Moments and the Autocorrelation Structure of the Exponential GARCH(p, q) Process

by

Changli He

Department of Economic Statistics

Stockholm School of Economics

Box 6501, SE-113 83 Stockholm, Sweden

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Abstract. In this paper the autocorrelation structure of the Exponential GARCH(p, q) process of Nelson (1991) is considered. Conditions for the existence of any arbitrary unconditional moment are given. Furthermore, the expressions for the kurtosis and the autocorrelations of squared observations are derived. The properties of the autocorrelation structure are discussed and compared to those of the standard GARCH(p, q) process. In particular, it is seen that, the EGARCH(p, q) model has a richer autocorrelation structure than the standard GARCH(p, q) one. The statistical theory is further illustrated by a few special cases such as the symmetric and the asymmetric EGARCH(2,2) models under the assumption of normal errors or non-normal errors. The autocorrelations computed from an estimated EGARCH(2,1) model of Nelson (1991) are highlighted.

Key Words. autocorrelation function of squared observations, conditional variance model, GARCH, time series, volatility

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1 Introduction

Recently, the moment structure of models of generalized autoregressive heteroskedasticity (GARCH) has attracted considerable attention. These models, first considered by Bollerslev (1986) and Taylor (1986), have become very popular in modelling and predicting volatility. The knowledge about the conditions of existence of moments and parametric definitions of moments is important when researchers want to know how well their estimated GARCH models match stylized facts found in a large number of financial series of sufficiently high sampling frequency. Typical stylized facts include little or no linear dependence in the series but high kurtosis of the observations and a slowly decaying autocorrelation function for the squared and centred observations.

Bollerslev (1986, 1988) already considered the fourth moment structure of the standard first-order GARCH model under the assumption of normality. He and Teräsvirta (1999a) gave a completely characterization of the moment structure of the general GARCH(p, q) model. But then, fewer results are available for another popular GARCH model, the exponential GARCH model of Nelson (1991). Breidt, Crato and Lima (1998) derived the autocorrelation function of the logarithms of squared observations of this model. This autocorrelation function may also be obtained as a limiting case from the so-called power ARCH model; see He and Teräsvirta (1999b) for discussion. He, Teräsvirta and Malmsten (1999) derived expressions for the moments, the kurtosis, and the autocorrelation function of the squared observations for a family of first-order EGARCH models, including that of Nelson (1991). On the other hand, corresponding results have not yet been obtained for higher-order EGARCH models.

The present paper intends to fill this gap. The results apply to Nelson's model as well as to the logarithmic GARCH model that Geweke (1986) and Pantula (1986) suggested.

The plan of the paper is as follows. In Section 2, conditions for the existence of any arbitrary moment of a family of EGARCH models are derived. The expression for the kurtosis is given as well. Section 3 contains the results on the autocorrelation function. In Section 4, the general theory is illustrated by considering a special case, the second-order EGARCH model. This Section also contains a numerical example. Section 5 contains some conclusions. The proofs are presented in the Appendix.

2 Existence of unconditional moments

We begin with a necessary and sufficient condition for existence of unconditional moments in the EGARCH(p,p) process. The results derived for the EGARCH(p,p) model apply directly to the EGARCH(p,q) model with $p \neq q$ because the latter is always nested in a model with $p = q$. Assume further that $p \geq 2$. This is a technical assumption due to the specific mixed moment notation used in this paper (certain moments are not defined for $p = 1$). The results for the lower-order EGARCH processes may be obtained from the EGARCH(2,2) model by setting the appropriate parameters equal to zero. The EGARCH model is defined as

$$\varepsilon_t = z_t h_t \tag{1}$$

where $\{z_t\}$ is a sequence of independent identically distributed random variables with zero mean and finite fourth moment, and h_t is a $\mathcal{F}_{t-1}$-measurable function, where $\mathcal{F}_{t-1}$ is the sigma-algebra generated by $\{z_{t-1},\, z_{t-2},\, z_{t-3},\, ...\}$ and positive with probability one. Furthermore,

$$\ln h_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i g(z_{t-i}) + \sum_{i=1}^{p} \beta_i \ln h_{t-i}^2 \tag{2}$$

where $g(z_t)$ is a well-defined function of z_t. Equations (1) and (2) define a class of Exponential GARCH(p,p) models. For example, setting $g(z_t) = \theta z_t + \omega(|z_t| - \mathsf{E}|z_t|)$ in (2)

yields the EGARCH(p,p) model proposed by Nelson (1991).

It is useful to rewrite (2) in the multiplicative form

$$h_t^2 = (\exp\{\alpha_0\} \prod_{i=1}^{p} \exp\{\alpha_i g(z_{t-i})\}) \prod_{i=1}^{p} h_{t-i}^{2\beta_i}. \tag{3}$$

This is because we are interested in the moment structure of $\{h_t^2\}$ instead of that of $\{\ln h_t^2\}$.

A useful representation for h_t^2 can be found by applying expression (3) recursively as follows. First, apply (3) to h_{t-1}^2 in $\prod_{i=1}^{p} h_{t-i}^{2\beta_i}$. This yields an expression which is dependent on h_{t-2}^2 but no longer on h_{t-1}^2. Applying (3) to h_{t-2}^2 in $\prod_{i=1}^{p} h_{t-i}^{2\beta_i}$ completes the next recursion. The appropriate expression for $\prod_{i=1}^{p} h_{t-i}^{2\beta_i}$ after the k-th recursion, $k > p$, which is a function of lags of $\prod_{i=1}^{p} h_{t-i-k}^{2\psi_{k+i}}$, where every ψ_{k+i} is a function of $\beta_1, ..., \beta_p$, appears in Lemma 2 of the Appendix. The general expression of $\prod_{i=1}^{p} h_{t-i}^{2\beta_i^*}$ where $\beta_i^* > 0$ is given by Lemma 4 in the Appendix.

In order to obtain the desired results we define a $(p \times p)$ matrix

$$\mathbf{\Gamma} = \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 & ... & \beta_{p-1} & \beta_p \\ 1 & 0 & 0 & ... & 0 & 0 \\ 0 & 1 & 0 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 & 0 \end{pmatrix}, \tag{4}$$

and set

$$\phi_k = \boldsymbol{\beta}' \mathbf{\Gamma}^{k-p-1} \boldsymbol{\phi},\ k > p, \tag{5}$$

where $\boldsymbol{\beta} = (\beta_1, ..., \beta_p)'$ and $\boldsymbol{\phi} = (\phi_p, ..., \phi_1)'$ such that

$$\begin{aligned} \phi_1 &= \beta_1 \\ \phi_j &= \sum_{i=1}^{j} \beta_i \phi_{j-i},\ j = 2, ..., p, \text{ with } \phi_0 = 1. \end{aligned} \tag{6}$$

Next, let

$$\lambda(\mathbf{\Gamma}) = \max\{|\lambda_i|\} \tag{7}$$

be the maximum absolute eigenvalue of the matrix $\mathbf{\Gamma}$ where λ_i, $i = 1, ..., p$, are the eigenvalues of matrix $\mathbf{\Gamma}$ and $\gamma(e^g, \omega) = \mathsf{E}(\exp\{\omega g(z_t)\})$ for any real number ω. Assume now that the EGARCH process started at some finite value infinitely many periods ago. It turns out that all unconditional moments of $\{\varepsilon_t\}$ exist and are independent of t as $k \to \infty$ if and only if all the eigenvalues of $\mathbf{\Gamma}$ in (4) lie inside the unit circle, that is, $\lambda(\mathbf{\Gamma}) < 1$. In that case, $\mathbf{\Gamma}^k \to \mathbf{0}$ and $\sum_{i=0}^{k} \mathbf{\Gamma}^i = (\mathbf{I} - \mathbf{\Gamma})^{-1}$ as $k \to \infty$. This is needed for the sequence $\{\phi_k\}$ defined in (5) to converge to a finite value, which, in turn, is required for any unconditional moment of the EGARCH(p, p) process (1) and (2) to exist. From Lemmata 1 and 2 in the Appendix we obtain

Theorem 1 *Assume that $\gamma(e^g, m\alpha_i\varphi_i) < \infty$ for an arbitrary positive integer m and that $\nu_{2m} = \mathsf{E}(|z_t|^{2m}) < \infty$. For the EGARCH(p,p) model (1) with (2), the 2m-th unconditional moment exists if and only if*

$$\lambda(\mathbf{\Gamma}) < 1. \tag{8}$$

Under this condition the 2m-th moment of ε_t can be expressed as

$$\mu_{2m} = \mathsf{E}\varepsilon_t^{2m} = (\exp\{m\alpha_0(1 - \sum_{i=1}^{p} \beta_i)^{-1}\}) \prod_{i=1}^{\infty} \gamma(e^g, m\varphi_i) \tag{9}$$

such that

$$\begin{aligned} \varphi_1 &= \alpha_1 \\ \varphi_j &= \sum_{i=1}^{j} \alpha_i \phi_{j-i}, \ \phi_0 = 1, \ j = 2, ..., p \\ \varphi_j &= \sum_{i=1}^{p} \alpha_i \phi_{j-i}, \ j > p \end{aligned} \tag{10}$$

where ϕ_k are defined in (5)-(6).

Proof. See Appendix.

Remark 1. Typically, an unconditional moment of $\{\varepsilon_t\}$ is expressed as an infinite product of unconditional moments of $\{z_t\}$ computed from the fractional exponential function of $g(z)$. From (5) and (6) it follows that φ_k in (9) can be rewritten as $\varphi_k = \sum_{i=1}^{p} \alpha_i \boldsymbol{\beta}' \boldsymbol{\Gamma}^{k-p-1-i} \boldsymbol{\phi}$ for $k > 2p$. When condition (8) holds, φ_k converges to a finite value, and this is sufficient for the product $\prod_{i=1}^{n} \gamma(e^g, m\varphi_i)$ to converge to a finite nonzero value as $n \to \infty$. Numerical computation of the moments $\gamma(e^g, m\varphi_i)$ may be difficult in the general case in that we have to approximate the infinite product in formula (9) by a product with a finite number of terms. In some special cases such as the example following Corollary 1 below, an analytical expression for the infinite product may be available in Corollary 3.

Remark 2. Nelson (1991) studied the stationarity conditions of $\{\ln h_t^2\}$ using an ARMA representation of (2)

$$\ln h_t^2 = \alpha_0 + \frac{1 + \sum_{i=1}^{p} \alpha_i L^i g(z_{t-i})}{1 - \sum_{i=1}^{p} \beta_i L^i} g(z_{t-1}).$$

Assuming that the lag polynomials $1 - \sum_{i=1}^{p} \beta_i L^i$ and $1 + \sum_{i=1}^{p} \alpha_i L^i$ have no common roots, $\{\ln h_t^2\}$ is stationary if and only if all the roots of $1 - \sum_{i=1}^{p} \beta_i L^i$ lie outside the unit circle. Obviously, condition (8) in Theorem 1 is equivalent to this condition, which in fact is the one Nelson (1991) derived.

Remark 3. Compared with the standard GARCH(p, p) process of Bollerslev (1986), condition (8) guarantees the existence of any moment for the EGARCH(p, p) model. For comparison, the existence of the second moment of the GARCH(p, p) model is necessary but far from sufficient for the existence of the fourth moment, see He and Teräsvirta (1999a) for a discussion. Assuming that $\nu_4 < \infty$, condition (8) for existence of the unconditional moments depends only on the eigenvalues of matrix $\boldsymbol{\Gamma}$ expressed in $\beta_1, ..., \beta_p$,

but conditions for existence of the fourth moment of the GARCH(p,p) model depend not only on the eigenvalues of $\mathbf{\Gamma}$ (note that the definition of $\mathbf{\Gamma}$ for a standard GARCH process is not the same as here): they depend on the values of α_i's and β_i's as well. Clearly, condition (8) is much easier to check than the condition for the existence of the fourth moment for higher order GARCH models.

An expression for the kurtosis of ε_t is stated as a corollary.

Corollary 1 *For the EGARCH(p,p) model (1) with (2) when $\nu_4 < \infty$, $\gamma(e^g, \varphi_i)$ and $\gamma(e^g, 2\varphi_i)$ exist for any i and (8) holds, the kurtosis of ε_t exists and is given by*

$$\kappa_4 = \mathsf{E}\varepsilon_t^4/(\mathsf{E}\varepsilon_t^2)^2 = \kappa_4(z_t) \prod_{i=1}^{\infty} \frac{\gamma(e^g, 2\varphi_i)}{(\gamma(e^g, \varphi_i))^2} \tag{11}$$

where $\kappa_4(z_t) = \nu_4/\nu_2^2$ is the kurtosis of z_t.

From Jensen's inequality and Corollary 1, we have

Corollary 2 *Consider the EGARCH(p,p) model (1) and (2) when $\nu_4 < \infty$. Assume, furthermore, that $\gamma(e^g, \varphi_i)$ and $\gamma(e^g, 2\varphi_i)$ exist for any i and that condition (8) holds. Then*

$$\kappa_4 > \kappa_4(z_t). \tag{12}$$

The unconditional distribution of ε_t is thus leptokurtic.

To illustrate Corollary 1 we give the following analytic example below:

Corollary 3 *Consider the EGARCH(2,2) model (1) and (2) is obtained by setting $g(z_t) = \theta z_t$ and assume $z_t \sim nid(0,1)$ in (2) and that (8) holds. It then the kurtosis of this process equals*

$$\kappa_4 = 3\exp\{\frac{\theta^2[(\alpha_1^2+\alpha_2^2)(1-\beta_2)+2\alpha_1\alpha_2\beta_1]}{1-\beta_2-\beta_1^2-\beta_2^2-\beta_1^2\beta_2+\beta_2^3}\} > 3 \tag{13}$$

for any $\theta \neq 0$.

Proof. See Appendix.

