similar to the actual path of Federal Reserve policy.[19]

One can think of a number of reasons why policymakers are not certain about the

[18] Contributions by Craine (1979) and Söderström (1999) show that the Brainard result does not apply to all types of multiplicative parameter uncertainty: uncertainty about the impact of policy leads to less aggressive policy, whereas uncertainty about the adjustment dynamics of the economy leads to more aggressive policy than under certainty equivalence.

[19] Apart from uncertainty about model parameters, one can also imagine other sources of uncertainty that complicate the policymaker's situation. Rudebusch (1998) investigates the effects of several sources of uncertainty in the same model framework: multiplicative parameter uncertainty, uncertainty about the quality of incoming data, and uncertainty about the means of parameters. In doing so, he does not use the standard methods of dynamic optimization, but instead simulates the economy a number of times for each configuration of decision rules to find the optimal restricted Taylor rule. This method is more flexible than the optimization techniques used in this paper, but also more time-demanding.

parameters in a model of the economy (see, e.g., Holly and Hughes Hallett, 1989). Parameters could be genuinely random, as agents adjust their behavior over time. The source of such randomness would then need to be found in more complete models of, for example, price-setting and investment behavior, that is, in the underlying equations of a reduced-form system. Alternatively, the parameters could be fixed in reality, but estimated by policymakers over finite samples, thus leading to randomness in point estimates. Finally, the model could be a linear approximation of a non-linear model, so that parameters vary in a well-defined but imperfectly known manner.

In this section, the second type of parameter uncertainty will be assumed, so the parameter matrices A and B vary stochastically over time, with known means and variances, but I disregard issues of learning and experimentation by assuming that the realizations of parameters are drawn from the same known distribution over time.[20] Consequently, I will continue to use the econometric estimates from Table 1, which were obtained assuming that parameters are non-stochastic.

The state-space formulation of the general model is then

$$x_{t+1} = A_{t+1}x_t + B_{t+1}i_t + \varepsilon_{t+1}, \tag{17}$$

where A_{t+1} and B_{t+1} are stochastic, with means A and B, variance matrices Σ_A and Σ_B, and covariance matrix Σ_{AB}. It is assumed that all parameters are independent of each other, so in the unrestricted model Σ_{AB} is zero, whereas in the restricted model, it is non-zero, since $B_1^y = -C_1^y$.

The central bank faces the same control problem

$$J(x_t) = \min_{i_t} \{x_t' Q x_t + \phi E_t J(x_{t+1})\} \tag{18}$$

but now subject to (17), leading to the policy rule

$$i_t = \tilde{f} x_t. \tag{19}$$

Now, however, the reaction function depends not only on the parameter means, but also on their variances. Appendix B shows that the solution to the central bank's problem is given by

$$\begin{aligned} \tilde{f} &= -\left[B'(\tilde{V} + \tilde{V}')B + 2\tilde{v}_{11}\Sigma_B^{11} + 2\tilde{v}_{55}\Sigma_B^{55}\right]^{-1} \\ &\times \left[B'(\tilde{V} + \tilde{V}')A + 2\tilde{v}_{11}\Sigma_{AB}^{11\prime}\right], \end{aligned} \tag{20}$$

[20] See Sack (1998b) or Wieland (1998) for analyses of learning and experimentation in models of monetary policy. However, to quote former Vice-Chairman of the Federal Reserve Board, Alan Blinder (1998, p. 11), "You don't conduct experiments on a real economy solely to sharpen your econometric estimates." See also Sargent (1998) for a discussion of this issue.

