Testing for common cointegrating rank in dynamic panels

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Abstract: The panel cointegration test of Larsson et al (1998) tests for the maximum number of cointegrating relations in a dynamic panel given the assumption of a common cointegrating rank. This paper presents a test for this assumption. The test is based on the test statistic of Larsson et al (1998) and a new panel test based on the principal component estimator of cointegrating relations of Harris (1997). The asymptotic distribution is derived and shown to be standard normal. An extensive Monte Carlo simulation shows that the test has good small sample size and power properties. In the consumption function example in Larsson et al (1998) the assumption of common cointegrating rank amongst 23 OECD countries is shown to hold.
Key Words: Cointegration; Consumption; Panel data; Rank test.
JEL-Classification: C12; C13; C15; C22; C23; D12.

1. Introduction

It is sometimes plausible to belive that economic theory postulates that e.g. different countries, should have some common features. Common features could be the number of common trends or that the cointegrating relations should lie in the same space. E.g. King et al (1991) gives a real business cycle model with output, consumption and investment where logs of consumption and investment should cointegrate with cointegrating vector (1,-1). The logs of output and investment should also cointegrate, having the cointegrating vector (1,-1). By testing, simultaneously, the implications of the theory on more than one country increased power is gained, or if the interest is in estimation, more precise estimates. This is the main reason for considering panels instead of the non-panel setting. One have to, of course, belive in the assumption that all the countries share some properties. When considering testing for the cointegrating rank a common miss-use of language is to say that one tests for common cointegrating rank when one is actually testing for the maximum cointegrating rank among the individual ones. The purpose in this paper is to present a test sequence that actually tests if the number of cointegrating relations are common or not.

The test sequence is as follows. First use a test that test for maximum cointegrating rank, e.g. the LR-bar statistic of Larsson, Lyhagen and Löthgren (1998). As this test is based on the sum of the individual test statistics, the LR-bar statistics would diverge to infinity with the sample size, if a cointegrating rank less than the maximum of the individual ones is tested for. Hence, the test tests for the maximum cointegrating rank. Given the maximum rank, the second stage is to test against one cointegrating relation less. For this purpose, a panel test is derived which tests the hypothesis of r cointegrating vectors against $r-1$. The test is the standardised sum of the test statistic proposed by Harris (1997). It is shown that, asymptotically, the test is standard normal distributed. Further, it has the property that it tests for the smallest rank, i.e. testing a hypothesis of larger rank than the smallest would, asymptotically, be rejected. If the second test does not reject the hypothesis of r cointegrating relations, then the test sequence indicates a common cointegrating rank.

The paper is as follows. The next section presents some commonly used panel cointegration tests while the test sequence for test of common cointegrating rank is in the third section. A Monte Carlo simulation is performed in the fourth section to evaluate some small sample properties. A conclusion ends the paper.

2. Testing for the cointegrating rank in panels

There are a number of panel unit root/cointegration tests available, e.g. Levin and Lin (1992, 1993), Pedroni (1995), Im, Pesaran and Shin (1997), Kao (1999) for the univariate setting and Larsson, Lyhagen and Löthgren (1998), Groen and Kleibergen (1999) and Larsson and Lyhagen (1999) for multivariate ones. Most tests are sum of the individual ones, or as in the case of Groen and Kleibergen (1999) the asymptotic distribution is the sum of the asymptotic distribution of the individual ones although the test statistic is not.

The data generating process of the p dimensional time series Y_{it} for group i out of N is

$$Y_{it} = \sum_{k=1}^{k_i} \Pi_{ik} Y_{i,t-k} + \varepsilon_{it} \tag{2.1}$$

where $Y_{i,-k_i+1}, ..., Y_{i0}$ are considered fixed and the errors ε_{it} are independently distributed as $N_p(0, \Omega_i)$. Due to the Granger representation theorem, an error correction representaion exists, see e.g. Johansen (1995):

$$\Delta Y_{it} = \Pi_i Y_{it-1} + \sum_{k=1}^{k_i-1} \Gamma_{ik} \Delta Y_{i,t-k} + \varepsilon_{it} \tag{2.2}$$

It is possible to decompose Π_i into $\Pi_i = \alpha_i \beta_i'$ where α_i and β_i are matrices of order $p \times r$ where r is the rank of Π_i. The main interest is to decide upon r.

For technical reasons, we will in this work only treat the simpler case where all Π matrices are equal, i.e. $\Pi_i = \Pi = \alpha\beta'$ for all i. Because for any invertible $r \times r$ matrix M, $\alpha_i \beta_i' = \widetilde{\alpha}_i \widetilde{\beta}_i'$, where $\widetilde{\alpha}_i \equiv \alpha_i M$ and $\widetilde{\beta}_i \equiv \beta_i M^{-1}$, this restriction in effect means that all α_i span the same space, and similarly for β_i. However, observe that the restriction is only motivated for the sake of formulating the central limit theorems 2.2 and 2.7 below. The estimation procedure will work without imposing all Π_i equal.

2.1. The LR-bar test

Consider the trace test of Johansen (1988)

$$LR_T(H(r)|H(p)) = -2 \ln Q_{iT}(H(r)|H(p))$$

which is asymptotically distributed as

$$Z_k \equiv tr\left\{ \int_0^1 (dW) W' \left[\int_0^1 WW' \right]^{-1} \int_0^1 W (dW)' \right\}, \tag{2.3}$$

where W is a $k = (p - r)$ dimensional Brownian motion. If the true rank is larger than the tested one the test has asymptotic power one, i.e. the test statistic converges to infinity.

Define the LR-bar statistic as the average of the N individual trace statistics $LR_{iT}\left(H\left(r\right)|H\left(p\right)\right)$ as

$$\overline{LR}_{NT}\left(H\left(r\right)|H\left(p\right)\right) = \frac{1}{N}\sum_{i=1}^{N} LR_{iT}\left(H\left(r\right)|H\left(p\right)\right), \tag{2.4}$$

The standardised LR-bar statistic for the panel cointegration rank test is

$$\Upsilon_{\overline{LR}}\left(H\left(r\right)|H\left(p\right)\right) = \frac{\sqrt{N}\left(\overline{LR}_{NT}\left(H\left(r\right)|H\left(p\right)\right) - E\left(Z_k\right)\right)}{\sqrt{Var\left(Z_k\right)}}. \tag{2.5}$$

where $E\left(Z_k\right)$ and $Var\left(Z_k\right)$ is the mean and variance of the asymptotic trace statistic defined by (2.3). To derive the asymptotic distribution we need some assumptions.