3 Autocorrelation function for the squared observations

In this section we derive the n-th order autocorrelation function of $\{\varepsilon_t^2\}$ for the EGARCH(p,p) model (1) with (2). To fix notation, define the n-th order autocorrelation of $\{\varepsilon_t^2\}$ as

$$\rho_n = \rho(\varepsilon_t^2, \varepsilon_{t-n}^2) = \frac{\mathsf{E}(\varepsilon_t^2\varepsilon_{t-n}^2) - (\mathsf{E}\varepsilon_t^2)^2}{\mathsf{E}\varepsilon_t^4 - (\mathsf{E}\varepsilon_t^2)^2} \tag{14}$$

for any $n \geq 1$. Having already obtained expressions for $\mathsf{E}\varepsilon_t^4$ and $\mathsf{E}\varepsilon_t^2$ we must find an appropriate one for $\mathsf{E}(\varepsilon_t^2\varepsilon_{t-n}^2)$ for $n \geq 1$. The expression is worked out and characterized in Lemmata 4-7. Let $\widetilde{\gamma}(e^g, \phi_{n-i}\alpha_{j+i} + \varphi_{jn}^*) = \mathsf{E}(\exp\{(\phi_{n-i}\alpha_{j+i} + \varphi_{jn}^*)g(z_{t-(n+j)})\})$ for some i and j such that $n \geq 1$. Furthermore, let $\overline{\gamma}_{np}(e^g, \varphi_n) = \mathsf{E}(z_{t-n}^2 \exp\{\varphi_n g(z_{t-n})\}$ for any positive integer n, where φ_{jn}^* for $n \geq 1$ and $j \geq 1$ are defined in Lemmata 5-7. Note that φ_{jn}^* are somewhat similar to φ_j but different in that φ_{jn}^* initially depend on n. Theorem 1 and Lemmata 4-7 allow us to express the autocorrelations of $\{\varepsilon_t^2\}$ for the EGARCH(p,p) process as follows:

Theorem 2 *Assume that $\nu_4 < \infty$ and (8) holds and that $\gamma(\cdot)$, $\overline{\gamma}(\cdot)$ and $\widetilde{\gamma}(\cdot)$ exist. Then the n-th order autocorrelation of $\{\varepsilon_t^2\}$ for the EGARCH(p,p) model (1) and (2) has the form*

$$\rho_n = \frac{\nu_2\overline{\gamma}_{np}(e^g, \varphi_n)M_1(n)M_2(n) - \nu_2^2(\prod_{i=1}^{\infty}(\gamma(e^g, \varphi_i))^2}{\nu_4 \prod_{i=1}^{\infty}\gamma(e^g, 2\varphi_i) - \nu_2^2(\prod_{i=1}^{\infty}\gamma(e^g, \varphi_i))^2}, \quad n \geq 1 \tag{15}$$

where $M_1(n)$, $n \geq 1$, is a function of $\gamma(\cdot)$ and $M_2(n)$, $n \geq 1$, a function of $\widetilde{\gamma}(\cdot)$ defined in Lemma 7.

Proof. See Appendix.

Remark 4. The mixed moment $\widetilde{\gamma}(\cdot)$ is required for higher-order EGARCH models. Because it is not defined for the EGARCH(1,1) model the formula for the autocorrelation

structure of $\{\varepsilon_t^2\}$ for the EGARCH(1,1) model is simpler than expression (15), see He, Teräsvirta and Malmsten (1999).

Remark 5. Properties of the autocorrelation function $\{\rho_n\}$, $n = 1, 2, ...$, can be established through expression (15). Some of them are listed below:

- $\rho_n \to 0$ as $n \to \infty$. In fact, $\overline{\gamma}_{np}(e^g, \varphi_n) \to \nu_2$, $M_1(n) \to (\prod_{i=1}^{\infty} (\gamma(e^g, \varphi_i))^2$ and $M_2(n) \to 1$ as $n \to \infty$.

- Autocorrelation ρ_n is a function of n in the terms of the powers of matrix $\mathbf{\Gamma}$ as well as $\phi_1, ..., \phi_p$. It is seen that the first p autocorrelations depend on $M_1(n)M_2(n)$ and do not need to decay monotonically. When $n > p$ the autocorrelations decay because the power of $\mathbf{\Gamma}$ increases with n. However, ρ_n for $n > p$ do not satisfy the Yule-Walker equations for the autocorrelation of $\{\varepsilon_t^2\}$ as is the case with the GARCH(p, p) model does. See, for example, the EGARCH(1,1) model in He, Teräsvirta and Malmsten (1999).

- Eigenvalues (a positive dominant eigenvalue, or a negative one, or a pair of complex eigenvalues) in $\mathbf{\Gamma}$ determine the shape of ρ_n in (15). See the EGARCH(2,1) example in Figure 1. The existence of an eigenvalue λ_i , $i \in \{1, ..., p\}$, of matrix $\mathbf{\Gamma}$ in (4), such that $\lambda_i \approx 1$, indicates that the autocorrelation is persistent; otherwise the autocorrelations decay quickly with increasing n.

- It is possible that $\rho_n < 0$ for some n. This is because for EGARCH models no parameter constraints in (2) are required to guarantee that the conditional variance be positive with probability one. For the GARCH(p, p) process, such conditions given in Nelson and Cao (1992) lead to $\rho_n > 0$, $n \geq 1$; see also He and Teräsvirta (1999c) for illustration.

4 Special case: EGARCH(2,2)

To illustrate the general result we consider the EGARCH(2,2) process.

Corollary 4 *Assume that $\nu_4 < \infty$ and condition (8) holds and that $\gamma(\cdot)$, $\overline{\gamma}(\cdot)$, $\widetilde{\gamma}(\cdot) < \infty$ in the EGARCH(2,2) model (1) with (2). Then the n-th order autocorrelation function of the squared observations for this process is obtained by setting $p = 2$ in equation (15). In the expression of this autocorrelation function,*

$$M_1(n) = (\prod_{i=1}^{n-1} \gamma(e^g, \varphi_i))(\prod_{i=2}^{\infty} \gamma(e^g, \varphi_{in}^*)) \tag{16}$$

$$M_2(n) = \widetilde{\gamma}(e^g, \alpha_2 \phi_{n-1} + \varphi_{1n}^*) \tag{17}$$

for any $n \geq 1$ such that

$$\begin{aligned} \varphi_{1n}^* &= \alpha_1(\phi_n + 1), \\ \varphi_{2n}^* &= \alpha_1 \phi_{2n}^* + \alpha_2 \phi_{1n}^*, \\ \varphi_{jn}^* &= \sum_{i=1}^{2} \alpha_i \phi_{(j+1)-i,n}^*, \ j \geq 3 \end{aligned} \tag{18}$$

with

$$\begin{aligned} \phi_{1n}^* &= \phi_n + 1, \\ \phi_{2n}^* &= \beta_1(\phi_n + 1) + \beta_2 \phi_{n-1}, \\ \phi_{jn}^* &= \boldsymbol{\beta}' \boldsymbol{\Gamma}^{j-2-1} \boldsymbol{\phi}_n^*, \ j \geq 3 \end{aligned} \tag{19}$$

where $\boldsymbol{\phi}_n^* = (\phi_{2n}^*, \phi_{1n}^*)'$, $\boldsymbol{\beta} = (\beta_1, \beta_2)'$, and $\boldsymbol{\Gamma} = \begin{pmatrix} \beta_1 & \beta_2 \\ 1 & 0 \end{pmatrix}$. Furthermore, ϕ_i and φ_i are defined in (5)-(6) and (10) for any $i \geq 1$.

Proof. See Appendix.

From Corollary 4 it follows that the first-order autocorrelation has the form

$$\rho_1 = \frac{\nu_2(\overline{\gamma}_{12}(e^g,\alpha_1))\widetilde{\gamma}(e^g,\alpha_2+\alpha_1(\beta_1+1))\prod_{i=2}^{\infty}\gamma(e^g,\varphi_{i1}^*) - \nu_2^2(\prod_{i=1}^{\infty}\gamma(e^g,\varphi_i))^2}{\nu_4\prod_{i=1}^{\infty}\gamma(e^g,2\varphi_i) - \nu_2^2(\prod_{i=1}^{\infty}\gamma(e^g,\varphi_i))^2} \tag{20}$$

where φ_{i1}^* is determined by setting $n=1$ in (19) and noting that

$$\begin{aligned} \phi_{11}^* &= \beta_1+1, \\ \phi_{21}^* &= \beta_1(\beta_1+1)+\beta_2, \\ \phi_{j1}^* &= \boldsymbol{\beta}'\boldsymbol{\Gamma}^{j-2-1}\boldsymbol{\phi}_1^*,\ j\geq 3 \end{aligned} \tag{21}$$

with $\boldsymbol{\phi}_1^* = (\phi_{21}^*,\phi_{11}^*)'$. Subsequently, the second-order autocorrelation

$$\rho_2 = \frac{\nu_2\gamma(e^g,\alpha_1)\overline{\gamma}_{22}(e^g,\alpha_1\beta_1+\alpha_2)\widetilde{\gamma}(e^g,\beta_1\alpha_2+\alpha_1(\beta_1^2+\beta_2+1))\prod_{i=2}^{\infty}\gamma(e^g,\varphi_{i2}^*) - \nu_2^2(\prod_{i=1}^{\infty}\gamma(e^g,\varphi_i))^2}{\nu_4\prod_{i=1}^{\infty}\gamma(e^g,2\varphi_i) - \nu_2^2(\prod_{i=1}^{\infty}\gamma(e^g,\varphi_i))^2} \tag{22}$$

where φ_{i2}^* is determined by setting $n=2$ in (19) and noting that

$$\begin{aligned} \phi_{12}^* &= \phi_2+1, \\ \phi_{22}^* &= \beta_1(\phi_2+1)+\beta_2\phi_1, \\ \phi_{j2}^* &= \boldsymbol{\beta}'\boldsymbol{\Gamma}^{j-2-1}\boldsymbol{\phi}_2^*,\ j\geq 3, \end{aligned} \tag{23}$$

with $\boldsymbol{\phi}_2^* = (\phi_{22}^*,\phi_{12}^*)'$. Similarly, we can compute higher-order autocorrelations through expressions (16)-(19) in Corollary 4.

Appropriate choices of $g(z_t)$ yield the autocorrelation functions for various EGARCH (2,2) models. Such cases are considered in the next subsections. Note that we always assume that conditions in Corollary 4 hold in the following corollaries.

4.1 An analytic example: Symmetric EGARCH(2,2) model

It follows from Corollaries 3 and 4 that an analytic expression of the autocorrelations for the symmetric EGARCH(2,2) model is available. It is given in the following corollary.

Corollary 5 *Consider a symmetric EGARCH(2,2) model (1) and (2) and assume that* $g(z_t) = \theta z_t$ *and* $z_t \sim nid(0,1)$ *in (2). The autocorrelation function of* $\{\varepsilon_t^2\}$ *then has the form*

$$\rho_n = \frac{(1+\theta^2\varphi_n^2)\exp\{0.5\theta^2\gamma(n)/(1-\beta_2-\beta_1^2-\beta_2^2-\beta_1^2\beta_2+\beta_2^3)\}\exp\{0.5\theta^2\sum_{i=1}^{n}\varphi_i^2\}-1}{3\exp\{\theta^2[(\alpha_1^2+\alpha_2^2)(1-\beta_2)+2\alpha_1\alpha_2\beta_1]/(1-\beta_2-\beta_1^2-\beta_2^2-\beta_1^2\beta_2+\beta_2^3)\}-1} \tag{24}$$

where

$$\begin{aligned}\gamma(n) &= [(\alpha_1^2+\alpha_2^2)(1-\beta_2)+2\alpha_1\alpha_2\beta_1](\phi_n^2+2\phi_n-1)\\ &\quad +2[(\alpha_1^2+\alpha_2^2)\beta_1\beta_2+\alpha_1\alpha_2(1-\beta_1^2-\beta_2^2)][\phi_{n-1}(\phi_n-1)]\\ &\quad +[(\alpha_1^2\beta_2^2+\alpha_2^2)(1-\beta_2)-\alpha_2^2\beta_1^2(1+\beta_2)+2\alpha_1\alpha_2\beta_1\beta_2^2]\phi_{n-1}^2. \end{aligned} \tag{25}$$

Proof. See Appendix.

The autocorrelation function remains positive as it dampens out as a function of n, if the dominant eigenvalue of $\mathbf{\Gamma}$ in (19) is positive. On the other hand, it alternates in sign as it dampens out, when the dominant eigenvalue of $\mathbf{\Gamma}$ is negative. Furthermore, the autocorrelation function displays a damped sine wave, when a complex pair of eigenvalues of $\mathbf{\Gamma}$ dominates; see Figure 1 for an illustration.

The corresponding results for the EGARCH(1,1) model are obtained by setting $\alpha_2 = \beta_2 = 0$ in (24). The autocorrelation function thus has the form

$$\rho_n = \frac{(1+\theta^2\alpha_1^2\beta_1^{2(n-1)})\exp\{\theta^2\alpha_1^2\beta_1^n/(1-\beta_1^2)\}-1}{3\exp\{\theta^2\alpha_1^2/(1-\beta_1^2)\}-1}, \quad n \geq 1, \tag{26}$$

as given in He, Teräsvirta and Malmsten (1999).

4.2 Asymmetric EGARCH(2,2) model under normal errors

It follows from Corollary 4 and Theorem A1.1 of Nelson (1991) that we are able to compute the autocorrelation function of the squared observations for an asymmetric EGARCH(2,2) model under normal errors. The result is contained in the following corollary.