Table 5: Optimal reaction functions under parameter uncertainty

	Restricted		Unrestricted		Empirical
	$\lambda = 0$	$\lambda = 1$	$\lambda = 0$	$\lambda = 1$	
y_t	4.801	4.615	1.288	1.339	0.465
y_{t-1}	−0.661	−0.679	−0.106	−0.149	−0.005
y_{t-2}	−0.826	−0.871	−0.251	−0.267	−0.146
y_{t-3}	−0.372	−0.329	−0.107	−0.108	0.046
y_{t-4}					−0.069
π_t	2.377	1.601	0.565	0.510	0.086
π_{t-1}	0.795	0.454	0.214	0.159	−0.010
π_{t-2}	0.419	0.119	0.234	0.206	0.122
π_{t-3}	0.414	0.296	0.151	0.148	0.003
π_{t-4}					−0.102
i_{t-1}	−0.011	−0.006	−0.191	−0.167	0.929
i_{t-2}	0.311	0.312	0.233	0.237	−0.291
i_{t-3}	−0.120	−0.120	−0.078	−0.085	0.290
i_{t-4}					−0.007

Optimal reaction function (the vector $\tilde{f}$ in equation (19)) from restricted and unrestricted models under multiplicative parameter uncertainty, and estimated empirical reaction function from Table 1.

where

$$\begin{aligned} \tilde{V} &= Q + \phi(A + B\tilde{f})'\tilde{V}(A + B\tilde{f}) \\ &+ \phi\tilde{v}_{11}\left(\Sigma_A^{11} + 2\Sigma_{AB}^{11}\tilde{f} + \tilde{f}'\Sigma_B^{11}\tilde{f}\right) + \phi\tilde{v}_{55}\left(\Sigma_A^{55} + \tilde{f}'\Sigma_B^{55}\tilde{f}\right), \end{aligned} \tag{21}$$

and where Σ_{AB}^{ij} is the covariance matrix of the ith row of A with the jth row of B.

Using the estimated parameter standard errors from the different models as a measure of the uncertainty concerning individual parameters, but assuming all covariances across parameters to be zero, we can plug in the parameter mean and variance estimates from Table 1 into the modified reaction function (19)–(21). The resulting reaction functions are given in Table 5. Comparing with the certainty equivalence case in Table 4, the coefficients under multiplicative parameter uncertainty are considerably smaller, leading to less aggressive policy, following the Brainard intuition. Policy is still more aggressive in the restricted than in the unrestricted model, which, in turn, is more aggressive than the empirical policy behavior.

The optimal responses of policy over time under parameter uncertainty are shown in Figures 5 and 6. In the restricted model of Figure 5, parameter uncertainty makes optimal policy much less volatile in response to a shock, especially for the case where $\lambda = 0$. At least for the first periods, however, the optimal response is considerably

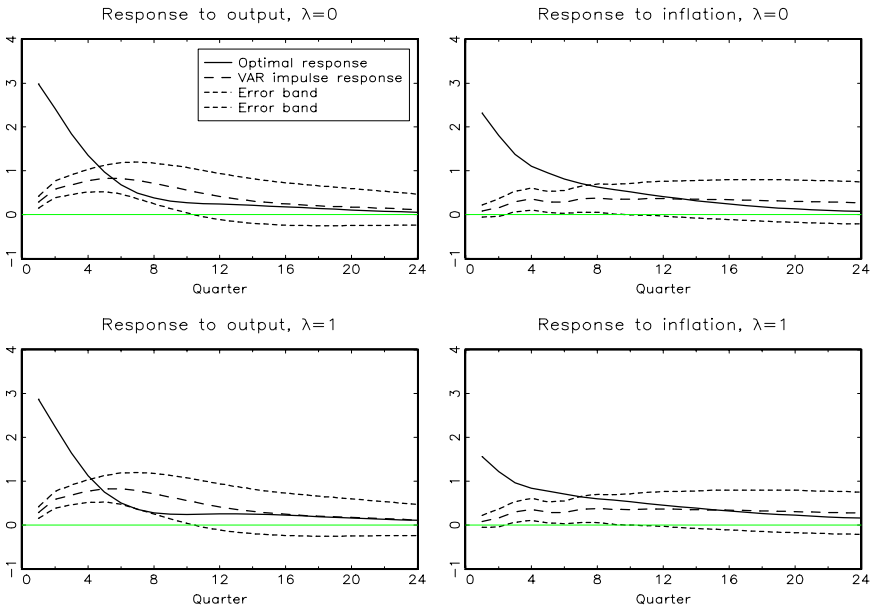

Figure 5: Estimated and optimal policy response to shocks in restricted model under parameter uncertainty