Assumption 1: The data generating process is assumed covariance stationary for each group, $i = 1, ..., N$.

Assumption 2: The disturbances are independent between groups.

We also need the following lemma. Observe that we do not prove it here, so it is merely a conjecture. However, proofs of corresponding results in the unit root case exist, cf Larsson (1998a,b). Moreover, an intermediate result, relating the VAR process of general order to a VAR process of order one, is given in Larsson (1999).

Lemma 2.1. *For some* $\kappa > 0$, $E\left(LR_{iT}\left(H\left(r\right)|H\left(p\right)\right)\right) = E\left(Z_k\right) + O\left(T^{-\kappa}\right)$ *and* $Var\left(LR_{iT}\left(H\left(r\right)|H\left(p\right)\right)\right) = Var\left(Z_k\right) + O\left(T^{-\kappa}\right)$.

The asymptotic distribution of the LR-bar statistic is given in the theorem below. For the result of the theorem to hold, we need the following "uniformity" assumption. This assumption basically ensures us that not too many of the Y_{it} series in the panel are too close to $I\left(2\right)$. (Cf the conditions for the Granger representation theorem, theorem 4.2 of Johansen (1995).) The matrix $\alpha_\perp$ is defined as a $p \times (p - r)$matrix such that $\alpha'_\perp \alpha = 0$ and $(\alpha, \alpha_\perp)$ is of full rank, and for each i, we put $\Gamma_i \equiv I_p - \sum_{i=1}^{k_i - 1} \Gamma_{ik}$. We use the matrix norm $\|A\|^2 = \text{tr}\left(A'A\right)$.

Assumption 3: $\sup_i \left\|\left(\alpha'_\perp \Gamma_i \beta_\perp\right)^{-1}\right\| < \infty$.

Theorem 2.2. *For some $\kappa > 0$, under Assumptions 1, 2 and 3 and the null of r cointegrating vectors, the standardised LR-bar statistic, $\Upsilon_{\overline{LR}}(H(r)H(p)) \xrightarrow{d} N(0,1)$ as T and $N \to \infty$ in such a way that $\sqrt{N}T^{-\kappa} \to 0$.*

Proof. See appendix 2. ■

Note that $T \to \infty$ is needed for each of the individual test statistics to converge to their asymptotic distribution while $N \to \infty$ is needed for the central limit theorem. From lemma 2.1 we see that $\sqrt{N}T^{-\kappa} \to 0$ makes the error of using the asymptotic mean instead of the small sample to vanish as T and $N \to \infty$.

Moreover, note that this is a one tailed test.

2.2. The PC-bar test

Harris (1997) proposes a test for testing the hypothsis of r cointegraing vectors against $r-1$. Define the three new variables based on the p dimensional time series y_t,

$$\begin{aligned} \beta' y_t &= z_t && (2.6) \\ \beta'_{\perp}\Delta y_t &= w_t && (2.7) \\ \zeta_t &= \begin{bmatrix} z_t \\ w_t \end{bmatrix} && (2.8) \end{aligned}$$

and the moment matrix

$$S_{yy} = T^{-1}\sum_{t=1}^{T} y_t y_t' \tag{2.9}$$

The principal component estimator of the cointegrating vectors are the eigenvectors corresponding to the r smallest eigenvalues of

$$|\lambda I_p - S_{yy}| = 0 \tag{2.10}$$

A convenient normalisation is $\hat{\beta}'\hat{\beta} = I_r$. An estimator of $\beta_{\perp}$ is the remaining $p-r$ eigenvectors. As the distribution of $\hat{\beta}$ is affected by nuisance parameters, a new variable, y_t^*, is defined according to

$$y_t^* = y_t - \hat{\beta}\left(\hat{\beta}'\hat{\beta}\right)^{-1}\hat{\Omega}_{zw}\hat{\Omega}_{ww}^{-1}\hat{w}_t - \hat{\beta}_{\perp}\left(\hat{\beta}'_{\perp}\hat{\beta}_{\perp}\right)^{-1}\hat{\Delta}_{w\zeta}\hat{S}_{\zeta\zeta}^{-1}\hat{\zeta}_t$$

where

$$\hat{\Omega}_{ab} = \sum_{j=-T+1}^{T-1} k\left(\frac{j}{m}\right)\hat{\Gamma}_{ab}(j) \tag{2.11}$$

$$\hat{\Delta}_{ab} = \sum_{j=0}^{T-1} k\left(\frac{j}{m}\right)\hat{\Gamma}_{ab}(j) \tag{2.12}$$

$$\hat{\Gamma}_{ab}(j) = T^{-1}\sum_{t=1}^{T} a_{t-j}b_t \tag{2.13}$$

a and b are any time series, $k(.)$ is a lag window and m is the band width parameter. Using y_t^* instead of y_t the distibution of $\hat{\beta}$ is free of nuisance parameters. Defining

$$\hat{S}_t = \sum_{j=1}^{t} z_j^* \tag{2.14}$$

the test statistic is

$$\hat{c} = T^{-2}\sum_{t=1}^{T} \hat{S}_t'\hat{\Omega}_{zz}^{*-1}\hat{S}_t \tag{2.15}$$

The distribution, under the null of r cointegrating vectors, is

$$\hat{c}_r \xrightarrow{d} X = \int_0^1 V(s)' V(s)\, ds \tag{2.16}$$

where, using the r and the $p-r$ dimensional standard Brownian motion W_1 and W_2 respectively,

$$V(s) = W_1(s) - \int dW_1 W_2' \left(\int W_2 W_2'\right)^{-1} \int_0^s W_2(r)\, dr \tag{2.17}$$

In analogy to the standardised LR-bar statistic the standardised PC-bar statistic is

$$PC_r = \frac{\sqrt{N}\left(\bar{c}_k(r) - E(X_k)\right)}{\sqrt{Var(X_k)}} \tag{2.18}$$

where $\bar{c}_k(r)$ is the mean of the individual test statistics and $E(X_k)$ and $Var(X_k)$ are the expectations and the variance respectively of the variable X that has the same distribution as the asymptotic distribution of $\hat{c}_r$. Further, a couple of lemmas are needed.