Corollary 6 *Consider the asymmetric EGARCH(2,2) model (1) and (2) is obtained by setting* $g(z_t) = \theta z_t + \omega(|z_t| - \mathsf{E}|z_t|)$ *and assuming* $z_t \sim nid(0,1)$ *in (2). The autocorrelation function of* $\{\varepsilon_t^2\}$ *then has the form*

$$\rho_n = \frac{\sqrt{\frac{2}{\pi}}\exp\{-\sqrt{\frac{2}{\pi}}\omega \sum_{i=1}^{n} \varphi_i\}\gamma(n)D_n(\cdot)\prod_{i=1}^{n-1}\Phi_i(\varphi_i)\prod_{i=2}^{\infty}\Phi_{in}(\varphi_{in}^*) - \prod_{i=1}^{\infty}\Phi_i^2(\varphi_i)}{3\prod_{i=1}^{\infty}\Phi_i(2\varphi_i) - \prod_{i=1}^{\infty}\Phi_i^2(\varphi_i)}. \tag{27}$$

In expression (27),

$$\begin{aligned}
\gamma(n) &= \exp\{-\frac{\sqrt{\frac{2}{\pi}}\omega}{1-(\beta_1+\beta_2)}[(\phi_n - 1)(\alpha_1+\alpha_2) + (\alpha_2(1-\beta_1) + \alpha_1\beta_2)\phi_{n-1}]\} \quad (28)\\
D_n(\cdot) &= D_{-3}[-\varphi_n(\omega+\theta)]\exp\{0.25\varphi_n^2(\omega+\theta)^2\}\\
&\quad + D_{-3}[-\varphi_n(\omega-\theta)]\exp\{0.25\varphi_n^2(\omega-\theta)^2\} \quad (29)
\end{aligned}$$

and $\Phi_i(x_i)$ *in (27) is a function of* x_i*, defined as*

$$\begin{aligned}
\Phi_i(x_i) &= \Phi(x_i(\omega+\theta))\exp\{0.5x_i^2(\omega+\theta)^2\}\\
&\quad + \Phi(x_i(\omega-\theta))\exp\{0.5x_i^2(\omega-\theta)^2\} \quad (30)
\end{aligned}$$

when $x_i = \varphi_i$*, or* φ_{in}^* *or* $2\varphi_i$*, where* $\Phi(\cdot)$ *is the cumulative distribution function of the standard normal random variable and* $D_q[\cdot]$ *is the parabolic cylinder function (Gradshteyn and Ryzhik, 1980).*

Proof. See Appendix.

As may be expected, the autocorrelation function of the squared observations for the asymmetric EGARCH(2,2) model depends on the asymmetry parameter ω.

4.3 Asymmetric EGARCH(2,2) model under non-normal errors

We set $g(z_t) = \theta z_t + \omega(|z_t| - \mathsf{E}\,|z_t|)$ (Nelson, 1991) in (2) and assume that the $\{z_t\}$ are a sequence of i.i.d. random variables from the GED (Generalized Error Distribution (Harvey (1981), Box and Tiao (1973)) with mean zero and variance one. Thus z_t has the density

$$f(z) = \frac{\upsilon \exp\{-0.5\,|z/\lambda|^{\upsilon}\}}{\lambda 2^{(1+1/\upsilon)}\Gamma(1/\upsilon)},\ \text{-}\infty < z < \infty,\ 0 < \upsilon \leq \infty, \tag{31}$$

where $\Gamma(\cdot)$ is the gamma function, and $\lambda = [2^{(-2/\upsilon)}\Gamma(1/\upsilon)/\Gamma(3/\upsilon)]^{1/2}$. Parameter υ is a tail-thickness parameter, see Nelson (1991) for a discussion. Following from Corollary 4 and Theorem A1.2 of Nelson (1991) that we have the following corollary:

Corollary 7 *Consider an asymmetric EGARCH(2,2) model (1) and (2) and assume that* $g(z_t) = \theta z_t + \omega(|z_t| - \mathsf{E}\,|z_t|)$ *and* $z_t \sim GED(\upsilon)$ *with* $\mathsf{E}z_t = 0$, $\mathsf{E}z_t^2 = 1$ *and* $\upsilon > 1$ *in (2). Then the autocorrelation function of* $\{\varepsilon_t^2\}$ *when* $\{\varepsilon_t\}$ *when follows an asymmetric EGARCH(2,2) model, equals*

$$\rho_n = \frac{\nu_2 2^{2/\upsilon}\lambda^2 \exp\{-\omega\Gamma(\lambda,\upsilon)\sum_{i=1}^{n}\varphi_i\}\gamma(n)S_{1n}(\varphi_n)\prod_{i=1}^{n-1}S_{2i}(\varphi_i)\prod_{i=2}^{\infty}S_{2in}(\varphi_{in}^*) - \nu_2^2\prod_{i=1}^{\infty}S_{2i}^2(\varphi_i)}{\nu_4\prod_{i=1}^{\infty}S_{2i}(2\varphi_i) - \nu_2^2\prod_{i=1}^{\infty}S_{2i}^2(\varphi_i)} \tag{32}$$

where

$$\begin{aligned}
\Gamma(\lambda,\upsilon) &= \lambda 2^{1/\upsilon}\Gamma(2/\upsilon)/\Gamma(1/\upsilon) \\
\gamma(n) &= \exp\{-\frac{\omega\Gamma(\lambda,\upsilon)}{1-(\beta_1+\beta_2)}[(\phi_n - 1)(\alpha_1+\alpha_2) + (\alpha_2(1-\beta_1) + \alpha_1\beta_2)\phi_{n-1}]\} \\
S_{1n}(\varphi_n) &= \sum_{k=0}^{\infty}(2^{1/\upsilon}\lambda\varphi_n)^k[(\omega+\theta)^k + (\omega-\theta)^k]\frac{\Gamma((k+3)/\upsilon)}{2\Gamma(1/\upsilon)\Gamma(k+1)}
\end{aligned} \tag{33}$$

$$
\begin{aligned}
S_{2i}(x_i) &= \sum_{k=0}^{\infty} (2^{1/\upsilon}\lambda x_i)^k[(\omega+\theta)^k + (\omega-\theta)^k]\frac{\Gamma((k+1)/\upsilon)}{2\Gamma(1/\upsilon)\Gamma(k+1)} \\
\text{for } x_i &= \varphi_i,\ \varphi_{in}^*,\ 2\varphi_i.
\end{aligned}
$$

Proof. See Appendix.

Note that the sums $S_{1n}(\cdot)$ and $S_{2i}(\cdot)$ in (33) are convergent if $\upsilon > 1$. If $\upsilon < 1$, we refer to Nelson (1991) for a discussion.

4.4 An empirical example

Nelson (1991) employed an EGARCH(2,1) model to analyze the daily returns for the value-weighted market index from the CRSP tapes for July 1962-December 1987. To accommodate the possibly asymmetric relation between stock returns and volatility Nelson chose $g(z_t) = \theta z_t + \omega(|z_t| - \mathsf{E}\,|z_t|)$. To allow flexible parametric families of distributions, Nelson assumed that $z_t \sim$iid with the GED(υ) density in (31). The estimated parameters are given in Table 1. Since $\widehat{\upsilon} = 1.5763 > 1$ the corresponding ν_4 and $\gamma(e^g, \cdot)$ exist according to the estimated model. The estimated GED($\widehat{\upsilon}$) density function has a more peaked centre part and a fatter tail than the standard normal density. The eigenvalues of the estimated $\widehat{\mathbf{\Gamma}}$ are 0.99957 and 0.92981, which indicates the existence of all the unconditional moments of ε_t for the estimated EGARCH(2,1) model. The positive dominant eigenvalue of $\widehat{\mathbf{\Gamma}}$ is very close to one, so that the estimated autocorrelation decays very slowly. Unfortunately, since $\lambda(\widehat{\mathbf{\Gamma}})$ (= 0.99957) is very closed to one, it is numerically difficult to compute the values of the kurtosis and the autocorrelations from the estimated model. We alleviate the difficulty by computing those results from the estimated model after a slight reduction in the value of $\widehat{\beta}_1$. The reduced parameter estimate $\widehat{\beta}_1(\mathrm{R}) = 1.92925$ and the corresponding $\widehat{\lambda}(\widehat{\mathbf{\Gamma}}(\mathrm{R})) = 0.9976$. The numerical results are given in Table 2. We find that the kurtosis for the reduced estimated model is very high, and $\widehat{\rho}_1(\mathrm{R})$ and $\widehat{\rho}_2(\mathrm{R})$ are lower than $\widehat{\rho}_4(\mathrm{R})$.

But it seems that $\widehat{\rho}_n(\mathrm{R})$ decays very quickly.

5 Final remarks

We have derived a complete characterization of the moment structure of a general EGARCH (p,q) process. The results of the paper are useful, for example, if we want to compare the higher-order EGARCH model with the standard higher-order GARCH model. In particular, we have already see that the autocorrelation function of the squared observations for the EGARCH(2,2) model of Nelson (1991) has a richer autocorrelation structure than that of the standard GARCH(2,2) model.

The statistical theory in this paper also applies to the general logarithmic GARCH (LGARCH) (p,q) model:

$$\varepsilon_t = z_t h_t, \tag{1'}$$

$$\ln h_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \ln g(z_{t-i}) h_{t-i}^2 + \sum_{i=1}^{p} \beta_i \ln h_{t-i}^2. \tag{2'}$$

Setting $g(z_t) = z_t^2$ in (2') gives the LGARCH(p,q) model Pantula (1986) and Geweke (1986) previously proposed. In general substituting $\alpha_i + \beta_i$ and $g^{\alpha_i}(z_{t-i})$ for β_i and $\exp\{\alpha_i g(z_{t-i})\}$, $i = 1, 2, ..., \max\{p,q\}$, respectively, throughout this paper, yields the corresponding results for the general LGARCH(p,q) model (1') and (2'). The first-order case has been treated in He, Teräsvirta and Malmsten (1999).

References

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics* **31**, 307-327.

Bollerslev, T. (1988). On the correlation structure for the generalized autoregressive conditional heteroskedastic process. *Journal of Time Series Analysis* ***9***, 121-131.

Bollerslev, T., R.F. Engle and D.B. Nelson (1994). ARCH models. In: R.F. Engle and D.L. McFadden, eds. *Handbook of econometrics*, Vol. **4**, 2959-3038. Amsterdam: Elsevier.

Breidt, F.J., N. Crato and P. de Lima (1998). The detection and estimation of long memory in stochastic volatility. *Journal of Econometrics* ***83***, 325-348.

Box, G. E. P. and G. C. Tiao (1973). *Bayesian Inference in Statistical Analysis.* Reading, MA: Addison-Wesley Publishing Co.

Engle, R.F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica* **50**, 987-1006.

Gradshteyn, I. S. and I. M. Ryzhik (1980). *Tables of Integrals, Series and Products.* New York: Academic Press.

Geweke, J. (1986). Modelling persistence of conditional variances-Comment. *Econometric Reviews* **5**, 57-61.

Harvey, A. C. (1981). *The Econometric Analysis of Time Series.* Oxford: Philip Allan.

He, C. and T. Teräsvirta (1999a). Fourth moment structure of the GARCH(p, q) process. *Econometric Theory* **15**, 824-846.

He, C. and T. Teräsvirta (1999b). Higher-order dependence in the general Power ARCH process and a special case. Stockholm School of Economics, Working Paper Series in Economics and Finance No. 315.

He, C. and T. Teräsvirta (1999c). Properties of the autocorrelation function of squared observations for second order GARCH processes under two sets of parameter constraints. *Journal of Time Series Analysis* ***20,*** 23-30.

He, C., T. Teräsvirta and H. Malmsten (1999). Fourth moment structure of a family of first-order Exponential GARCH models. Stockholm School of Economics, Working Paper Series in Economics and Finance No. 345.

Nelson, D.B. (1991). Conditional heteroskedasticity in asset returns: a new approach. *Econometrica* **59**, 347-370.

Nelson, D.B. and C.Q. Cao (1992). Inequality constraints in the univariate GARCH model. *Journal of Business & Economic Statistics* **10**, 229-235.

Pantula, S.G. (1986). Modelling persistence of conditional variances-Comment. *Econometric Reviews* **5**, 71-74.

Taylor, S. (1986). *Modelling financial time series.* New York: Wiley.

Appendix.

Lemma 1 *Under (4), (5) and (6), for any $k > p$,*

$$\phi_k = \sum_{i=1}^{p} \beta_i \phi_{k-i} \tag{A.1}$$

holds.

Proof. Note that by (4) and (6)

$$\boldsymbol{\Gamma}^p \begin{pmatrix} \phi_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \boldsymbol{\Gamma}^{p-1} \begin{pmatrix} \phi_1 \\ \phi_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \boldsymbol{\Gamma}^{p-2} \begin{pmatrix} \phi_2 \\ \phi_1 \\ \phi_0 \\ \vdots \\ 0 \end{pmatrix} = \cdots = \boldsymbol{\Gamma} \begin{pmatrix} \phi_{p-1} \\ \phi_{p-2} \\ \vdots \\ \phi_1 \\ \phi_0 \end{pmatrix} = \begin{pmatrix} \phi_p \\ \phi_{p-1} \\ \vdots \\ \phi_2 \\ \phi_1 \end{pmatrix} \tag{A.2}$$

holds. Applying (A.2) and (5) gives

$$\begin{aligned} \phi_k &= \boldsymbol{\beta}' \boldsymbol{\Gamma}^{(k-1)-p-1} \boldsymbol{\Gamma} \begin{pmatrix} \phi_p \\ \phi_{p-1} \\ \vdots \\ \phi_2 \\ \phi_1 \end{pmatrix} = \boldsymbol{\beta}' \boldsymbol{\Gamma}^{(k-1)-p-1} \begin{pmatrix} \beta_1 \phi_p + \cdots + \beta_p \phi_1 \\ \beta_1 \phi_{p-1} + \cdots + \beta_p \\ \cdots \\ \beta_1 \phi_2 + \cdots + \beta_3 \\ \beta_1 \phi_1 + \beta_2 \end{pmatrix} \\ &= \beta_1 \boldsymbol{\beta}' \boldsymbol{\Gamma}^{(k-1)-p-1} \begin{pmatrix} \phi_p \\ \phi_{p-1} \\ \vdots \\ \phi_2 \\ \phi_1 \end{pmatrix} + \beta_2 \boldsymbol{\beta}' \boldsymbol{\Gamma}^{(k-1)-p-1} \begin{pmatrix} \phi_{p-1} \\ \phi_{p-2} \\ \vdots \\ \phi_1 \\ \phi_0 \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
& +\beta_3 \boldsymbol{\beta}' \boldsymbol{\Gamma}^{(k-1)-p-1} \begin{pmatrix} \phi_{p-2} \\ \phi_{p-3} \\ \vdots \\ \phi_0 \\ 0 \end{pmatrix} + \cdots + \beta_p \boldsymbol{\beta}' \boldsymbol{\Gamma}^{(k-1)-p-1} \begin{pmatrix} \phi_1 \\ \phi_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\
= \quad & \beta_1 \phi_{k-1} + \beta_2 \phi_{k-2} + \cdots + \beta_p \phi_{k-p}.
\end{aligned}
$$

■

Lemma 2 *Let $k > p$. Under (3), the product $\prod_{i=1}^{p} h_{t-i}^{2\beta_i}$ on the right-hand side of (3) can be expressed in terms of the product $\prod_{i=1}^{p} h_{t-i-k}^{2\psi_{k+i}}$ such that*

$$
\prod_{i=1}^{p} h_{t-i}^{2\beta_i} = (\prod_{i=1}^{k} \exp\{\alpha_0 \phi_i\})(\prod_{i=1}^{k} \exp\{\phi_i \boldsymbol{\alpha}' \mathbf{g}_{t-i}\}) \prod_{i=1}^{p} h_{t-i-k}^{2\psi_{k+i}} \tag{A.3}
$$

where $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_p)'$ and, furthermore,

$$
\begin{aligned}
\psi_{k+1} &= \phi_{k+1}, \\
\psi_{k+2} &= \beta_2 \phi_k + ... + \beta_p \phi_{k+1-(p-1)}, \\
\psi_{k+3} &= \beta_3 \phi_k + ... + \beta_p \phi_{k+1-(p-2)}, \\
& ..... \\
\psi_{k+p} &= \beta_p \phi_k,
\end{aligned} \tag{A.4}
$$

$\mathbf{g}_t = (g(z_{t-1}), \cdots, g(z_{t-p}))'$ and, finally, ϕ_k is defined by (5).