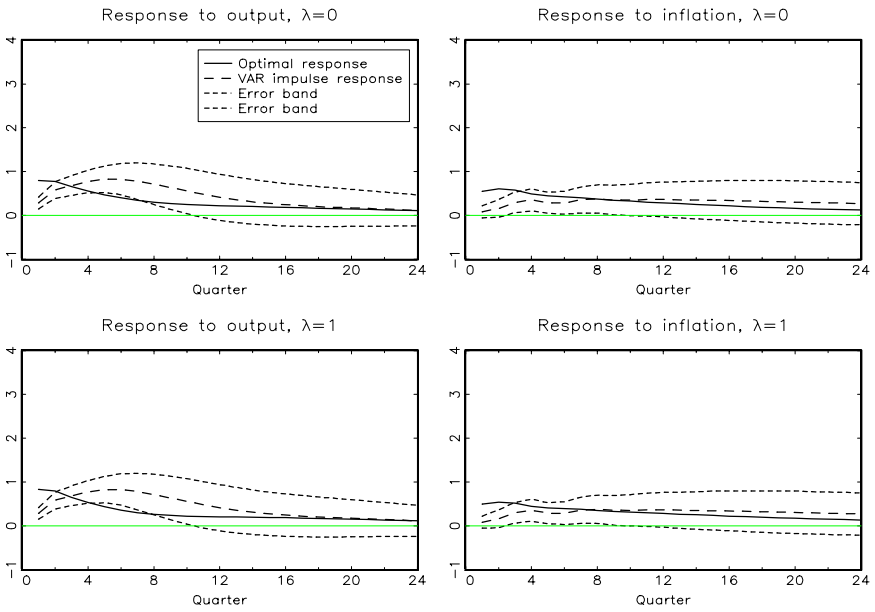

Figure 6: Estimated and optimal policy response to shocks in unrestricted model under parameter uncertainty

stronger than the empirical impulse response. The unrestricted model in Figure 6 is also less volatile than under certainty equivalence, and now implies an optimal response which is very similar to the observed response. The optimal response lies outside the confidence bands of the impulse response functions only during the first periods; in later periods, it is very close to the observed behavior.

Consequently, taking parameter uncertainty into account, at least in this configuration of uncertainty, leads to less aggressive policy for both models.[21] The unrestricted policy response is now very close to the empirically observed response, whereas the restricted model still implies too aggressive behavior as compared with the empirical impulse responses.

5 The implied path of the funds rate

As a final experiment, we can calculate the implied optimal path of the federal funds rate over the sample period by applying the different reaction functions to the actual data for the U.S. economy. Comparing the resulting path with the actual path of the funds rate gives a further illustration of the results of previous sections.

Letting the central bank respond in an 'optimal' manner to output, inflation, and past values of the funds rate, assuming that the weights of output and inflation in the loss function are equal (so $\lambda = 1$), the implied paths of the funds rate from 1959 to 1998 are shown in Figure 7. The two top graphs show the implied paths from the restricted and unrestricted models under certainty equivalence and the actual funds rate path, and the two bottom graphs show the paths under parameter uncertainty. The standard deviations of the funds rate in the different models and in the actual path are shown in Table 6, along with the mean squared deviation of the optimal path from the actual funds rate path.