Lemma 2.3. *Conditional on W_2, the $V(s)$ process is a normal process with expectation 0. Furthermore, for $s \leq t$ the covariance function is, denoting expectation w.r.t. W_2 by E_2,*

$$E_2\left\{V(s)V(t)'\right\} = \rho(s,t) I_r$$

where

$$\rho(s,t) \equiv s - A_{p-r}(s,t)$$

with

$$A_{p-r}(s,t) \equiv E_2\left\{\int_0^s W_2(v)' dv \left(\int W_2 W_2'\right)^{-1} \int_0^t W_2(u) du\right\}.$$

Moreover, $0 < E\{\rho(s,s)\} \leq s$ for all s.

Proof. See appendix 2. ■

As a consequence of this lemma, the $V(s)$ process is a normal process also unconditionally, with expectation 0 and finite covariance function. Hence, it is not difficult to see that all moments of X exist. The following lemma is useful for evaluating the first two central moments.

Lemma 2.4. *For $p \geq r$,*

$$E(X) = r\left(\frac{1}{2} - b_{p-r}\right), \tag{2.19}$$

$$Var(X) = r^2 d_{p-r,2} + r d_{p-r,1}, \tag{2.20}$$

where $b_0 = 0$, $d_{01} = 1/3$, $d_{02} = 0$ and for $p > r$,

$$\begin{aligned}
d_{p-r,1} &\equiv \frac{1}{3} - 8\int_0^1 s \int_s^1 a_{p-r}(s,t)\, dtds + 4\int_0^1 \int_s^1 E\left\{A_{p-r}(s,t)^2\right\}, \\
d_{p-r,2} &\equiv 2\int_0^1 \int_s^1 E\left\{A_{p-r}(s,s) A_{p-r}(t,t)\right\} dtds - b_{p-r}^2, \\
b_{p-r} &\equiv \int_0^1 a_{p-r}(s)\, ds, \\
a_{p-r}(s,t) &\equiv E\left\{A_{p-r}(s,t)\right\}.
\end{aligned}$$

Moreover,

$$b_{p-r} = 2E\left\{\int_0^1 (1-v) W_2(v)' \left(\int W_2 W_2'\right)^{-1} \int_0^t W_2(u)\, dudv\right\}. \tag{2.21}$$

Proof. See appendix 2. ■

Observe that the lemma immedeately yields $E(X) = r/2$ and $Var(X) = r/3$ if $p - r = 0$. In the case $p - r = 1$, it is possible to find numerical results (see the lemma below). When $p - r > 1$, it seems hard to the moments numerically by other means than simulation. Note that, as in Chan and Wei (1988) it may be seen that, defining $S_t \equiv \sum_{i=1}^{t} \varepsilon_i$, where $\{\varepsilon_t\}$ is a sequence of $(p-r)$-variate standard normals, the r.h.s. of (2.21) is the limit of

$$2T^{-2}E\left[\sum_{i=1}^{T}\left\{(T-i)\,S_{i-1}'\left(\sum_{j=1}^{T}S_{t-1}S_{t-1}'\right)^{-1}\sum_{k=1}^{i}S_{k-1}\right\}\right],$$

as $T \to \infty$. Clearly, using this fact, b_{p-r} may be found by simulation. Alternatively, $E(X)$ may be simulated directly from the definitions. Using the latter approach, because of the structure of the results in the previous lemma only the combinations $(r,p) = (1, 1+n)$, $n = 2, \ldots$ need to be simulated. This is true also for $Var(X)$, where a formula in the same style as (2.21) *could* be found, but this one would be so complicated that we strongly recommend the direct approach in this case.

In the special case $p - r = 1$, we have found the following result:

Lemma 2.5. *The constant b_1 is given by*

$$b_1 = \int_0^\infty \frac{1}{x(\cosh x)^{3/2}}\left\{\frac{1}{2}\cosh x - \frac{3}{2}\frac{\sinh x}{x} + \frac{2}{x^2}(\cosh x - 1)\right\}dx.$$

Moreover,

$$Var(X) = d_{12}r^2 + d_{11}r,$$

where

$$\begin{aligned} d_{12} &\equiv \int_0^\infty x^3(\cosh x)^{-1/2}k_1(x)\,dx - b_1^2, \\ d_{11} &\equiv \frac{1}{3} + 2\int_0^\infty x(\cosh x)^{-1/2}k_2(x)\,dx, \end{aligned}$$

with b_1 as above,

$$\begin{aligned} & k_1(x) \\ = \; & \frac{1}{48x^8\cosh^2 x}\left\{48 - 111x^2 + 7x^4 - 48\left(-4+3x^2\right)\cosh x + 432x\sinh x\right. \\ & \left.(-240 + 51x^2 + 7x^4)\cosh(2x) + 2x\left(39 - 17x^2\right)\sinh(2x)\right\} \end{aligned}$$

and

$$
\begin{aligned}
& k_2(x) \\
= \; & \frac{1}{48x^6 \cosh^2 x}\Big\{240 - 87x^2 - x^4 - 48\left(4+x^2\right)\cosh x + 432x \sinh x \\
& - \left(48 - 75x^2 + x^4\right)\cosh(2x) - 18x\left(5+x^2\right)\sinh(2x)\Big\}.
\end{aligned}
$$

Proof. See appendix 2. ■

From the results of this lemma, we numerically find (using Mathematica)

$$
\begin{aligned}
b_1 &\approx 0.14183 \\
d_{12} &\approx 0.0071266, \\
d_{11} &\approx 0.18048.
\end{aligned}
$$

Consequently, we have in the special case $p - r = 1$

$$
\begin{aligned}
E(X) &= r\left(\frac{1}{2} - b_1\right) \approx 0.35817r, \\
Var(X) &= 0.0071266r^2 + 0.18048r.
\end{aligned}
$$

The mean and variance of X_k up to $p = 6$ are displayed in tables (8.2) and (8.3).

Tables (8.2) and (8.3) in here

The next lemma is needed for our central limit theorem below. Strictly speaking, as was the case with lemma 2.1, it is merely a conjecture.

Lemma 2.6. *For some* $\kappa > 0$, $E(\bar{c}_k(r)) = E(X_k) + O(T^{-\kappa})$ *and* $Var(\bar{c}_k(r)) = Var(X_k) + O(T^{-\kappa})$.

The main result in this section is the following theorem. To formulate it, we need

Assumption 4: $\sup_i \|\alpha' \Gamma_i \beta_\perp\| < \infty$ and $\sup_i \|\alpha'_\perp \Gamma_i \beta_\perp\| < \infty$.

Theorem 2.7. *For some* $\kappa > 0$, *under Assumptions 1, 2 and 4 and the null of* r *cointegrating vectors, the standardised PC-bar statistic* $PC_r \xrightarrow{d} N(0,1)$ *as* T *and* $N \to \infty$ *in such a way that* $\sqrt{N}T^{-\kappa} \to 0$.