Proof. Let $1 \leq k \leq p$. Applying (3) to h_{t-1}^2 in $\prod_{i=1}^{p} h_{t-i}^{2\beta_i}$ gives

$$
\begin{aligned}
\prod_{i=1}^{p} h_{t-i}^{2\beta_i} &= \exp\{\beta_1 \alpha_0\} \exp\{\beta_1 \boldsymbol{\alpha}' \mathbf{g}_{t-1}\} \prod_{i=1}^{p} h_{t-i-1}^{2\beta_i \beta_1} \prod_{i=2}^{p} h_{t-i}^{2\beta_i} \\
&= \exp\{\beta_1 \alpha_0\} \exp\{\beta_1 \boldsymbol{\alpha}' \mathbf{g}_{t-1}\} \prod_{i=1}^{p-1} h_{t-i-1}^{2(\beta_i \beta_1 + \beta_{i+1})} h_{t-p-1}^{2\beta_1 \beta_p}.
\end{aligned} \tag{A.5}
$$

Define $\phi_1 = \beta_1$ and rewrite the right-hand side of (A.5) as

$$
\prod_{i=1}^{p} h_{t-i}^{2\beta_i} = \exp\{\phi_1 \alpha_0\} \exp\{\phi_1 \boldsymbol{\alpha}' \mathbf{g}_{t-1}\} \prod_{i=1}^{p-1} h_{t-i-1}^{2(\beta_i \phi_1 + \beta_{i+1})} h_{t-p-1}^{2\phi_1 \beta_p}. \tag{A.6}
$$

Applying (3) to h_{t-2}^2 on the right-side of (A.6) gives

$$\begin{aligned}
&\prod_{i=1}^{p} h_{t-i}^{2\beta_i} \\
= \;& \exp\{\phi_1\alpha_0\}\exp\{\phi_1\boldsymbol{\alpha}'\mathbf{g}_{t-1}\}\exp\{(\beta_1\phi_1+\beta_2)\alpha_0\}\exp\{(\beta_1\phi_1+\beta_2)\boldsymbol{\alpha}'\mathbf{g}_{t-2}\} \\
&\times \prod_{i=1}^{p-2} h_{t-i-2}^{2[\beta_i(\beta_1\phi_1+\beta_2)+(\phi_1\beta_{i+1}+\beta_{i+2})]} h_{t-p-1}^{2[\beta_{p-1}(\beta_1\phi_1+\beta_2)+\phi_1\beta_p]} h_{t-p-2}^{2\beta_p(\beta_1\phi_1+\beta_2)}. \quad \text{(A.7)}
\end{aligned}$$

Defining $\phi_2 = \beta_1\phi_1 + \beta_2$ and rewriting the right-hand side of (A.7) yield

$$\begin{aligned}
\prod_{i=1}^{p} h_{t-i}^{2\beta_i} \;=\; & \exp\{\phi_1\alpha_0\}\exp\{\phi_1\boldsymbol{\alpha}'\mathbf{g}_{t-1}\}\exp\{\phi_2\alpha_0\}\exp\{\phi_2\boldsymbol{\alpha}'\mathbf{g}_{t-2}\} \\
&\times \prod_{i=1}^{p-2} h_{t-i-2}^{2[\beta_i\phi_2+(\phi_1\beta_{i+1}+\beta_{i+2})]} h_{t-p-1}^{2[\beta_{p-1}\phi_2+\phi_1\beta_p]} h_{t-p-2}^{2\beta_p\phi_2}.
\end{aligned}$$

Continuing the iteration until $k = p$ and defining $\phi_p = \beta_1\phi_{p-1} + \beta_2\phi_{p-2} + \cdots + \beta_p\phi_0$ and $\phi_0 = 1$, the product $\prod_{i=1}^{p} h_{t-i}^{2\beta_i}$ can be expressed as

$$\begin{aligned}
&\prod_{i=1}^{p} h_{t-i}^{2\beta_i} \\
= \;& \prod_{i=1}^{p}\exp\{\phi_i\alpha_0\}\prod_{i=1}^{p}\exp\{\phi_i\boldsymbol{\alpha}'\mathbf{g}_{t-i}\} \\
&\times h_{t-p-1}^{2\sum_{i=1}^{p}\beta_i\phi_{p-(i-1)}} \times h_{t-p-2}^{2\sum_{i=2}^{p}\beta_i\phi_{p-(i-2)}} \times\cdots\times h_{t-p-(p-1)}^{2(\beta_{p-1}\phi_p+\beta_p\phi_{p-1})} \times h_{t-p-p}^{2\beta_p\phi_p}. \quad \text{(A.8)}
\end{aligned}$$

Let $k > p$. (i) Assume $k = p+1$. We shall prove that (A.3) and (A.4) hold. From Lemma 1 it follows that

$$\phi_{p+1} = \sum_{i=1}^{p}\beta_i\phi_{p-(i-1)}. \quad \text{(A.9)}$$

Applying (3) to $h_{t-(p+1)}^2$ on the right-hand side of (A.8) and rewriting the corresponding result using (A.9) yield

$$\begin{aligned}
\prod_{i=1}^{p} h_{t-i}^{2\beta_i} \;=\; & \prod_{i=1}^{p+1}\exp\{\phi_i\alpha_0\}\prod_{i=1}^{p+1}\exp\{\phi_i\boldsymbol{\alpha}'\mathbf{g}_{t-i}\} \\
&\times h_{t-(p+1)-1}^{2\sum_{i=1}^{p}\beta_i\phi_{p-(i-2)}} \times h_{t-(p+1)-2}^{2\sum_{i=2}^{p}\beta_i\phi_{p-(i-3)}} \times\cdots \\
&\times h_{t-(p+1)-(p-1)}^{2(\beta_{p-1}\phi_{p+1}+\beta_p\phi_p)} \times h_{t-(p+1)-p}^{2\beta_p\phi_{p+1}}. \quad \text{(A.10)}
\end{aligned}$$

From the right-hand side of (A.10) and Lemma 1 we find

$$\begin{aligned}
\psi_{(p+1)+1} &= \sum_{i=1}^{p} \beta_i \phi_{p-(i-2)} = \phi_{p+2} \\
\psi_{(p+1)+2} &= \sum_{i=2}^{p} \beta_i \phi_{p-(i-3)} \\
&\cdots\cdots \\
\psi_{(p+1)+(p-1)} &= \beta_{p-1}\phi_{p+1} + \beta_p \phi_p \\
\psi_{(p+1)+p} &= \beta_p \phi_{p+1}.
\end{aligned} \tag{A.11}$$

From (A.10) and (A.11) it follows that (A.3) and (A.4) are valid for $k = p+1$. (ii) Assume now that (A.3) and (A.4) are true for $k = m > p+1$. This implies

$$\prod_{i=1}^{p} h_{t-i}^{2\beta_i} = \prod_{i=1}^{m} \exp\{\phi_i \alpha_0\} \prod_{i=1}^{m} \exp\{\phi_i \boldsymbol{\alpha}' \mathbf{g}_{t-i}\} \prod_{i=1}^{p} h_{t-i-m}^{2\psi_{m+i}} \tag{A.12}$$

where

$$\begin{aligned}
\psi_{m+1} &= \phi_{m+1}, \\
\psi_{m+2} &= \beta_2 \phi_m + ... + \beta_p \phi_{m+1-(p-1)}, \\
\psi_{m+3} &= \beta_3 \phi_m + ... + \beta_p \phi_{m+1-(p-2)}, \\
&\cdots\cdots \\
\psi_{m+p} &= \beta_p \phi_m.
\end{aligned} \tag{A.13}$$

We prove by induction that (A.3) and (A.4) are valid for $k = m+1$. Applying (3) to $h_{t-(m+1)}^2$ on the right-hand side of (A.12) while noting that $\psi_{m+1} = \phi_{m+1}$, gives

$$\begin{aligned}
\prod_{i=1}^{p} h_{t-i}^{2\beta_i} &= \prod_{i=1}^{m+1} \exp\{\phi_i \alpha_0\} \prod_{i=1}^{m+1} \exp\{\phi_i \boldsymbol{\alpha}' \mathbf{g}_{t-i}\} \prod_{i=1}^{p} h_{t-i-(m+1)}^{2\beta_i \psi_{m+1}} \prod_{i=2}^{p} h_{t-i-m}^{2\psi_{m+i}} \\
&= \prod_{i=1}^{m+1} \exp\{\phi_i \alpha_0\} \prod_{i=1}^{m+1} \exp\{\phi_i \boldsymbol{\alpha}' \mathbf{g}_{t-i}\} \\
&\quad \times h_{t-1-(m+1)}^{2(\beta_1 \psi_{m+1} + \psi_{m+2})} \times h_{t-2-(m+1)}^{2(\beta_2 \psi_{m+1} + \psi_{m+3})} \times h_{t-3-(m+1)}^{2(\beta_3 \psi_{m+1} + \psi_{m+4})} \\
&\quad \times \cdots \times h_{t-(p-1)-(m+1)}^{2(\beta_{p-1} \psi_{m+1} + \psi_{m+p})} \times h_{t-p-(m+1)}^{2\beta_p \psi_{m+1}}.
\end{aligned} \tag{A.14}$$

From (A.14), (A.13) and Lemma 1 we obtain

$$\begin{aligned}
\beta_1\psi_{m+1}+\psi_{m+2} &= \phi_{m+2} \\
\beta_2\psi_{m+1}+\psi_{m+3} &= \beta_2\phi_{m+1}+\beta_3\phi_m+\cdots+\beta_p\phi_{m+1-(p-2)} \\
\beta_3\psi_{m+1}+\psi_{m+4} &= \beta_3\phi_{m+1}+\beta_4\phi_m+\cdots+\beta_p\phi_{m+1-(p-3)} \\
&\cdots\cdots \\
\beta_{p-1}\psi_{m+1}+\psi_{m+p} &= \beta_{p-1}\phi_{m+1}+\beta_p\phi_m \\
\beta_p\psi_{m+1} &= \beta_p\phi_{m+1}
\end{aligned} \tag{A.15}$$

From (A.14) and (A.15) we see that (A.3) and (A.4) are valid for $k=m+1$. ■

Proof of Theorem 1. We prove the case $m=1$. Without loss of generality, the results apply to the case of any positive integer number m by raising both sides of equation (A.16) below to the power m.

Let $k>p$. Applying Lemma 2 to $\prod_{i=1}^{p} h_{t-i}^{2\beta_i}$ on the right-hand side of (3) yields

$$h_t^2 = \prod_{i=0}^{k}\exp\{\phi_i\alpha_0\}\prod_{i=0}^{k}\exp\{\phi_i\boldsymbol{\alpha}'\mathbf{g}_{t-i}\}\prod_{i=1}^{p}h_{t-i-k}^{2\psi_{k+i}} \tag{A.16}$$

where ϕ_i and ψ_{k+i} are given in (5), (6) and (A.4). Multiplying by z_t^2 and taking expectations on both sides of (A.16) leads to

$$\mathsf{E}\varepsilon_t^2 = \nu_2\exp\{\alpha_0\sum_{i=0}^{k}\phi_i\}\mathsf{E}[\prod_{i=0}^{k}\exp\{\phi_i\boldsymbol{\alpha}'\mathbf{g}_{t-i}\}\prod_{i=1}^{p}h_{t-i-k}^{2\psi_{k+i}}].$$

As $k\to\infty$,

$$\mathsf{E}\varepsilon_t^2 = \nu_2\exp\{\alpha_0\sum_{i=0}^{\infty}\phi_i\}\mathsf{E}\prod_{i=0}^{\infty}\exp\{\phi_i\boldsymbol{\alpha}'\mathbf{g}_{t-i}\} \tag{A.17}$$

if and only if $\lambda(\boldsymbol{\Gamma})<1$ because $\lim_{k\to\infty}\psi_{k+i}=0$ if and only if $\lambda(\boldsymbol{\Gamma})<1$.