It is immediately clear, from both Figure 7 and Table 6, that the restricted model implies considerably more interest rate volatility than the unrestricted model, especially in the certainty equivalence case (note that the scales on the vertical axes in Figure 7 differ across graphs).[22] The unrestricted model under certainty equivalence and the restricted model under parameter uncertainty are remarkably

[21] In some configurations of uncertainty in the restricted model, with much emphasis on uncertainty concerning the B_s^π-coefficients, the optimal policy under parameter uncertainty is actually *more* aggressive than under certainty equivalence. See Söderström (1999) for a discussion of this result within a simpler one-lag version of the Svensson model.

[22] A serious flaw of the methodology applied is also obvious from Figure 7: it allows for negative values of the nominal interest rate. For models taking the zero-bound on nominal interest rates into account, see, e.g., Fuhrer and Madigan (1997) or Orphanides and Wieland (1998).

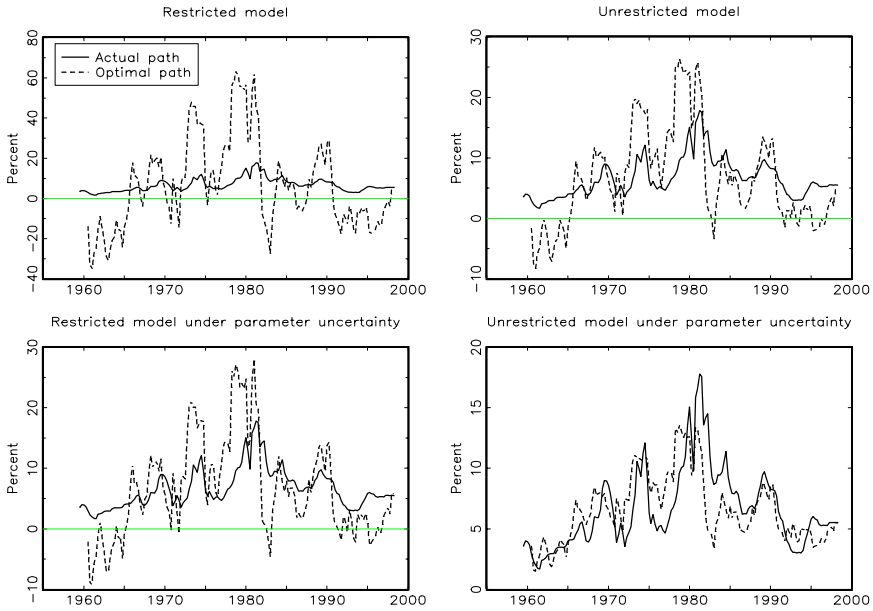

Figure 7: Actual and optimal interest rate paths, 1959–98

similar in their policy paths, although they are far from the actual path. The only reasonable approximation of the true policy path comes from the unrestricted model under parameter uncertainty. As seen in the bottom right-hand graph of Figure 7, the implied optimal path of policy is very similar to the actual path; according to Table 6, optimal policy is even less volatile than actual policy, although it has a tendency to lead actual policy in the response to macroeconomic developments.

It is remarkable how close we can get to mimicking the actual behavior of the Federal Reserve by introducing parameter uncertainty into an unrestricted optimizing model, without including an interest rate smoothing objective into the central bank's loss function. As noted by Sack (1998a), such an assumption of interest rate smoothing does not seem to be warranted solely because of the apparent propensity of central banks to smooth their interest rate instrument. Instead, such behavior can equally plausibly be the result of simple optimizing behavior of the central bank, taking into account the dynamic properties of the economy and the effects of uncertainty on policy.