Proof. See appendix 2. ■

The first assumption implies that variables integrated of an order of most one are considered. The second assumption is a standard regression assumption that may be relaxed. Lemma 2.3 guarantees the existence of the first two moments which is needed for the central limit theorem to apply. Note the convergence conditions on N and T, which are the same as those appearing in the CLT for the LR-bar statistic.

3. Testing for common cointegrating rank

The testing procedure for test of common cointegrating rank is proposed to be carried out as follows. Test for the maximum rank with the test proposed by Larsson, Lyhagen and Löthgren (1998). Given the maximum rank test the hypothesis of r cointegrating relations against $r-1$, with the panel test proposed above.

The standardised LR-bar statistic has the property that it estimates the maximum rank amongst the N individual ones. This is easily seen when considering $N-1$ groups with rank r^* and one group with rank r^*+1. Testing the hypothesis of $r=r^*$ gives $N-1$ well behaved test statistics in the sum, but the last one tends to infinity with sample size, hence, the mean and so also the standardised test statistic tends to infinity. If the null of $r=r^*$ cointegrating vectors is true then the asymptotic size is, say, α^1. The same analysis applies to the standardised PC-bar statistic. If (at least) one of the individual ranks is less than the hypothesed, the test asymptotically rejects the null. Under the null of common cointegrating rank this test has an asymptotic size of α^2. From this it is clear that the test sequence of first using the LR-bar statistic and then the standardised PC-bar statistic gives an asymptotic power of one.

To be able to discuss the asymptotic size of our procedure, we will need the following lemma:

Lemma 3.1. *Under the null of r cointegrating vectors, and under assumptions 1 and 2, the LR-bar and PC-bar statistics are asymptotically independent.*

Proof. See appendix 2. ■

Now, if the null of common cointegrating rank is true the LR-bar statistic would, asymptotically, reject the null a fraction α^1 of the times. In all these instances, the standardised PC-bar would, asymptotically, give a rejection. In the remaining $(1-\alpha^1)$ fraction of times, the PC-bar would asymptotically give rejection in a fraction α^2 of times. Hence, because of lemma 3.1, the asymptotic size is

$$\alpha = \alpha^1 + \left(1 - \alpha^1\right)\alpha^2. \tag{3.1}$$

If one sets $\alpha^1 = \alpha^2 = \alpha^*$,

$$\alpha = 2\alpha^* - \alpha^{*2}$$

and solving for α^* gives

$$\alpha^* = \frac{2 - \sqrt{4 - 4\alpha}}{2}$$

i.e. if an overall asymptotic size of 5% is desired, each test should have an asymptotic size of 2.532%. Note that for small α, a Taylor expansion of the square root gives us the approximation $\alpha^* \approx \alpha/2$.

4. Monte Carlo simulations

To evaluate the test for common cointegrating rank in terms of size and power, we have conducted a small Monte Carlo simulation. The number of replicates is 10000 which gives sufficiently accurate results. Sample sizes considered are $T = 50$, 100 and 200 and $N = 2$, 5, 10 and 25. The data generating process is

$$\begin{aligned} y_t &= \beta_1 y_{t-1} + \beta_2 x_t + e_{1t} && (4.1) \\ x_t &= x_{t-1} + e_{1t} && (4.2) \end{aligned}$$

which may be reformulated in ECM form as

$$\Delta\begin{pmatrix} y \\ x \end{pmatrix}_t = \begin{pmatrix} \beta_1 - 1 & \beta_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y \\ x \end{pmatrix}_{t-1} + \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}_t.$$

When $\beta_1 = 1$ and $\beta_2 = 0$ a system of zero rank is generated while $\beta_1 = 0$ and $\beta_2 = 1$ gives rank 1. For the simulations of the size all cross sections have rank 1. The power simulations involves N_0 cross sections of rank zero and N_1 of rank 1 where $N = N_0 + N_1$. For $N = 2$, 5 and 10 all combinations of N_0 and N_1 are considered while for $N = 25$ the values are $N_0 = 1, ..., 10, 15, 20, 24$.

The results for the size is in Table 8.4 and shows that for smaller values of N the size is slightly above the nominal 5% level while the oposite is true for larger values of N. For given N there is no clear effect on size when increasing T. The effect of increasing N raises the power for a given ratio of N_0/N, see Tables 8.5-8.6. Increasing N_0 for fixed N increases the power until a point whereafter it decreases. The maximum power is for the larger values of the ratio N_0/N and when T increases maximum power is gained by even larger values of N_0/N. Increasing T always increases power. Overall, the testing procedure has satisfactory size and power properties.

5. The Consumption function revisited

Larsson et al (1998) investigated the type of consumption function considered by Davidson et al (1978) for a sample of 23 OECD countries by the use of the LR-bar statistic. Data is quarterly and ranging from 1960 to 1994. The variables are the logarithm of consumption per capita, c_{it}, logarithm of the real disposable income per capita, y_{it}, and the rate of inflation, Δp_{it}, is measured by the change of the logarithm consumption deflator. The definitions follow from Pesaran et al (1999). See Larsson et al (1998) for further details of the analysis conducted. The variable vector for country i is

$$Y_{it} = (c_{it}, y_{it}, \Delta p_{it})'$$

In Table 8.8 the LR-bar and the PC-bar test statistics are shown together with the individuial statistics which they are based on. Although most individual test for the rank gives a values of 1, the panel test gives rank 2 as a result which might be due to that the assumption of common rank is invalid. Using the test procedure for common cointegrating rank a test statistic of 0.41 is gained for the PC-bar statistic. Comparing that with the critical value of 1.955 when testing on the 5% level, the hypothesis of common cointegrating rank is not rejected. The individual PC tests support this conclusion as if a cointegrating rank of 2 is assumed, only one (Turkey) out the 23 countries used in the study rejects the null of rank 2. The result in this section implies that the analysis in Larsson et al (1998) is based on the valid assumption of common cointegrating rank amongst the 23 OECD countries studied.