Note that $\{z_t\}$ is a sequence of i.i.d. variables. Thus,

$$\mathsf{E}\prod_{i=0}^{\infty}\exp\{\phi_i\boldsymbol{\alpha}'\mathbf{g}_{t-i}\} = \mathsf{E}\exp\{\alpha_1\phi_0 g(z_{t-1})\}$$

$$
\begin{aligned}
&\times \mathsf{E}\exp\{(\alpha_1\phi_1+\alpha_2\phi_0)g(z_{t-2})\}\times\cdots \\
&\times \mathsf{E}\exp\{(\alpha_1\phi_{p-1}+\alpha_2\phi_{p-2}+\cdots+\alpha_p\phi_0)g(z_{t-p})\} \qquad (A.18)\\
&\times \mathsf{E}\exp\{(\alpha_1\phi_{(p+1)-1}+\alpha_2\phi_{(p+1)-2}+\cdots+\alpha_p\phi_{(p+1)-p})g(z_{t-(p+1)})\} \\
&\times\cdots \\
&\times \mathsf{E}\exp\{(\alpha_1\phi_{k-1}+\alpha_2\phi_{k-2}+\cdots+\alpha_p\phi_{k-p})g(z_{t-p})\} \\
&\times\cdots
\end{aligned}
$$

Defining φ_i, $i \geq 1$, as the coefficients of $g(z_{t-i})$, respectively, in (A.18), we see that (10) holds.

Next we show that, under (5) and (6),

$$\sum_{i=0}^{\infty}\phi_i = (1-\sum_{i=1}^{p}\beta_i)^{-1}$$

if and only if $\lambda(\mathbf{\Gamma}) < 1$. It follows from (5) that

$$
\begin{aligned}
\sum_{i=0}^{\infty}\phi_i &= \sum_{i=0}^{p}\phi_i + \sum_{i=p+1}^{\infty}\phi_i \\
&= \sum_{i=0}^{p}\phi_i + \boldsymbol{\beta}'\sum_{i=p+1}^{\infty}\mathbf{\Gamma}^{i-p-1}\boldsymbol{\phi} \\
&= \sum_{i=0}^{p}\phi_i + \boldsymbol{\beta}'(\mathbf{I}-\mathbf{\Gamma})^{-1}\boldsymbol{\phi} \\
&= (1-\sum_{i=1}^{p}\beta_i)^{-1}
\end{aligned}
$$

if and only if $\lambda(\mathbf{\Gamma}) < 1$, since

$$(\mathbf{I}-\mathbf{\Gamma})^{-1}$$

$$
= (1-\sum_{i=1}^{p}\beta_i)^{-1}\begin{pmatrix}
1 & \sum_{i=2}^{p}\beta_i & \sum_{i=3}^{p}\beta_i & \cdots & \cdots & \beta_{p-1}+\beta_p & \beta_p \\
1 & 1-\beta_1 & \sum_{i=3}^{p}\beta_i & & & & \\
1 & 1-\beta_1 & 1-(\beta_1+\beta_2) & \ddots & & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & & \\
 & & & & \ddots & \beta_{p-1}+\beta_p & \beta_p \\
1 & 1-\beta_1 & 1-(\beta_1+\beta_2) & & \cdots & 1-\sum_{i=1}^{p-2}\beta_i & \beta_p \\
1 & 1-\beta_1 & 1-(\beta_1+\beta_2) & & \cdots & 1-\sum_{i=1}^{p-2}\beta_i & 1-\sum_{i=1}^{p-1}\beta_i
\end{pmatrix}.
$$

■

Lemma 3 *Assume that $p=2$. Then*

$$
\begin{aligned}
&\sum_{k=0}^{\infty}(\boldsymbol{\beta}'\boldsymbol{\Gamma}^k\boldsymbol{\phi})^2 \\
= \;& \frac{\beta_2^4+4\beta_1^2\beta_2^2+4\beta_1^4\beta_2+2\beta_1^2\beta_2^3-\beta_2^5+\beta_1^6+3\beta_1^4\beta_2^2-2\beta_1^2\beta_2^4+\beta_1^6\beta_2-\beta_1^4\beta_2^3}{1-\beta_2-\beta_1^2-\beta_2^2-\beta_1^2\beta_2+\beta_2^3}
\end{aligned}
\tag{A.19}
$$

if and only if $\lambda(\boldsymbol{\Gamma})<1$.

Proof. Write

$$
\sum_{k=0}^{\infty}(\boldsymbol{\beta}'\boldsymbol{\Gamma}^k\boldsymbol{\phi})^2=\sum_{k=0}^{\infty}\boldsymbol{\beta}'\boldsymbol{\Gamma}^k\mathbf{A}\boldsymbol{\Gamma}^k\boldsymbol{\phi}=\boldsymbol{\beta}'\mathbf{B}\boldsymbol{\phi}
$$

where $\mathbf{A}=\boldsymbol{\phi}\boldsymbol{\beta}'$ and $\mathbf{B}=\sum_{k=0}^{\infty}\boldsymbol{\Gamma}^k\mathbf{A}\boldsymbol{\Gamma}^k$. Furthermore,

$$
\begin{aligned}
vec(\mathbf{B}) &= \sum_{k=0}^{\infty}vec(\boldsymbol{\Gamma}^k\mathbf{A}\boldsymbol{\Gamma}^k) \\
&= \sum_{k=0}^{\infty}(\boldsymbol{\Gamma}'^k\otimes\boldsymbol{\Gamma}^k)vec(\mathbf{A}) \\
&= \sum_{k=0}^{\infty}(\boldsymbol{\Gamma}'\otimes\boldsymbol{\Gamma})^k vec(\mathbf{A}) \\
&= (\mathbf{I}'\otimes\mathbf{I}-\boldsymbol{\Gamma}'\otimes\boldsymbol{\Gamma})^{-1}vec(\mathbf{A})
\end{aligned}
$$

if and only if $\lambda(\boldsymbol{\Gamma})<1$, where $vec(\cdot)$ is the vectorization of a matrix and $\otimes$ is the Kronecker

product of the matrices. We have

$$\mathbf{B} = \frac{1}{1-\beta_2-\beta_1^2-\beta_2^2-\beta_1^2\beta_2+\beta_2^3}\begin{pmatrix} \beta_1(\beta_2+\beta_1^2+\beta_2^2+\beta_1^3-\beta_2^3) & \beta_2(\beta_2+\beta_1^2-\beta_2^2) \\ \beta_1^2+\beta_2^2+\beta_1^2\beta_2-\beta_2^3 & \beta_1\beta_2 \end{pmatrix}.$$

Then (A.19) holds. ■

Proof of Corollary 3. It follows both from Theorem A1.1 of Nelson (1991) and Corollary 1 that the kurtosis of the process equals

$$\kappa_4 = 3\exp\{\theta^2 \sum_{i=1}^{\infty} \varphi_i^2\}.$$

Next we shall prove when $p = 2$

$$\sum_{i=1}^{\infty} \varphi_i^2 = \frac{(\alpha_1^2+\alpha_2^2)(1-\beta_2)+2\alpha_1\alpha_2\beta_1}{1-\beta_2-\beta_1^2-\beta_2^2-\beta_1^2\beta_2+\beta_2^3} \tag{A.20}$$

if and only if $\lambda(\mathbf{\Gamma}) < 1$. Write

$$\begin{aligned} \sum_{i=1}^{\infty} \varphi_i^2 &= \sum_{i=1}^{4} \varphi_i^2 + \sum_{i=5}^{\infty} (\sum_{j=1}^{2} \alpha_j \boldsymbol{\beta}'\mathbf{\Gamma}^{i-2-1-j}\boldsymbol{\phi})^2 \\ &= \sum_{i=1}^{4} \varphi_i^2 + \sum_{i=5}^{\infty} (\alpha_1 \boldsymbol{\beta}'\mathbf{\Gamma}^{i-4}\boldsymbol{\phi})^2 + \sum_{i=5}^{\infty} (\alpha_2 \boldsymbol{\beta}'\mathbf{\Gamma}^{i-5}\boldsymbol{\phi})^2 \\ &\quad +2\sum_{i=5}^{\infty} \alpha_1\alpha_2(\boldsymbol{\beta}'\mathbf{\Gamma}^{i-4}\boldsymbol{\phi}\boldsymbol{\beta}'\mathbf{\Gamma}^{\mathbf{i-5}}\boldsymbol{\phi}). \end{aligned} \tag{A.21}$$

Applying Lemma 3 to the last three terms of the right-hand side of (A.21) gives (A.20). Then equation (13) holds. ■

Lemma 4 *Let $k > p$. Under (3), the product $\prod_{i=1}^{p} h_{t-i}^{2\beta_i^*}$ where $\beta_i^* > 0$ for $i = 1, ..., p$ can be expressed in terms of the product $\prod_{i=1}^{p} h_{t-i-k}^{2\psi_{k+i}^*}$ such that*

$$\prod_{i=1}^{p} h_{t-i}^{2\beta_i^*} = (\prod_{i=1}^{k} \exp\{\alpha_0\phi_i^*\})(\prod_{i=1}^{k} \exp\{\phi_i^*\boldsymbol{\alpha}'\mathbf{g}_{t-i}\}) \prod_{i=1}^{p} h_{t-i-k}^{2\psi_{k+i}^*} \tag{A.22}$$

where

$$\phi_k^* = \boldsymbol{\beta}'\mathbf{\Gamma}^{k-p-1}\boldsymbol{\phi}^* \tag{A.23}$$

and

$$\begin{aligned}
\psi^*_{k+1} &= \phi^*_{k+1}, \\
\psi^*_{k+2} &= \beta_2\phi^*_k + ... + \beta_p\phi^*_{k+1-(p-1)}, \\
\psi^*_{k+3} &= \beta_3\phi^*_k + ... + \beta_p\phi^*_{k+1-(p-2)}, \\
& ..... \\
\psi^*_{k+p} &= \beta_p\phi^*_k.
\end{aligned} \tag{A.24}$$

In (A.23), $\boldsymbol{\beta}$ *and* Γ *are defined in (5) whereas* $\boldsymbol{\phi}^* = (\phi^*_p, ..., \phi^*_1)'$ *with*

$$\begin{aligned}
\phi^*_1 &= \beta^*_1, \\
\phi^*_j &= \sum_{i=1}^{j-1} \beta_i\phi^*_{j-i} + \beta^*_j, \ j = 2, ..., p.
\end{aligned} \tag{A.25}$$

Proof. The result follows by appropriately substituting β^*_i for β_i, $i = 1, ..., p$, in the proof of Lemma 2. ■

In Lemmata 5-7 we give an expression for the cross-moment $\mathsf{E}\varepsilon^2_t\varepsilon^2_{t-n}$ for $n \geq 1$.

Lemma 5 *Assume that (8) holds,* $\gamma(\cdot) < \infty$, $\beta^*_{in} > 0$ *for* $i = 1, ..., p$, *and* $1 \leq n \leq p$. *Then*

$$\begin{aligned}
& \mathsf{E}(\prod_{j=1}^{n} \exp\{(\phi_{n-j} \sum_{i=1}^{p-j} \alpha_{j+i})g(z_{t-(n+i)})\})(\prod_{i=1}^{p} h^{2\beta^*_{in}}_{t-i-(n-1)}) \\
= \ & (\prod_{i=1}^{\infty} \exp\{\alpha_0\phi^*_{in}\})(\prod_{j=1}^{p-n} \tilde{\gamma}(e^g, \sum_{i=1}^{n} \phi_{n-i}\alpha_{j+i} + \varphi^*_{jn})) \\
& \times(\prod_{j=1}^{n-1} \tilde{\gamma}(e^g, \sum_{i=1}^{j} \phi_{n-i}\alpha_{p-j+i} + \varphi^*_{p-j,n}))(\prod_{i=p}^{\infty} \gamma(e^g, \varphi^*_{in}))
\end{aligned} \tag{A.26}$$

where

$$\begin{aligned}
\varphi^*_{jn} &= \sum_{i=1}^{j} \alpha_i\phi^*_{(j+1)-i,n} \ \textit{for } j = 1, ..., p \\
\varphi^*_{jn} &= \sum_{i=1}^{p} \alpha_i\phi^*_{(j+1)-i,n} \ \textit{for } j > p
\end{aligned} \tag{A.27}$$

such that

$$\begin{aligned}
\phi_{1n}^* &= \beta_{1n}^*, \\
\phi_{jn}^* &= \sum_{i=1}^{j-1} \beta_i \phi_{j-i,n}^* + \beta_{jn}^*, j = 2, ..., p, \\
\phi_{jn}^* &= \boldsymbol{\beta}' \boldsymbol{\Gamma}^{j-p-1} \boldsymbol{\phi}_n^* \ for \ j > p
\end{aligned} \tag{A.28}$$

with $\boldsymbol{\phi}_n^* = (\phi_{pn}^*, ..., \phi_{1n}^*)'$.

Proof. Rewrite

$$\begin{aligned}
& \mathsf{E}(\prod_{j=1}^{n} \exp\{(\phi_{n-j} \sum_{i=1}^{p-j} \alpha_{j+i}) g(z_{t-(n+i)})\})(\prod_{i=1}^{p} h_{t-i-(n-1)}^{2\beta_{in}^*}) \\
= \ & \mathsf{E}[\prod_{j=1}^{p-n} \exp\{(\sum_{i=1}^{n} \phi_{n-i} \alpha_{j+i}) g(z_{t-(n+j)})\} \\
& \times \prod_{j=1}^{n-1} \exp\{(\sum_{i=1}^{j} \phi_{n-i} \alpha_{p-j+i}) g(z_{t-p-(n-j)})\}(\prod_{i=1}^{p} h_{t-i-(n-1)}^{2\beta_{in}^*})].
\end{aligned} \tag{A.29}$$

Applying Lemma 4 to $\prod_{i=1}^{p} h_{t-i-(n-1)}^{2\beta_{in}^*}$ on the right-hand side of (A.29) gives (A.26)-(A.28).

■

Lemma 6 *Assume that (8) holds,* $\gamma(\cdot) < \infty$, $\beta_{in}^* > 0$ *for* $i = 1, ..., p$ *and* $n \geq p+1$. *Then*

$$\begin{aligned}
& \mathsf{E}(\prod_{j=1}^{p-1} \exp\{(\sum_{i=1}^{p-j} \phi_{n-i} \alpha_{j+i}) g(z_{t-(n+j)})\})(\prod_{i=1}^{p} h_{t-i-(n-1)}^{2\beta_{in}^*}) \\
= \ & (\prod_{i=1}^{\infty} \exp\{\alpha_0 \phi_{in}^*\})(\prod_{j=1}^{p-1} \widetilde{\gamma}(e^g, \sum_{i=1}^{p-j} \phi_{n-i} \alpha_{j+i} + \varphi_{jn}^*))(\prod_{i=p}^{\infty} \gamma(e^g, \varphi_{in}^*))
\end{aligned} \tag{A.30}$$

where φ_{jn}^* *and* ϕ_{jn}^* *for* $j \geq 1$ *are given in* (A.27) *and* (A.28).