6 Final remarks

The results of this paper indicate that the restrictions introduced in the simple macroeconomic model of Svensson (1997a,b) are responsible for the model's failure

Table 6: Comparison of optimal and actual funds rate paths

	Certainty equivalence		Parameter uncertainty
Standard deviation: levels			
Actual		3.265	
Restricted model	22.426		8.158
Unrestricted model	7.998		2.801
Standard deviation: differences			
Actual		1.058	
Restricted model	9.201		3.569
Unrestricted model	2.899		1.018
Mean squared deviation from actual			
Restricted model	415.116		41.914
Unrestricted model	38.484		6.347

Standard deviations of optimal and actual funds rate and mean squared deviations of optimal from actual funds rate. In the derivation of the optimal funds rate, $\lambda = 1$.

to match the observed policy behavior. The coefficients of the optimal decision rule are considerably larger than those of empirical reaction functions, leading to excessive interest rate variability in response to shocks. In contrast, an unrestricted VAR model leads to less volatility in the policy instrument, and, taking parameter uncertainty into account, policy predictions which are very close to the observed behavior of the Federal Reserve, as suggested by Sack (1998a).

From the formal hypothesis tests, the restriction on the output equation seems more at fault than that on the inflation equation. However, additional experiments indicate that the extreme interest rate volatility emanates from the inflation restrictions rather than the output restrictions. In any case, more work on the exact specification seems warranted if one is to come up with a model that better fits the empirical facts.

A Solving the control problem

From equation (13), the central bank solves the problem

$$J(x_t) = \min_{i_t} \{x_t' Q x_t + \phi E_t J(x_{t+1})\} \tag{22}$$

subject to

$$x_{t+1} = Ax_t + Bi_t + \varepsilon_{t+1}. \tag{23}$$

Since the objective function is quadratic and the constraint linear, the value function will be of the form

$$J(x_t) = x_t' V x_t + w. \tag{24}$$

Using the transition law to eliminate the next period's state, the Bellman equation is

$$x_t' V x_t + w = \min_{i_t} \{x_t' Q x_t + \phi(Ax_t + Bi_t)' V (Ax_t + Bi_t) + \phi w\}. \tag{25}$$

The first-order condition for the minimization problem is then[23]

$$B'VBi_t = -B'VAx_t, \tag{26}$$

leading to the optimal interest rate

$$\begin{aligned} i_t &= -(B'VB)^{-1} B'VAx_t \\ &= fx_t. \end{aligned} \tag{27}$$

Substituting the decision rule into the Bellman equation (25), we get

$$\begin{aligned} x_t' V x_t + w &= x_t' Q x_t + \phi\left[(Ax_t + Bfx_t)' V (Ax_t + Bfx_t) + w\right] \\ &= x_t' \left[Q + \phi(A + Bf)' V (A + Bf)\right] x_t + \phi w. \end{aligned} \tag{28}$$

Thus V is determined by the Ricatti equation

$$V = Q + \phi(A + Bf)' V (A + Bf), \tag{29}$$

where

$$f = -(B'VB)^{-1} B'VA. \tag{30}$$

[23] Use the rules $\partial x'Ax/\partial x = (A + A')x$, $\partial y'Bz/\partial y = Bz$, and $\partial y'Bz/\partial z = B'y$, and the fact that V is symmetric. See, e.g., Ljungqvist and Sargent (1997).

B The stochastic control problem

From (18), the bank's problem under parameter uncertainty is

$$J(x_t) = \min_{i_t} \{x_t' Q x_t + \phi E_t J(x_{t+1})\} \tag{31}$$

subject to

$$x_{t+1} = A_{t+1} x_t + B_{t+1} i_t + \varepsilon_{t+1}. \tag{32}$$

The value function will still be

$$J(x_t) = x_t' \tilde{V} x_t + \tilde{w}, \tag{33}$$

but now with expected value

$$E_t J(x_{t+1}) = (E_t x_{t+1})' \tilde{V} (E_t x_{t+1}) + \text{tr}\left(\tilde{V} \Sigma_{t+1|t}\right) + \tilde{w}, \tag{34}$$

where the expected value of x_{t+1} is given by

$$E_t x_{t+1} = A x_t + B i_t, \tag{35}$$

and where $\Sigma_{t+1|t}$ is the covariance matrix of x_{t+1}, evaluated at t, and 'tr' denotes the trace operator.