6. Conclusions

The panel cointegration test of Larsson et al (1998) tests for the number of cointegrating relations in a dynamic panel. The test is based on the key assumption of a common cointegrating rank. This paper constructs a test for this assumption-which combines two test statistics, the LR-bar of Larsson et al (1998), and a new panel test, PC-bar, based on the principal component analysis of cointegrating relations of Harris (1997). The asymptotic distribution is derived and shown to be standard normal when the standardized with the asymptotic mean and standard deviation. Expressions for the asymptotic mean and variance of the Harris (1997) test statistic are derived and are found only to depend on the number of variables and the number of cointegrating relations. The test sequence is as follows: 1) use the LR-bar to estimate the number of cointegrating relations and 2) use the panel principal component test to test if the number of cointegrating relations is less than the maximum. If the numbers of cointegrating relations are not the same, it is noted that the LR-bar test gives the maximum number. Similarly, the PC-bar test gives the minimum number. Hence, if the two tests coincide, the null of the same number of cointegrating relations cannot be rejected. The significance level is easily derived as it is proven that the two tests are asymptotic independent. An extensive Monte Carlo simulation shows that the test has good finite sample size and power properties even for small sample sizes such as $T = 50$ and $N = 2$. In the last section the consumption function example in Larsson et al (1998) the assumption of common cointegrating rank amongst 23 OECD countries are shown to hold.

7. References

Billingsley, P. (1968) *Convergence of probability measures,* New York: Wiley.

N. H. Chan and Wei C. Z. (1988) Limiting distributions of least squares estimates of unstable autoregressive processes, *The Annals of Statistics* 16 , 367-401.

Davidson, J.E.H., D.F. Hendry, F. Srba and S. Yeo, (1978) Econometric modelling of the aggregate time-series relationships between consumers' expenditure and income in the United Kingdom, *Economic Journal* 88, 661-692.

Groen, J.J.J. and Kleibergen, F.R. (1999) Likelihood-based cointegration analysis in panels of vector error correction models, Tinbergen Institute, Discussion Paper TI 99-055/4.

Harris, D. (1997) Principal component analysis of cointegrated time series, *Econometric Theory* 13, 529-557.

Im, K.S., Pesaran M.H. and Shin, Y. 1997, Testing for unit roots in heterogeneous panels Trinity College, mimeo.

Johansen, S., (1988) Statistical analysis of cointegrating vectors, *Journal of Economic Dynamics and Control* 12, 231-254.

Johansen, S. (1995) *Likelihood-based Inference in Cointegrated Vector Autoregressive Models,* Oxford: Oxford University Press.

Kao (1999) Spurious regression and residual-based tests for cointegration in panel data, *Journal of Econometrics* 90, 1-44.

King, R.G., Plosser, J.H., Stock, J.H. and Watson, M.W. (1991) Stochastic trends and economics fluctuations, *The American Economic Review* 81, 819-840.

Larsson, R (1998a) The order of the error term for moments of the log likelihood ratio unit root test in an autoregressive process, *Annals of the Institute of Statistical Mathematics* 50, 29-48.

Larsson, R. (1998b) Bartlett corrections for unit root test statistics, *Journal of Time Series Analysis* 19, 425-438.

Larsson, R (1999) The order of the asymptotic error term for moments of the log likelihood ratio test for cointegration. Unpublished manuscript.

Larsson, R. and Lyhagen, J. (1999) Likelihood-Based Inference in Multivariate Panel Cointegration Models, Working Paper Series in Economics and Finance No. 331, Stockholm School of Economics.

Larsson, R., Lyhagen, J. and Löthgren, M. (1998) Likelihood-Based Cointegration Tests in Heterogeneous Panels, Working Paper Series in Economics and Finance No. 250, Stockholm School of Economics.

Levin, A. and Lin, C.-F. (1992) Unit root test in panel data: Asymptotics and finit-sample properties, Discussion Paper 92-23, University of California at San Diego.

Levin, A. and Lin, C.-F. (1993) Unit root test in panel data: New results, Discussion Paper 94-56, University of California at San Diego.

Magnus, J.R. (1978) The moments of products of quadratic forms in normal variables. *Statistica Nederlandica* 32, 201-210.

Magnus, J.R. and H. Neudecker (1988) *Matrix Differential Calculus with Applications in Statistics and Economics,* Wiley.

Pedroni (1995) Panel cointegraton; Asymptotic and finite sample properties of pooled time series tests with an application to the PPP Hypothesis, Working paper in Economics 95-013, Indiana University.

Pesaran, M.H., Shin, Y. and Smith, R.P. (1999) Pooled mean group estimation of dynamic heterogenous panels. *Journal of the American Statistical Association* 94, 621-634.

Phillips, P.C.B. and Moon, H.R. (1999) Linear regression limit theory for nonstationary panel data, *Econometrica* 67, 1057-1112.

8. Appendix 1: Numerical values

$p-r$	1	2	3	4	5	6
b_{p-r}	0.142	0.234	0.294	0.333	0.361	0.382
$d_{p-r,1}$	0.180	0.0991	0.0575	0.0355	0.0231	0.0159
$d_{p-r,2}$	0.00713	0.00628	0.00434	0.00286	0.00190	0.00136

Table 8.1: b_{p-r} and c_{p-r}

	p					
r	1	2	3	4	5	6
1	0.500	0.358	0.266	0.206	0.167	0.139
2		1.000	0.716	0.532	0.412	0.333
3			1.500	1.075	0.798	0.619
4				2.000	1.433	1.064
5					2.500	1.791
6						3.000

Table 8.2: Mean of X_k

	p					
r	1	2	3	4	5	6
1	0.333	0.188	0.105	0.0619	0.0384	0.0250
2		0.666	0.389	0.223	0.132	0.0825
3			1.000	0.606	0.354	0.212
4				1.333	0.836	0.497
5					1.666	1.081
6						2.000

Table 8.3: Variance of X_k

	N			
T	2	5	10	25
50	0.0645	0.0559	0.0502	0.0441
100	0.0720	0.0575	0.0543	0.0432
200	0.0689	0.0617	0.0480	0.0428

Table 8.4: Size of the test for common cointegrating rank when true rank is one.

	N	2	5			
	N_{r_0}	1	1	2	3	4
	N_{r_1}	1	4	3	2	1
T	50	0.321	0.284	0.468	0.620	0.706
	100	0.479	0.413	0.661	0.818	0.910
	200	0.636	0.557	0.827	0.942	0.982

Table 8.5: Power of the test for common cointegrating rank, N=2 and 5.

	N					10				
	N_{r_0}	1	2	3	4	5	6	7	8	9
	N_{r_1}	9	8	7	6	5	4	3	2	1
T	50	0.265	0.437	0.569	0.680	0.765	0.827	0.875	0.914	0.789
	100	0.351	0.578	0.741	0.853	0.920	0.960	0.980	0.990	0.996
	200	0.477	0.757	0.899	0.964	0.988	0.995	0.998	1.000	1.000

Table 8.6: Power of the test for common cointegrating rank, N=10.