Proof. Let $k > p$. Applying Lemma 4 to $(\prod_{i=1}^{p} h_{t-i-(n-1)}^{2\beta_{in}^*})$ on the left-hand side of (A.30) yields

$$\begin{aligned}
& \mathsf{E}(\prod_{j=1}^{p-1} \exp\{(\sum_{i=1}^{p-j} \phi_{n-i} \alpha_{j+i}) g(z_{t-(n+j)})\})(\prod_{i=1}^{p} h_{t-i-(n-1)}^{2\beta_{in}^*}) \\
= \ & \mathsf{E}[(\prod_{j=1}^{p-1} \exp\{(\sum_{i=1}^{p-j} \phi_{n-i} \alpha_{j+i}) g(z_{t-(n+j)})\})(\prod_{i=1}^{k} \exp\{\alpha_0 \phi_{in}^*\}) \\
& \times (\prod_{i=1}^{k} \exp\{\phi_{in}^* \boldsymbol{\alpha}' \mathbf{g}_{t-n-i}\})(\prod_{i=1}^{p} h_{t-(n-1)-i-k}^{2\psi_{(k+i)n}^*})]
\end{aligned} \tag{A.31}$$

where the corresponding ϕ^*_{in} and $\psi^*_{(k+i)n}$ can be defined by substituting β^*_{in} for β^*_i in (A.23)-(A.25) such that

$$\phi^*_{kn} = \boldsymbol{\beta}'\boldsymbol{\Gamma}^{k-p-1}\boldsymbol{\phi}^*_n \tag{A.32}$$

with $\boldsymbol{\phi}^*_n = (\phi^*_{pn}, ..., \phi^*_{1n})'$ where

$$\begin{aligned} \phi^*_{1n} &= \beta^*_{1n}, \\ \phi^*_{jn} &= \sum_{i=1}^{j-1} \beta_i \phi^*_{j-i,n} + \beta^*_{jn}, \ j = 2, ..., p. \end{aligned} \tag{A.33}$$

and

$$\begin{aligned} \psi^*_{k+1,n} &= \phi^*_{k+1,n}, \\ \psi^*_{k+2,n} &= \beta_2\phi^*_{kn} + ... + \beta_p\phi^*_{k+1-(p-1),n}, \\ \psi^*_{k+3,n} &= \beta_3\phi^*_{kn} + ... + \beta_p\phi^*_{k+1-(p-2),n}, \\ & \quad ..... \\ \psi^*_{k+p,n} &= \beta_p\phi^*_{kn}. \end{aligned} \tag{A.34}$$

From (A.32)-(A.33) we see that (A.28) holds.

Assume that $\lambda(\boldsymbol{\Gamma}) < 1$ holds. As $k \to \infty$, the left-hand side of (A.31) becomes

$$\begin{aligned} & \mathsf{E}(\prod_{j=1}^{p-1} \exp\{(\sum_{i=1}^{p-j} \phi_{n-i}\alpha_{j+i})g(z_{t-(n+j)})\})(\prod_{i=1}^{p} h^{2\beta^*_{in}}_{t-i-(n-1)}) \\ = \ & \mathsf{E}[(\prod_{j=1}^{p-1} \exp\{(\sum_{i=1}^{p-j} \phi_{n-i}\alpha_{j+i})g(z_{t-(n+j)})\})(\prod_{i=1}^{\infty} \exp\{\alpha_0\phi^*_{in}\}) \\ & \times(\prod_{i=1}^{\infty} \exp\{\varphi^*_{in}g(z_{t-(n+i)})\})] \\ = \ & \prod_{i=1}^{\infty} \exp\{\alpha_0\phi^*_{in}\} \\ & \times\mathsf{E}[(\prod_{j=1}^{p-1} \exp\{(\sum_{i=1}^{p-j} \phi_{n-i}\alpha_{j+i})g(z_{t-(n+j)})\}) \prod_{i=1}^{p} \exp\{\varphi^*_{in}g(z_{t-(n+i)})\})] \\ & \times\mathsf{E}(\prod_{i=p}^{\infty} \exp\{\varphi^*_{in}g(z_{t-(n+i)})\}) \\ = \ & \prod_{i=1}^{\infty} \exp\{\alpha_0\phi^*_{in}\} \end{aligned}$$

$$
\begin{aligned}
&\times \mathsf{E}[\prod_{j=1}^{p-1} \exp\{(\sum_{i=1}^{p-j} \phi_{n-i}\alpha_{j+i} + \varphi^*_{jn}) g(z_{t-(n+j)})\}] \\
&\times \mathsf{E}(\prod_{i=p}^{\infty} \exp\{\varphi^*_{in} g(z_{t-(n+i)})\}) \\
= \quad & \prod_{i=1}^{\infty} \exp\{\alpha_0 \phi^*_{in}\}(\prod_{j=1}^{p-1} \widetilde{\gamma}(e^g, \sum_{i=1}^{p-j} \phi_{n-i}\alpha_{j+i} + \varphi^*_{jn}))(\prod_{i=p}^{\infty} \gamma(e^g, \varphi^*_{in})) \qquad \text{(A.35)}
\end{aligned}
$$

such that

$$
\begin{aligned}
\varphi^*_{1n} &= \alpha_1 \phi^*_{1n} \\
\varphi^*_{2n} &= \alpha_1 \phi^*_{2n} + \alpha_2 \phi^*_{1n} \\
&\cdots\cdots \qquad \text{(A.36)} \\
\varphi^*_{pn} &= \alpha_1 \phi^*_{pn} + \alpha_2 \phi^*_{2n} + \cdots + \alpha_p \phi^*_{1n} \\
\varphi^*_{kn} &= \sum_{i=1}^{p} \alpha_i \boldsymbol{\beta}' \boldsymbol{\Gamma}^{k-p-i} \boldsymbol{\phi}^*_n, \text{ for } k > p.
\end{aligned}
$$

It follows from (A.35) and (A.36) that (A.26) and (A.27) hold. ■

Lemma 7 *Assume that (8) holds and* $\gamma(\cdot), \overline{\gamma}(\cdot), \widetilde{\gamma}(\cdot) < \infty$. *Then*

$$
\begin{aligned}
\mathsf{E}(\varepsilon_t^2 \varepsilon_{t-n}^2) &= \nu_2(\overline{\gamma}_{np}(e^g, \varphi_n))(\prod_{i=1}^{n-1} \exp\{\alpha_0 \phi_i\})(\prod_{i=1}^{\infty} \exp\{\alpha_0 \phi^*_{in}\}) \\
&\quad \times M_1(n) M_2(n) \qquad \text{(A.37)}
\end{aligned}
$$

such that

$$
M_1(n) = \prod_{i=1}^{n-1} \gamma(e^g, \varphi_i) \prod_{i=p}^{\infty} \gamma(e^g, \varphi^*_{in}) \ \textit{for } n \geq 1 \qquad \text{(A.38)}
$$

$$
M_2(n) = (\prod_{j=1}^{p-n} \widetilde{\gamma}(e^g, \sum_{i=1}^{n} \phi_{n-i}\alpha_{j+i} + \varphi^*_{jn}))(\prod_{j=1}^{n-1} \widetilde{\gamma}(e^g, \sum_{i=1}^{j} \phi_{n-i}\alpha_{p-j+i} + \varphi^*_{p-j,n})) \ \textit{for } 1 \leq n \leq p \qquad \text{(A.39)}
$$

$$
M_2(n) = \prod_{j=1}^{p-1} \widetilde{\gamma}_{jp}(e^g, \sum_{i=1}^{p-j} \phi_{n-i}\alpha_{j+i} + \varphi^*_{jn}) \ \textit{for } n \geq p+1 \qquad \text{(A.40)}
$$

with

$$
\beta^*_{1n} = \phi_n + 1 \ \textit{for any } n \geq 1 \qquad \text{(A.41)}
$$

$$\beta^*_{in} = \sum_{j=1}^{n} \beta_{j+(i-1)}\phi_{n-j} \ for\ 1 \le n \le p-(i-1)\ and\ 2 \le i \le p-1 \tag{A.42}$$

$$\beta^*_{in} = \sum_{j=i}^{p} \beta_j\phi_{(n+i-1)-j} \ for\ p-(i-2) \le n \le p\ and\ 2 \le i \le p-1 \tag{A.43}$$

$$\beta^*_{pn} = \beta_p\phi_{n-1} \ for\ 1 \le n \le p \tag{A.44}$$

$$\beta^*_{in} = \sum_{j=1}^{p+1-i} \beta_{i+j-1}\phi_{n-j} \ for\ n > p\ and\ 2 \le i \le p \tag{A.45}$$

where ϕ_i, φ_i, ϕ^*_{in} and φ^*_{in} for any i and n are defined in (5)-(6), (10), (A.28) and (A.27).

Proof. First consider the product $h_t^2 h_{t-n}^2$ for $1 \le n \le p$. Applying (3) to h_t^2 in $h_t^2 h_{t-n}^2$ and using (A.5)-(A.8) in the first part of the proof of Lemma 2 we obtain, for $1 \le n \le p$,

$$\begin{aligned} h_t^2 h_{t-n}^2 &= \left(\prod_{i=0}^{n-1} \exp\{\alpha_0\phi_i\boldsymbol{\alpha}'\mathbf{g}_{t-i}\}\right) \times \left(\prod_{j=1}^{p-(n-1)} h_{t-j-(n-1)}^{2\sum_{i=1}^{n}\beta_{j+i-1}\phi_{n-i}}\right) \\ &\quad \times \left(\prod_{j=1}^{n-1} h_{t-p-j}^{2\sum_{i=1}^{j}\beta_{p-j+i}\phi_{n-i}}\right) h_{t-n}^2. \end{aligned} \tag{A.46}$$

(i) Consider $j = 1$ in the second product term $\prod_{j=1}^{p-(n-1)} h_{t-j-(n-1)}^{2\sum_{i=1}^{n}\beta_{j+i-1}\phi_{n-i}}$ on the right-hand side of (A.46). This term multiplied by h_{t-n}^2 has the form

$$h_{t-n}^{2(\sum_{i=1}^{n}\beta_i\phi_{n-i}+1)}$$

which implies

$$\beta^*_{1n} = \sum_{i=1}^{n} \beta_i\phi_{n-i} + 1 = \phi_n + 1 for\ 1 \le n \le p.$$

Thus, (A.41) holds for $1 \le n \le p$.

(ii) Set $j = 1$ in the third product term $\prod_{j=1}^{n-1} h_{t-p-j}^{2\sum_{i=1}^{j}\beta_{p-j+i}\phi_{n-i}}$ on the right-hand side of (A.46). This yields $h_{t-p-1}^{2\beta_p\phi_{n-1}}$ which shows that

$$\beta^*_{pn} = \beta_p\phi_{n-1} for\ 1 \le n \le p.$$

Thus, (A.44) holds.

(iii) Consider the remaining case: $j = 2, ..., p-(n-1)$ in the second and $j = 2, ..., n-1$ in the third product term on the right-hand side of (A.46). Thus,

$$h_{t-j-(n-1)}^{2\sum_{i=1}^{n}\beta_{j+i-1}\phi_{n-i}} \quad for\ 2 \quad \leq \quad j \leq p-(n-1) \tag{A.47}$$

$$h_{t-p-j}^{2\sum_{i=1}^{j}\beta_{p-j+i}\phi_{n-i}} \quad for\ 2 \quad \leq \quad j \leq n-1. \tag{A.48}$$

Formulas (A.47) and (A.48) indicate that β_{in}^*, $i = 2, ..., p-(n-1)$ can be defined by setting $j = 2, ..., p-(n-1)$ in (A.47) and β_{in}^*, $i = p-(n-2) \leq i \leq p-1$, can be defined by setting $j = n-1, ..., 2$ in (A.48). Equivalently, for $2 \leq i \leq p-1$,

$$\beta_{in}^* = \begin{cases} \sum_{j=1}^{n} \beta_{j+i-1}\phi_{n-j}, & 1 \leq n \leq p-(i-1) \\ \sum_{j=1}^{p} \beta_j \phi_{(n+i-1)-j}, & p-(i-2) \leq n \leq p. \end{cases}$$

As a result, (A.42) and (A.43) hold.

Next, let $n > p$. It follows from Lemma 2 that the corresponding representation of $h_t^2 h_{t-n}^2$ can be obtained by setting $k = n-1$ in (A.3) and (A.4). This gives

$$h_t^2 h_{t-n}^2 = (\prod_{i=0}^{n-1} \exp\{\alpha_0 \phi_i \boldsymbol{\alpha}' \mathbf{g}_{t-i}\}) \prod_{i=1}^{p} h_{t-i-(n-1)}^{2\psi_{(n-1)+i}} h_{t-n}^2$$

where

$$\begin{aligned} \psi_{(n-1)+1} + 1 &= \phi_n + 1 = \beta_{1n}^* \\ \psi_{(n-1)+2} &= \beta_2\phi_{n-1} + \cdots + \beta_p\phi_{n-(p-1)} = \beta_{2n}^* \\ \psi_{(n-1)+3} &= \beta_3\phi_{n-1} + \cdots + \beta_p\phi_{n-(p-2)} = \beta_{3n}^* \\ &\cdots\cdots \\ \psi_{(n-1)+p} &= \beta_p\phi_{n-1} = \beta_{pn}^*. \end{aligned}$$

This shows that β_{1n}^* in (A.41) for $n \geq p+1$ and β_{in}^* for $n > p$ and $2 \leq i \leq p$ in (A.45) hold.