Following Holly and Hughes Hallet (1989), the (i,j)th element of $\Sigma_{t+1|t}$ is given by

$$\Sigma^{ij}_{t+1|t} = x_t' \Sigma^{ij}_A x_t + 2 x_t' \Sigma^{ij}_{AB} i_t + i_t' \Sigma^{ij}_B i_t + \Sigma^{ij}_\varepsilon, \tag{36}$$

where Σ^{ij}_{AB} is the covariance matrix of the ith row of A with the jth row of B. Since at t, y_{t+1} and π_{t+1} are the only stochastic variables in x_{t+1}, and these are assumed independent of each other, the only non-zero entries of $\Sigma_{t+1|t}$ are the matrices $\Sigma^{11}_{t+1|t}$ and $\Sigma^{55}_{t+1|t}$.

The (11×11) matrix Σ^{11}_A has diagonal elements

$$\left[\begin{array}{ccccccccccc} \sigma^2_{A^y_1} & \sigma^2_{A^y_2} & \sigma^2_{A^y_3} & \sigma^2_{A^y_4} & \sigma^2_{B^y_1} & \sigma^2_{B^y_2} & \sigma^2_{B^y_3} & \sigma^2_{B^y_4} & \sigma^2_{C^y_2} & \sigma^2_{C^y_3} & \sigma^2_{C^y_4} \end{array} \right], \tag{37}$$

and other elements equal to zero, and, likewise, the diagonal of Σ^{55}_A is

$$\left[\begin{array}{ccccccccccc} \sigma^2_{A^\pi_1} & \sigma^2_{A^\pi_2} & \sigma^2_{A^\pi_3} & \sigma^2_{A^\pi_4} & \sigma^2_{B^\pi_1} & \sigma^2_{B^\pi_2} & \sigma^2_{B^\pi_3} & \sigma^2_{B^\pi_4} & \sigma^2_{C^\pi_2} & \sigma^2_{C^\pi_3} & \sigma^2_{C^\pi_4} \end{array} \right]. \tag{38}$$

The variances Σ^{11}_B and Σ^{55}_B are simply $\sigma^2_{C^y_1}$ and $\sigma^2_{C^\pi_1}$, and both Σ^{11}_{AB} and Σ^{55}_{AB} are zero in the general setup, assuming parameters are uncorrelated with each other. In the

Svensson model, however, the restriction $B_1^y = -C_1^y$ implies that Σ_{AB}^{11} is an (11×1) vector given by

$$\Sigma_{AB}^{11} = \left[\begin{array}{cccccccccccc} 0 & 0 & 0 & 0 & -\sigma^2_{C_1^y} & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]', \tag{39}$$

whereas Σ_{AB}^{55} is still a vector of zeros. Finally, the covariances of the shocks are given by $\Sigma_{\varepsilon}^{11} = \sigma_y^2$ and $\Sigma_{\varepsilon}^{55} = \sigma_\pi^2$.

The only non-zero elements of $\Sigma_{t+1|t}$ are then

$$\begin{aligned} \Sigma_{t+1|t}^{11} &= \mathrm{Var}_t(y_{t+1}) \\ &= x_t' \Sigma_A^{11} x_t + 2 x_t' \Sigma_{AB}^{11} i_t + i_t' \Sigma_B^{11} i_t + \Sigma_{\varepsilon}^{11} \end{aligned} \tag{40}$$

and

$$\begin{aligned} \Sigma_{t+1|t}^{55} &= \mathrm{Var}_t(\pi_{t+1}) \\ &= x_t' \Sigma_A^{55} x_t + i_t' \Sigma_B^{55} i_t + \Sigma_{\varepsilon}^{55}. \end{aligned} \tag{41}$$

Consequently

$$\begin{aligned} \mathrm{tr}\left(\tilde{V} \Sigma_{t+1|t}\right) &= \tilde{v}_{11} \left(x_t' \Sigma_A^{11} x_t + 2 x_t' \Sigma_{AB}^{11} i_t + i_t' \Sigma_B^{11} i_t + \Sigma_{\varepsilon}^{11} \right) \\ &+ \tilde{v}_{55} \left(x_t' \Sigma_A^{55} x_t + i_t' \Sigma_B^{55} i_t + \Sigma_{\varepsilon}^{55} \right), \end{aligned} \tag{42}$$

where $\tilde{v}_{ij}$ is the (i,j)th element of $\tilde{V}$.