	N_{r_0}	1	2	3	4	5	6	7
	N_{r_1}	24	23	22	21	20	19	18
T	50	0.230	0.389	0.519	0.625	0.711	0.782	0.832
	100	0.299	0.499	0.655	0.770	0.852	0.906	0.945
	200	0.378	0.647	0.812	0.908	0.961	0.983	0.994
	N_{r_0}	8	9	10	15	20	22	24
	N_{r_1}	17	16	15	10	5	3	1
T	50	0.871	0.905	0.929	0.985	0.997	0.998	0.574
	100	0.969	0.982	0.990	1.000	1.000	1.000	0.977
	200	0.998	0.999	1.000	1.000	1.000	1.000	1.000

Table 8.7: Power of the test for common cointegrating rank, N=25.

Country	Lag (k_i)	$LR_{iT}\left(H\left(r\right)\vert H\left(3\right)\right)$ $r=0$	$r=1$	$r=2$	Rank, r_i	$PC^i_{r=2}$
Australia	1	79.16	12.87	1.16	2	0.145
Austria	1	86.02	21.26	4.99	3	0.210
Belgium	1	70.39	11.86	2.37	1	0.191
Canada	1	59.67	11.86	0.42	1	0.272
Denmark	1	24.88	8.05	2.21	1	0.0859
Finland	1	65.67	6.38	0.61	1	0.826
France	1	103.73	18.91	0.94	2	0.203
Greece	1	88.68	16.71	6.94	3	0.461
Iceland	2	28.64	8.46	1.83	1	0.432
Ireland	1	38.25	6.21	1.21	1	0.854
Italy	1	60.55	4.46	0.35	1	0.340
Japan	1	109.33	13.84	0.15	2	1.605
Luxembourg	1	51.76	9.29	2.45	1	0.845
Netherland	2	25.64	8.01	0.21	1	0.455
New Zealand	1	37.18	4.97	1.56	1	0.377
Norway	1	53.24	9.17	2.66	1	0.679
Portugal	1	60.15	14.51	1.52	2	0.548
Spain	1	60.13	4.86	0.59	1	0.692
Sweden	1	51.50	6.50	0.88	1	0.528
Switzerland	2	26.01	4.98	1.15	1	0.476
Turkey	1	39.25	7.67	0.59	1	6.248
U.K.	1	33.33	3.89	0.70	1	0.877
U.S.	1	68.15	10.40	0.52	1	0.343
Panel tests		$r=0$	$r=1$	$r=2$		
$\Upsilon_{\overline{LR}}\left(H\left(r\right)\vert H\left(3\right)\right)$		40.98	5.47	1.38		
$PC_{r=2}$						0.408

All tests are performed on the 5% level. For the country-by country LR tests the critical values are 24.08, 12.21, and 4.14 for testing r = 0, 1, and 2 respectively. The individual PC tests critical value is 1.95 if a rank of 2 is assumed. The panel tests has the same critical value of 1.955 which yields an averall significance level of 5% when testing for common cointegrating rank.

Table 8.8: Empirical result of the trace test and the PC test.

9. Appendix 2: Omitted proofs

Proof of theorem 2.2: For all i, define the lag polynomials

$$A_i(L) = (1-L) I_p - \Pi L - \sum_{i=1}^{k_i-1} \Gamma_{ik} (1-L) L^i,$$

so that (2.2) is equivalent to $A_i(L) Y_{it} = 0$. Furthermore, let

$$\Gamma_i \equiv I_p - \sum_{i=1}^{k_i-1} \Gamma_{ik}.$$

Moreover, for a Π matrix of rank $r < p$, we may write $\Pi = \alpha\beta'$ where α and β are $p \times r$ and of full rank, and define $\alpha_\perp$ as a $p \times (p-r)$ matrix such that $\alpha_\perp \alpha = 0$ and $(\alpha, \alpha_\perp)$ has full rank p, and similarly for $\beta_\perp$. We then have the following lemma:

Lemma 9.1. *If* $\det\{A_i(L)\} = 0$ *implies* $|L| > 1$ *or* $L = 1$ *and if* $\alpha'_\perp \Gamma_i \beta_\perp$ *has full rank, then we have the representation*

$$LR_{iT}(H(r)|H(p)) = Z_{0iT} + T^{-1/2} Z_{1iT} + o_P\left(T^{-1/2}\right),$$

as $T \to \infty$*, where* Z_{0T} *and* Z_{1iT} *are* $O_P(1)$*, and where all* Z_{0iT} *are iid. Indeed,*

$$Z_{0iT} \stackrel{d}{=} \operatorname{tr}\left\{ \left(\sum_{t=1}^{T} V_{t-1} V'_{t-1}\right)^{-1} \sum_{t=1}^{T} V_{t-1} u'_t \left(\sum_{t=1}^{T} u_t u'_t\right)^{-1} \sum_{t=1}^{T} u_t V'_{t-1} \right\}, \qquad (9.1)$$

where $\{u_t\}$ *is a sequence of independent* $(p-r)$*-variate normal random variables with expectation zero and unit covariance matrix, and where* $V_t \equiv \sum_{i=1}^{t} u_t$*. Moreover,*

$$Z_{1iT} = \operatorname{tr}(C_i X_{iT} + Y_{iT}),$$

where

$$C_i \equiv \beta_\perp \left(\alpha'_\perp \Gamma_i \beta_\perp\right)^{-1} \alpha'_\perp$$

and where the X_{iT} *and* Y_{iT} *are sequences of* $O_P(1)$ *random variables, which are independent across* i*. Furthermore, the sequences* $\|X_{iT}\|$ [1] *and* $\|Y_{iT}\|$ *are uniformly integrable in* T *for all* i*.*

[1] We use the norm $\|A\|^2 = \operatorname{tr}(A'A)$.