Finally, consider

$$
\begin{aligned}
& \mathsf{E}(\varepsilon_t^2 \varepsilon_{t-n}^2) \\
= \; & \mathsf{E}(z_t^2 z_{t-n}^2 h_t^2 h_{t-n}^2) \\
= \; & \mathsf{E}[z_t^2 z_{t-n}^2 (\prod_{i=0}^{n-1} \exp\{\alpha_0 \phi_i\})(\prod_{i=0}^{n-1} \exp\{\phi_i \boldsymbol{\alpha}' \mathbf{g}_{t-i}\})(\prod_{i=1}^{p} h_{t-i-(n-1)}^{2\beta_{in}^*})] \qquad \text{(A.49)}
\end{aligned}
$$

where β_{in}^* for $i = 1, ..., p$ and $n \geq 1$ are given in (A.41)-(A.45). Assume now that (8) holds. Then it follows from Lemma 4 that $\prod_{i=1}^{p} h_{t-i-(n-1)}^{2\beta_{in}^*}$ on the right-hand side of (A.49) can be expressed as a function of k. As $k \to \infty$, (A.49) converges to

$$
\begin{aligned}
& \mathsf{E}(\varepsilon_t^2 \varepsilon_{t-n}^2) \\
= \; & \mathsf{E}[z_t^2 z_{t-n}^2 (\prod_{i=0}^{n-1} \exp\{\alpha_0 \phi_i\})(\prod_{i=0}^{n-1} \exp\{\phi_i \boldsymbol{\alpha}' \mathbf{g}_{t-i}\}) \\
& \times (\prod_{i=1}^{\infty} \exp\{\alpha_0 \phi_{in}^*\})(\prod_{i=1}^{\infty} \exp\{\varphi_{in}^* g(z_{t-(n+i)})\})] \\
= \; & \nu_2 (\prod_{i=0}^{n-1} \exp\{\alpha_0 \phi_i\})(\prod_{i=1}^{\infty} \exp\{\alpha_0 \phi_{in}^*\}) \\
& \times \mathsf{E}[z_{t-n}^2 (\prod_{i=0}^{n-1} \exp\{\phi_i \boldsymbol{\alpha}' \mathbf{g}_{t-i}\})(\prod_{i=1}^{\infty} \exp\{\varphi_{in}^* g(z_{t-(n+i)})\})]. \qquad \text{(A.50)}
\end{aligned}
$$

Rewriting the right-hand side of (A.50) gives, when $n \geq p+1$,

$$
\begin{aligned}
\mathsf{E}(\varepsilon_t^2 \varepsilon_{t-n}^2) \; = \; & \nu_2 (\prod_{i=0}^{n-1} \exp\{\alpha_0 \phi_i\})(\prod_{i=1}^{\infty} \exp\{\alpha_0 \phi_{in}^*\}) \\
& \times \mathsf{E}[z_{t-n}^2 (\prod_{i=1}^{n} \exp\{\varphi_i g(z_{t-i})\}) \\
& \times (\prod_{j=1}^{p-1} \exp\{(\sum_{i=1}^{p-j} \phi_{n-i} \alpha_{j+i}) g(z_{t-(n+j)})\}) \\
& \times (\prod_{i=1}^{\infty} \exp\{\varphi_{in}^* g(z_{t-(n+i)})\})], \qquad \text{(A.51)}
\end{aligned}
$$

while $1 \leq n \leq p$ it has the form

$$
\begin{aligned}
\mathsf{E}(\varepsilon_t^2 \varepsilon_{t-n}^2) \; = \; & \nu_2 (\prod_{i=0}^{n-1} \exp\{\alpha_0 \phi_i\})(\prod_{i=1}^{\infty} \exp\{\alpha_0 \phi_{in}^*\}) \\
& \times \mathsf{E}[z_{t-n}^2 (\prod_{i=1}^{n} \exp\{\varphi_i g(z_{t-i})\})
\end{aligned}
$$

$$\times(\prod_{j=1}^{n}\exp\{(\phi_{n-j}\sum_{i=1}^{p-j}\alpha_{j+i})g(z_{t-(n+i)})\})$$
$$\times(\prod_{i=1}^{\infty}\exp\{\varphi_{in}^{*}g(z_{t-(n+i)})\})]. \tag{A.52}$$

Applying the results in Lemma 5 to (A.51) and defining $M_1(n)$ and $M_2(n)$ for $1 \le n \le p$ gives (A.37)-(A.39), whereas $n \ge p+1$, applying Lemma 6 to (A.52) and defining $M_1(n)$ and $M_2(n)$ for $n > p$ gives (A.37), (A.38) and (A.40). Noting that in (A.51) and (A.52) $\mathsf{E}(z_{t-n}^2\exp\{\varphi_n g(z_{t-n})\}) = \overline{\gamma}_{np}(e^g, \varphi_n)$, we then complete the proof of Lemma 7. ■

Proof of Theorem 2. We assume that (8) holds. To derive expression (15), we have to show that for any $n \ge 1$,

$$\prod_{i=0}^{n-1}\exp\{\alpha_0\phi_i\}\prod_{i=1}^{\infty}\exp\{\alpha_0\phi_{in}^{*}\} = \exp\{2\alpha_0(1-\sum_{i=1}^{p}\beta_i)^{-1}\} \tag{A.53}$$

holds. This follows from Lemmata 2 and 4-7. To do so we first show that for any $n \ge 1$

$$\sum_{i=1}^{\infty}\phi_{in}^{*} = (\sum_{i=1}^{p}\beta_{in}^{*})(1-\sum_{i=1}^{p}\beta_i)^{-1}. \tag{A.54}$$

Consider

$$\begin{aligned}\sum_{i=1}^{\infty}\phi_{in}^{*} &= \sum_{i=1}^{p}\phi_{in}^{*}+\sum_{i=p+1}^{\infty}\phi_{in}^{*}\\ &= \sum_{i=1}^{p}\phi_{in}^{*}+\boldsymbol{\beta}'\sum_{i=p+1}^{\infty}\boldsymbol{\Gamma}^{i-p-1}\boldsymbol{\phi}_n^{*}\\ &= \sum_{i=1}^{p}\phi_{in}^{*}+\boldsymbol{\beta}'(\mathbf{I}-\boldsymbol{\Gamma})^{-1}\boldsymbol{\phi}_n^{*}.\end{aligned} \tag{A.55}$$

Since $(\mathbf{I}-\boldsymbol{\Gamma})^{-1}$ is known (see Lemma 2), it is sufficient to consider the case $p = 2$. Then (A.55) becomes, by (A.33),

$$\begin{aligned}\sum_{i=1}^{\infty}\phi_{in}^{*} &= \phi_{1n}^{*}+\phi_{2n}^{*}+\frac{1}{1-(\beta_1+\beta_2)}\begin{pmatrix}\beta_1 & \beta_2\end{pmatrix}\begin{pmatrix}1 & \beta_2\\ 1 & 1-\beta_1\end{pmatrix}\begin{pmatrix}\phi_{2n}^{*}\\ \phi_{1n}^{*}\end{pmatrix}\\ &= \frac{\phi_{1n}^{*}+\phi_{2n}^{*}-\beta_1\phi_{1n}^{*}}{1-(\beta_1+\beta_2)}\\ &= \frac{\beta_{1n}^{*}+\beta_{2n}^{*}}{1-(\beta_1+\beta_2)}.\end{aligned} \tag{A.56}$$

Next we show that (A.53) holds for any $n \geq 1$.

(i) Let $n = 1$. Then (A.53) has the form

$$\begin{aligned}\exp\{\alpha_0\} \prod_{i=1}^{\infty} \exp\{\alpha_0 \phi_{i1}^*\} &= (\exp\{\alpha_0\})(\exp\{\alpha_0 \frac{\beta_{11}^* + \beta_{21}^*}{1-(\beta_1+\beta_2)}\}) \\ &= \exp\{\frac{2\alpha_0}{1-(\beta_1+\beta_2)}\}.\end{aligned}$$

Similarly, we check that (A.53) holds for $n = 2$.

(ii) Assume now that (A.53) holds for $n = m > 2$, that is,

$$\prod_{i=0}^{m-1} \exp\{\alpha_0 \phi_i\} \prod_{i=1}^{\infty} \exp\{\alpha_0 \phi_{im}^*\} = \exp\{2\alpha_0 (1 - \sum_{i=1}^{2} \beta_i)^{-1}\}. \tag{A.57}$$

We want to show that (A.53) holds for $n = m+1$. From (A.57) we have

$$\begin{aligned}& \sum_{i=0}^{m} \phi_i + \sum_{i=1}^{\infty} \phi_{i,m+1}^* \\ = & \sum_{i=0}^{m-1} \phi_i + \phi_m + \frac{\beta_{1,m+1}^* + \beta_{2,m+1}^*}{1-(\beta_1+\beta_2)} \\ = & \sum_{i=0}^{m-1} \phi_i + \frac{\beta_{1,m}^* + \beta_{2,m}^*}{1-(\beta_1+\beta_2)} + (\phi_m + \frac{\beta_{1,m+1}^* + \beta_{2,m+1}^*}{1-(\beta_1+\beta_2)} - \frac{\beta_{1,m}^* + \beta_{2,m}^*}{1-(\beta_1+\beta_2)}) \\ = & 2\alpha_0 (1 - \sum_{i=1}^{2} \beta_i)^{-1} + (\phi_m + \frac{\beta_{1,m+1}^* + \beta_{2,m+1}^*}{1-(\beta_1+\beta_2)} - \frac{\beta_{1,m}^* + \beta_{2,m}^*}{1-(\beta_1+\beta_2)}).\end{aligned} \tag{A.58}$$

Now, the second term on the right-hand side of (A.58) equals zero, because $\beta_{1,m+1}^* = \phi_{m+1} + 1$, $\beta_{2,m+1}^* = \beta_2 \phi_m$ and $\phi_{m+1} = \beta_1 \phi_m + \beta_2 \phi_{m-1}$. Thus, from (i) and (ii) we conclude that (A.53) holds for any $n \geq 1$. ■

Proof of Corollary 4. Results in Corollary 4 are obtained by setting $p = 2$ in Theorem 2. We only point out that setting $p = 2$ in (A.41)-(A.45) implies for any $n \geq 1$

$$\beta_{1n}^* = \phi_n + 1, \tag{A.59}$$

$$\beta_{2n}^* = \beta_2 \phi_{n-1}. \tag{A.60}$$

Then inserting (A.59) and (A.60) into (A.28) and (A.27) gives ϕ_{in}^* and φ_{jn}^*, so that (18)

and (19) hold. Furthermore, applying (A.59) and (A.60) to (A.39) and (A.40) yields

$$M_2(n) = \widetilde{\gamma}(e^g, \alpha_2\phi_{n-1} + \varphi^*_{1n}),\ \ n \geq 1.$$

Then (17) holds. ■

Proof of Corollary 5. It follows from Theorem A1.1 of Nelson (1991) that ρ_n in Corollary 5 has a simplified form

$$\rho_n = \frac{(1+\theta^2\varphi_n^2)\exp\{0.5\theta^2(\alpha_2^2\phi_{n-1}^2 + 2\alpha_2\phi_{n-1}\varphi_{1n}^{*2} + \sum_{i=1}^{n}\varphi_i^2 + \sum_{i=1}^{\infty}\varphi_{in}^{*2})\} - \exp\{\theta^2\sum_{i=1}^{\infty}\varphi_i^2\}}{3\exp\{2\theta^2\sum_{i=1}^{\infty}\varphi_i^2\} - \exp\{\theta^2\sum_{i=1}^{\infty}\varphi_i^2\}}. \tag{A.61}$$

The sum $\sum_{i=1}^{\infty}\varphi_i^2$ in (A.61) is defined by (A.20). We only need to derive an expression for $\sum_{i=1}^{\infty}\varphi_{in}^{*2}$ in (A.61), and then the proof of (24) is completed through simple algebra. By applying (18) and (19), write

$$\begin{aligned}
\sum_{i=1}^{\infty}\varphi_{in}^{*2} &= \sum_{i=1}^{3}\varphi_{in}^{*2} + \sum_{i=4}^{\infty}\varphi_{in}^{*2} \\
&= \sum_{i=1}^{3}\varphi_{in}^{*2} + \sum_{i=4}^{\infty}\Big(\sum_{j=1}^{2}\alpha_j\phi^*_{(i+1)-j,n}\Big)^2 \\
&= \sum_{i=1}^{3}\varphi_{in}^{*2} + \sum_{i=4}^{\infty}\Big(\sum_{j=1}^{2}\alpha_j\boldsymbol{\beta}'\boldsymbol{\Gamma}^{(i+1)-j-3}\boldsymbol{\phi}_n^*\Big)^2 \\
&= \sum_{i=1}^{3}\varphi_{in}^{*2} + \sum_{i=4}^{\infty}[(\alpha_1\boldsymbol{\beta}'\boldsymbol{\Gamma}^{i-3}\boldsymbol{\phi}_n^*)^2 + (\alpha_2\boldsymbol{\beta}'\boldsymbol{\Gamma}^{i-4}\boldsymbol{\phi}_n^*)^2 \\
&\quad + 2\alpha_1\alpha_2\boldsymbol{\beta}'\boldsymbol{\Gamma}^{i-3}\boldsymbol{\phi}_n^*\boldsymbol{\beta}'\boldsymbol{\Gamma}^{i-4}\boldsymbol{\phi}_n^*].
\end{aligned} \tag{A.62}$$