Using (33)–(35) and (42) in (31), the Bellman equation is

$$\begin{aligned} x_t' \tilde{V} x_t + \tilde{w} &= \min_{i_t} \Big\{ x_t' Q x_t + \phi (A x_t + B i_t)' \tilde{V} (A x_t + B i_t) \\ &+ \phi \tilde{v}_{11} \left(x_t' \Sigma_A^{11} x_t + 2 x_t' \Sigma_{AB}^{11} i_t + i_t' \Sigma_B^{11} i_t + \Sigma_{\varepsilon}^{11} \right) \\ &+ \phi \tilde{v}_{55} \left(x_t' \Sigma_A^{55} x_t + i_t' \Sigma_B^{55} i_t + \Sigma_{\varepsilon}^{55} \right) + \phi \tilde{w} \Big\}, \end{aligned} \tag{43}$$

so the first-order condition is[24]

$$B' (\tilde{V} + \tilde{V}') (A x_t + B i_t) + 2 \tilde{v}_{11} \left(\Sigma_{AB}^{11\,\prime} x_t + \Sigma_B^{11} i_t \right) + 2 \tilde{v}_{55} \Sigma_B^{55} i_t = 0, \tag{44}$$

leading to the optimal interest rate

$$i_t = \tilde{f} x_t, \tag{45}$$

[24]Note that in the setup with multiplicative parameter uncertainty, $\tilde{V}$ is not necessarily symmetric.

where

$$\begin{aligned}\tilde{f} &= -\left[B'(\tilde{V}+\tilde{V}')B + 2\tilde{v}_{11}\Sigma_B^{11} + 2\tilde{v}_{55}\Sigma_B^{55}\right]^{-1} \\ &\times \left[B'(\tilde{V}+\tilde{V}')A + 2\tilde{v}_{11}\Sigma_{AB}^{11\prime}\right]. \end{aligned} \tag{46}$$

Substituting back into the Bellman equation (43), we get

$$\begin{aligned}x_t'\tilde{V}x_t + \tilde{w} &= x_t'Qx_t + \phi\left[(Ax_t + B\tilde{f}x_t)'\tilde{V}(Ax_t + B\tilde{f}x_t)\right] \\ &+ \phi\tilde{v}_{11}\left(x_t'\Sigma_A^{11}x_t + 2x_t'\Sigma_{AB}^{11}\tilde{f}x_t + x_t'\tilde{f}'\Sigma_B^{11}\tilde{f}x_t + \Sigma_\varepsilon^{11}\right) \\ &+ \phi\tilde{v}_{55}\left(x_t'\Sigma_A^{55}x_t + x_t'\tilde{f}'\Sigma_B^{55}\tilde{f}x_t + \Sigma_\varepsilon^{55}\right) + \phi\tilde{w}, \end{aligned} \tag{47}$$

and it can be established that $\tilde{V}$ is determined by the Ricatti equation

$$\begin{aligned}\tilde{V} &= Q + \phi(A+B\tilde{f})'\tilde{V}(A+B\tilde{f}) \\ &+ \phi\tilde{v}_{11}\left(\Sigma_A^{11} + 2\Sigma_{AB}^{11}\tilde{f} + \tilde{f}'\Sigma_B^{11}\tilde{f}\right) + \phi\tilde{v}_{55}\left(\Sigma_A^{55} + \tilde{f}'\Sigma_B^{55}\tilde{f}\right). \end{aligned} \tag{48}$$

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