Proof. Consider a specific time series Y_{it} satisfying (2.2). Then, from Johansen (1995), the likelihood ratio test of $H(r) : \text{rank}(\Pi) \leq r$ against $H(p)$ is given by

$$LR_{iT}(H(r)|H(p)) = -T \sum_{j=r+1}^{p} \log\left(1 - \widehat{\lambda}_{ij}\right), \tag{9.2}$$

where $\widehat{\lambda}_{i1} > \widehat{\lambda}_{i2} > ... > \widehat{\lambda}_{ip}$ are the ordered solutions of the eigenvalue problem

$$|S_i(\lambda)| = 0, \quad S_i(\lambda) \equiv \lambda S_{11} - S_{10}S_{00}^{-1}S_{01}, \tag{9.3}$$

with, for $k_i > 1$,

$$S_{jk} \equiv M_{jk} - M_{j2}M_{22}^{-1}M_{2k}, \quad M_{jk} \equiv T^{-1}\sum_{t=1}^{T} Z_{jt}Z_{kt}', \tag{9.4}$$

letting $Z_{0t} \equiv \Delta Y_{i,t}$, $Z_{1t} \equiv Y_{i,t-1}$ and $Z_{2t} \equiv \left(\Delta Y_{i,t-1}', ..., \Delta Y_{i,t-m+1}'\right)'$. (The terms S_{jk} depend on i, but we suppress this in our notation.) In Larsson (1999) (cf also Johansen (1995), p. 159), it is proved that[2]

$$LR_{iT} = T\,\text{tr}\left\{\left(\overline{\beta}_{\perp}' S_{11}\overline{\beta}_{\perp}\right)^{-1}\overline{\beta}_{\perp}' S_{10}U_{iT}S_{01}\overline{\beta}_{\perp}\right\} + O_p\left(T^{-1}\right), \tag{9.5}$$

where

$$U_{iT} \equiv S_{00}^{-1} - S_{00}^{-1}S_{01}\beta\left(\beta' S_{10}S_{00}^{-1}S_{01}\beta\right)^{-1}\beta' S_{10}S_{00}^{-1}.$$

We now have the following lemma, which is analogous to lemma 10.3 of Johansen (1995). The proof, which is very similar to that of Johansen, hence not given here, is built upon the Granger representation (see th. 4.2 of Johansen (1995))

$$Y_{it} = C_i\sum_{j=1}^{t}\varepsilon_{ij} + C_i\varepsilon_{it} + C_i^*(L)\Delta\varepsilon_{it} + \overline{\beta}_{\perp}\beta_{\perp}' Y_{i0}, \tag{9.6}$$

where $C_i^*(L)$ is a lag polynomial (cf Johansen (1995)) and $C_i \equiv \beta_{\perp}\left(\alpha_{\perp}'\Gamma_i\beta_{\perp}\right)^{-1}\alpha_{\perp}'$ with $\Gamma_i \equiv I_p - \sum_{j=1}^{k_i-1}\Gamma_{ij}$.

[2] The result in Larsson (1998) has $\beta_{\perp}$ instead of $\overline{\beta}_{\perp}$, but it is easily seen that this makes no difference.

Lemma 9.2. *If $\alpha'_\perp \Gamma_i \beta_\perp$ is non-singular, then as $T \to \infty$,*

$$\begin{aligned}
S_{00} &\xrightarrow{P} \Sigma_{00}, \\
\beta' S_{11} \beta &\xrightarrow{P} \Sigma_{\beta\beta}, \\
\beta' S_{10} &\xrightarrow{P} \Sigma_{\beta 0}, \\
\overline{\beta}'_\perp S_{1\varepsilon} &\xrightarrow{w} \int G_i dW'_i, \\
T^{-1} \overline{\beta}'_\perp S_{11} \overline{\beta}_\perp &\xrightarrow{w} \int G_i G'_i, \\
\overline{\beta}'_\perp S_{11} \beta &= O_P(1).
\end{aligned}$$

where $S_{\varepsilon 1} \equiv S_{01} - \alpha\beta' S_{11}$ and $G_i \equiv \overline{\beta}'_\perp C_i W_i$ where the W_i, $i = 1, ..., N$ are independent p-dimensional Wiener processes with covariance matrices Ω, and where C_i is as above.

Now, putting $S_{\beta\beta} \equiv \beta' S_{11} \beta$,

$$S_{01}\beta = \alpha S_{\beta\beta} + S_{\varepsilon 1}\beta,$$

where, as is proved in Larsson (1999), $S_{\varepsilon 1}\beta$ is $O_P\left(T^{-1/2}\right)$. Hence,

$$U_{iT} = \widetilde{U}_{iT} + T^{-1/2}\left(R_{1iT} + R'_{1iT} + R_{2iT} + R'_{2iT}\right) + O_P\left(T^{-1}\right), \tag{9.7}$$

where

$$\begin{aligned}
\widetilde{U}_{iT} &\equiv S_{00}^{-1} - S_{00}^{-1}\alpha S_{\beta\beta}\left(S_{\beta\beta}\alpha' S_{00}^{-1}\alpha S_{\beta\beta}\right)^{-1} S_{\beta\beta}\alpha' S_{00}^{-1} \\
&= S_{00}^{-1} - S_{00}^{-1}\alpha\left(\alpha' S_{00}^{-1}\alpha\right)^{-1}\alpha' S_{00}^{-1}
\end{aligned}$$

and

$$\begin{aligned}
T^{-1/2} R_{1iT} &\equiv -S_{00}^{-1} S_{\varepsilon 1}\beta\left(S_{\beta\beta}\alpha' S_{00}^{-1}\alpha S_{\beta\beta}\right)^{-1} S_{\beta\beta}\alpha' S_{00}^{-1} \\
&= -S_{00}^{-1} S_{\varepsilon 1}\beta\left(\alpha' S_{00}^{-1}\alpha S_{\beta\beta}\right)^{-1}\alpha' S_{00}^{-1}, \\
T^{-1/2} R_{2iT} &\equiv -S_{00}^{-1}\alpha S_{\beta\beta}\left(S_{\beta\beta}\alpha' S_{00}^{-1}\alpha S_{\beta\beta}\right)^{-1} S_{\beta\beta}\alpha' S_{00}^{-1} S_{\varepsilon 1}\beta\left(S_{\beta\beta}\alpha' S_{00}^{-1}\alpha S_{\beta\beta}\right)^{-1} S_{\beta\beta}\alpha S_{00}^{-1} \\
&= -S_{00}^{-1}\alpha\left(\alpha' S_{00}^{-1}\alpha\right)^{-1}\alpha' S_{00}^{-1} S_{\varepsilon 1}\beta\left(\alpha' S_{00}^{-1}\alpha S_{\beta\beta}\right)^{-1}\alpha' S_{00}^{-1}.
\end{aligned}$$

Furthermore,

$$\widetilde{U}_{iT} = \alpha_\perp\left(\alpha'_\perp S_{00}\alpha_\perp\right)^{-1}\alpha'_\perp,$$

where the equality follows since left-hand multiplication by $(\alpha, S_{00}\alpha_\perp)'$ yields the same result on both sides. Now, from (9.6) and some manipulations (cf Larsson (1999) for details), it follows that

$$T\,\mathrm{tr}\left\{\left(\beta_\perp' S_{11}\beta_\perp\right)^{-1}\beta_\perp' S_{10}\widetilde{U}_{iT}S_{01}\beta_\perp\right\} = Z_{0iT} + T^{-1/2}\left(R_{3iT} + R_{3iT}'\right) + O_p\left(T^{-1}\right), \tag{9.8}$$

where the R_{3iT} term is $O_P(1)$. It is discussed in more detail below.