On the other hand, set

$$\mathbf{B}_n^* = \sum_{k=0}^{\infty}\boldsymbol{\Gamma}^k\mathbf{A}_n^*\boldsymbol{\Gamma}^k$$

where $\mathbf{A}_n^* = \boldsymbol{\phi}_n^*\boldsymbol{\beta}'$. Substituting ϕ_{2n}^* and ϕ_{1n}^* for ϕ_{2n} and ϕ_{1n}, respectively, in $\mathbf{A}$ and $\boldsymbol{\phi}$ of Lemma 3 yields

$$\begin{aligned}
&\mathbf{B}_n^* \\
= &\frac{1}{1-\beta_2-\beta_1^2-\beta_2^2-\beta_1^2\beta_2+\beta_2^3}
\end{aligned}$$

$$\times \begin{pmatrix} \beta_1\phi^*_{2n} + (\beta_1^2\beta_2 + \beta_2^2 - \beta_2^3)\phi^*_{1n} & (\beta_2 - \beta_2^2)\phi^*_{2n} + \beta_1\beta_2^2\phi^*_{1n} \\ (\beta_2 + \beta_1^2 - \beta_2^2)\phi^*_{2n} + (\beta_1 - \beta_1\beta_2 + \beta_1\beta_2^2 - \beta_1^3)\phi^*_{1n} & \beta_1\beta_2\phi^*_{2n} + (\beta_2 - \beta_2^2 - \beta_1^2\beta_2)\phi^*_{1n} \end{pmatrix}. \tag{A.63}$$

Assuming $\lambda(\mathbf{\Gamma}) < 1$ and applying (A.63) to the last three terms on the right-hand side of (A.62) leads to

$$\begin{aligned} \sum_{i=1}^{\infty} \varphi^{*2}_{in} &= \frac{1}{1 - \beta_2 - \beta_1^2 - \beta_2^2 - \beta_1^2\beta_2 + \beta_2^3}[(\alpha_1^2 - \alpha_1^2\beta_2 - \alpha_2^2\beta_2 + \alpha_2^2 + 2\alpha_1\beta_1\alpha_2) \\ &+(2\alpha_1^2 + 2\alpha_2^2 - 2\alpha_1^2\beta_2 + 4\alpha_1\alpha_2\beta_1 - 2\alpha_2^2\beta_2)\phi_n + (\alpha_1^2 + \alpha_2^2 \\ &-\alpha_1^2\beta_2 - \alpha_2^2\beta_2 + 2\alpha_1\alpha_2\beta_1)\phi_n^2 + (2\alpha_1^2\beta_1\beta_2 - 2\alpha_1\alpha_2\beta_2^3 \\ &+2\alpha_1\beta_1^2\alpha_2\beta_2 + 2\alpha_2^2\beta_1\beta_2 + 2\alpha_1\alpha_2\beta_2)\phi_n\phi_{n-1} + (2\alpha_1\beta_1^2\alpha_2\beta_2 + 2\alpha_1^2\beta_1\beta_2 \\ &+2\alpha_2^2\beta_1\beta_2 + 2\alpha_1\alpha_2\beta_2 - 2\alpha_1\alpha_2\beta_2^3)\phi_{n-1} \\ &+(2\alpha_1\beta_1\alpha_2\beta_2^2 - \alpha_2^2\beta_2^3 - \alpha_1^2\beta_2^3 + \alpha_2^2\beta_2^2 + \alpha_1^2\beta_2^2)\phi_{n-1}^2 \end{aligned} \tag{A.64}$$

Inserting (A.64) into (A.61), simplifying the corresponding results, and defining $\gamma(n)$ as in (25) we see that (24) holds. ■

Proof of Corollary 6. It follows from Corollary 4 and Theorem 1 and Theorem A1.1 of Nelson (1991) that

$$\begin{aligned} (\prod_{i=1}^{\infty} \gamma(e^g, \varphi_i))^2 &= [\prod_{i=1}^{\infty} \exp\{-\sqrt{\frac{2}{\pi}}\omega\varphi_i\}(\Phi(\varphi_i(\omega+\theta))\exp\{0.5\varphi_i^2(\omega+\theta)^2\} \\ &+\Phi(\varphi_i(\omega-\theta))\exp\{0.5\varphi_i^2(\omega-\theta)^2\})]^2 \\ &= \exp\{-\frac{2\sqrt{\frac{2}{\pi}}\omega(\alpha_1+\alpha_2)}{1-(\beta_1+\beta_2)}\}[\prod_{i=1}^{\infty} (\Phi(\varphi_i(\omega+\theta))\exp\{0.5\varphi_i^2(\omega+\theta)^2\} \\ &+\Phi(\varphi_i(\omega-\theta))\exp\{0.5\varphi_i^2(\omega-\theta)^2\}]^2. \end{aligned} \tag{A.65}$$

Similarly,

$$\begin{aligned} \prod_{i=1}^{\infty} \gamma(e^g, 2\varphi_i)) &= \exp\{-\frac{2\sqrt{\frac{2}{\pi}}\omega(\alpha_1+\alpha_2)}{1-(\beta_1+\beta_2)}\} \prod_{i=1}^{\infty} [\Phi(2\varphi_i(\omega+\theta))\exp\{2\varphi_i^2(\omega+\theta)^2\} \\ &+\Phi(2\varphi_i(\omega-\theta))\exp\{2\varphi_i^2(\omega-\theta)^2\}], \end{aligned} \tag{A.66}$$

and

$$
\begin{aligned}
& M_1(n)M_2(n) \\
= \; & [\prod_{i=1}^{n-1} \exp\{-\sqrt{\frac{2}{\pi}}\omega\varphi_i\}(\Phi(\varphi_i(\omega+\theta))\exp\{0.5\varphi_i^2(\omega+\theta)^2\} \\
& +\Phi(\varphi_i(\omega-\theta))\exp\{0.5\varphi_i^2(\omega-\theta)^2\})] \\
& \times[\prod_{i=2}^{\infty} \exp\{-\sqrt{\frac{2}{\pi}}\omega\varphi_{in}^*\}(\Phi(\varphi_{in}^*(\omega+\theta))\exp\{0.5\varphi_{in}^{*2}(\omega+\theta)^2\} \\
& +\Phi(\varphi_{in}^*(\omega-\theta))\exp\{0.5\varphi_{in}^{*2}(\omega-\theta)^2\}] \\
& \times[\exp\{-\sqrt{\frac{2}{\pi}}\omega(\alpha_2\phi_{n-1}+\varphi_{1n}^*)\}(\Phi((\alpha_2\phi_{n-1}+\varphi_{1n}^*)(\omega+\theta)) \\
& \times\exp\{0.5(\alpha_2\phi_{n-1}+\varphi_{1n}^*)^2(\omega+\theta)^2\} \\
& +\Phi((\alpha_2\phi_{n-1}+\varphi_{1n}^*)(\omega-\theta))\exp\{0.5(\alpha_2\phi_{n-1}+\varphi_{1n}^*)^2(\omega-\theta)^2\}] \\
= \; & \exp\{-\sqrt{\frac{2}{\pi}}\omega\sum_{i=1}^{n-1}\varphi_i\}\exp\{-\sqrt{\frac{2}{\pi}}\omega\alpha_2\phi_{n-1}\}\exp\{-\sqrt{\frac{2}{\pi}}\omega\sum_{i=1}^{\infty}\varphi_{in}^*\} \\
& \times[\prod_{i=1}^{n-1}(\Phi(\varphi_i(\omega+\theta))\exp\{0.5\varphi_i^2(\omega+\theta)^2\}+\Phi(\varphi_i(\omega-\theta))\exp\{0.5\varphi_i^2(\omega-\theta)^2\})] \\
& \times[\prod_{i=2}^{\infty}(\Phi(\varphi_{in}^*(\omega+\theta))\exp\{0.5\varphi_{in}^{*2}(\omega+\theta)^2\}+\Phi(\varphi_{in}^*(\omega-\theta))\exp\{0.5\varphi_{in}^{*2}(\omega-\theta)^2\}] \\
& \times[\Phi((\alpha_2\phi_{n-1}+\varphi_{1n}^*)(\omega+\theta))\exp\{0.5(\alpha_2\phi_{n-1}+\varphi_{1n}^*)^2(\omega+\theta)^2\} \\
& +\Phi((\alpha_2\phi_{n-1}+\varphi_{1n}^*)(\omega-\theta))\exp\{0.5(\alpha_2\phi_{n-1}+\varphi_{1n}^*)^2(\omega-\theta)^2\}]. \qquad \text{(A.67)}
\end{aligned}
$$

Applying formula (A1.4) in Nelson (1991) gives

$$
\begin{aligned}
\overline{\gamma}_{n2}(e^g,\varphi_n) \;=\; & \sqrt{\frac{2}{\pi}}\exp\{-\sqrt{\frac{2}{\pi}}\omega\varphi_n\}\times[D_{-3}[-\varphi_n(\omega+\theta)]\exp\{0.25\varphi_n^2(\omega+\theta)^2\} \\
& +D_{-3}[-\varphi_n(\omega-\theta)]\exp\{0.25\varphi_n^2(\omega-\theta)^2\}]. \qquad \text{(A.68)}
\end{aligned}
$$

Note that in (A.67),

$$
\begin{aligned}
& \exp\{-\sqrt{\frac{2}{\pi}}\omega\sum_{i=1}^{\infty}\varphi_{in}^*\} \\
= \; & \exp\{-\frac{\sqrt{\frac{2}{\pi}}\omega(\alpha_1+\alpha_2)(\phi_{2n}^*+\phi_{1n}^*-\beta_1\phi_{2n}^*)}{1-(\beta_1+\beta_2)}\} \qquad \text{(A.69)}
\end{aligned}
$$

if and only if $\lambda(\mathbf{\Gamma}) < 1$. Insert (A.69) into (A.67), separate the term of $\exp\{-\frac{2\sqrt{\frac{2}{\pi}}\omega(\alpha_1+\alpha_2)}{1-(\beta_1+\beta_2)}\}$ from the product of the right-hand sides of (A.67) and (A.68), define $\gamma(n)$, $D_n(\cdot)$ and $\Phi_i(x_i)$ as in (28), (29) and (30), respectively. Further simplification yields (27). ■

Proof of Corollary 7. Let $p = 2$, $g(z_t) = \theta z_t + \omega(z_t - \mathsf{E}|z_t|)$ in (2) and $\{z_t\}$ ~GED in (31). Applying Theorem A1.2 of Nelson (1991) to the results in Corollary 4 gives, for any $i \geq 1$,

$$\gamma(e^g, x_i) = S_{2i}(x_i)\exp\{-\Gamma(\lambda, \upsilon)\omega x_i\} \tag{A.70}$$

where $x_i = \varphi_i$ or φ_{in}^* or $2\varphi_i$,

$$S_{2i}(x_i) = \sum_{k=0}^{\infty} (2^{1/\upsilon}\lambda x_i)^k[(\omega+\theta)^k + (\omega-\theta)^k]\frac{\Gamma((k+1)/\upsilon)}{2\Gamma(1/\upsilon)\Gamma(k+1)}$$

and $\Gamma(\lambda, \upsilon) = \lambda 2^{1/\upsilon}\Gamma(2/\upsilon)/\Gamma(1/\upsilon)$. Furthermore, for any $n \geq 1$

$$\overline{\gamma}_{n2}(e^g, \varphi_n) = 2^{2/\upsilon}\lambda^2 S_{1n}(\varphi_n)\exp\{-\omega\Gamma(\lambda, \upsilon)\varphi_n\} \tag{A.71}$$

where

$$S_{1n}(\varphi_n) = \sum_{k=0}^{\infty} (2^{1/\upsilon}\lambda\varphi_n)^k[(\omega+\theta)^k + (\omega-\theta)^k]\frac{\Gamma((k+3)/\upsilon)}{2\Gamma(1/\upsilon)\Gamma(k+1)}.$$

Substitute $\Gamma(\lambda, \upsilon)$, $S_{2i}(x_i)$ and $S_{1n}(\varphi_n)$ for $\sqrt{\frac{2}{\pi}}$, $\Phi_i(x_i)$ and $D_n(\cdot)$ in (27) and separate the term

$$\gamma(n) = \exp\{-\frac{\omega\Gamma(\lambda, \upsilon)}{1-(\beta_1+\beta_2)}[(\phi_n - 1)(\alpha_1+\alpha_2) + (\alpha_2(1-\beta_1) + \alpha_1\beta_2)\phi_{n-1}]\}$$

from the product term $\overline{\gamma}_{n2}(e^g, \varphi_n)M_1(n)M_2(n)$. Some further manipulation gives (32). ■

Table 1. The EGARCH(2,1) model estimated for the CRSP excess return series, July 1962 to December 1987. Source: Nelson (1991) (standard errors in parentheses)

	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\widehat{\alpha}_1$	$\widehat{\theta}$	$\widehat{\omega}$	$\widehat{v}^1$	$\widehat{\lambda}(\widehat{\mathbf{\Gamma}})^2$
Parameter estimate	1.92938 (0.0145)	−0.92941 (0.0145)	−0.9782 (0.0062)	−0.1178 (0.0090)	0.1559 (0.0125)	1.5763 (0.0320)	0.99957
Reduced Parameter estimate	1.92925	−0.92941	−0.9782	−0.1178	0.1559	1.5763	0.9976.

$\widehat{v}^1$–the estimated tail-thickness parameter of the GED(v) distribution

$\widehat{\lambda}(\widehat{\mathbf{\Gamma}})^2$–the maximum absolute eigenvalue from the estimated $\widehat{\mathbf{\Gamma}}$. $\lambda(\mathbf{\Gamma}) < 1$ implies that the autocorrelations for the estimated model exists.

Table 2. Autocorrelation functions of $\{\varepsilon_t^2\}$ and kurtosis of $\{\varepsilon_t\}$ are computed from the reduced estimated model

Lag									
1	2	3	4	5	10	20	30	40	50
0.2597	0.2411	0.2232	0.3634	0.2001	0.0876	0.0130	0.0016	1.9×10^{-4}	2.3×10^{-5}
Kurtosis									
8.2×10^{80}									

Figure 1. The three types of autocorrelation functions of squared observations in Eq. (24) generated by an EGARCH(2,1) process.

(a) $\lambda(\mathbf{\Gamma}) = 0.99993$ $(\beta_1 = 0.5, \beta_2 = 0.4999, \alpha_1 = -0.1, \theta = 0.1)$

(b) $\lambda(\mathbf{\Gamma}) = -0.9659$ $(\beta_1 = -0.5, \beta_2 = 0.45, \alpha_1 = -0.9, \theta = 0.1)$

(c) $\lambda_{1,2} = 0.8 \pm 0.557i$ $(\beta_1 = 1.6, \beta_2 = -0.95, \alpha_1 = -0.9, \theta = 0.1)$