Next, we treat R_{1iT} and R_{2iT}. As for R_{1iT}, we find as in Larsson (1999) that (cf (9.30)) , denoting asymptotic equivalence by $\sim$,

$$T^{1/2}S_{\varepsilon 1}\beta \sim \widetilde{X}_{1iT}C_i' + \widetilde{Y}_{iT},$$

where $\widetilde{X}_{1iT}$ and $\widetilde{Y}_{iT}$ are $O_P(1)$ and independent over i, so that, by the lemma,

$$\begin{aligned} R_{1iT} &\sim -\Sigma_{00}^{-1}\left(\widetilde{X}_{1iT}C_i' + \widetilde{Y}_{iT}\right)\left(\alpha'\Sigma_{00}^{-1}\alpha\Sigma_{\beta\beta}\right)^{-1}\alpha'\Sigma_{00}^{-1} \\ &\equiv \widetilde{R}_{1iT}. \end{aligned}$$

Moreover, (9.6) and the lemma yield that because $G_i \equiv \overline{\beta}_\perp' C_i W_i$,

$$\begin{aligned} \overline{\beta}_\perp' S_{11}\beta &\sim \overline{\beta}_\perp' C_i \widetilde{X}_{2iT}, \\ \overline{\beta}_\perp' S_{10} &= \overline{\beta}_\perp' S_{1\varepsilon} + \overline{\beta}_\perp' S_{11}\beta\alpha' \\ &\sim \int G_i dW_i' + \overline{\beta}_\perp' C_i \widetilde{X}_{2iT}\alpha' = \overline{\beta}_\perp' C_i \widetilde{X}_{3iT} \end{aligned} \tag{9.9}$$

where the $\widetilde{X}_{2iT}$ are $O_P(1)$ and independent and

$$\widetilde{X}_{3iT} \equiv \int W_i dW_i' + \widetilde{X}_{2iT}\alpha',$$

also $O_P(1)$ and independent. Hence,

$$\begin{aligned} & T^{1/2}\overline{\beta}_\perp' S_{10}R_{1iT}S_{01}\overline{\beta}_\perp \\ \sim\; & T^{1/2}\overline{\beta}_\perp' C_i \widetilde{X}_{3iT}\widetilde{R}_{1iT}\widetilde{X}_{3iT}' C_i'\overline{\beta}_\perp \\ \sim\; & -\overline{\beta}_\perp' C_i \widetilde{X}_{3iT}\Sigma_{00}^{-1}\left(\widetilde{X}_{1iT}C_i' + \widetilde{Y}_{iT}\right)\left(\alpha'\Sigma_{00}^{-1}\alpha\Sigma_{\beta\beta}\right)^{-1}\alpha'\Sigma_{00}^{-1}\widetilde{X}_{3iT}' C_i'\overline{\beta}_\perp. \end{aligned}$$

Similarly, it may be seen that for some $O_P(1)$ variable $\widetilde{X}_{4iT}$, again independent over i,

$$\begin{aligned} & T^{1/2}\overline{\beta}_\perp' S_{10}R_{2iT}S_{01}\overline{\beta}_\perp \\ \sim\; & -\overline{\beta}_\perp' C_i \widetilde{X}_{4iT}\Sigma_{00}^{-1}\left(\widetilde{X}_{1iT}C_i' + \widetilde{Y}_{iT}\right)\left(\alpha'\Sigma_{00}^{-1}\alpha\Sigma_{\beta\beta}\right)^{-1}\alpha'\Sigma_{00}^{-1}\widetilde{X}_{4iT}' C_i'\overline{\beta}_\perp. \end{aligned}$$

As for the term R_{3iT} of (9.8), it gets no contribution from error terms of the expansion of $\beta_{\perp}^{'}S_{11}\beta_{\perp}$, because these turn out to be $O_P\left(T^{-1}\right)$, but a refinement of (9.9) yields

$$\begin{aligned}
\overline{\beta}_{\perp}^{'}S_{10} &= \overline{\beta}_{\perp}^{'}C_i\widetilde{X}_{3iT} + T^{-1/2}R_{3iT} + O_P\left(T^{-1}\right), \\
R_{3iT} &\equiv \overline{\beta}_{\perp}^{'}C_i\widetilde{X}_{5iT} + \overline{\beta}_{\perp}^{'}\widetilde{Y}_{5iT},
\end{aligned}$$

where $\widetilde{X}_{5iT}$ and $\widetilde{Y}_{5iT}$ are $O_P\left(1\right)$ and independent over i.

Hence, via (9.5), (9.7), (9.8) and the lemma,

$$\begin{aligned}
LR_{iT} &= Z_{0iT} + 2T^{-1/2}\operatorname{tr}\left\{\left(\beta_{\perp}^{'}S_{11}\beta_{\perp}\right)^{-1}\beta_{\perp}^{'}S_{10}R_{1iT}S_{01}\beta_{\perp}\right\} \\
&\quad + 2T^{-1/2}\operatorname{tr}\left\{\left(\beta_{\perp}^{'}S_{11}\beta_{\perp}\right)^{-1}\beta_{\perp}^{'}S_{10}R_{2iT}S_{01}\beta_{\perp}\right\} \\
&\quad + 2T^{-1/2}\operatorname{tr}\left\{\left(\beta_{\perp}^{'}S_{11}\beta_{\perp}\right)^{-1}R_{3iT}\widetilde{U}_{iT}S_{01}\beta_{\perp}\right\} \\
&\quad + O_P\left(T^{-1}\right) \\
&= Z_{0iT} + T^{-1/2}Z_{1iT} + o_P\left(T^{-1/2}\right),
\end{aligned}$$

where

$$Z_{1iT} = \operatorname{tr}\left(C_iX_{iT} + Y_{iT}\right),$$

with

$$\begin{aligned}
X_{iT} &\equiv -\widetilde{X}_{1iT}^{\prime}\Sigma_{00}^{-1}\widetilde{X}_{3iT}^{\prime}\left(\int W_iW_i^{\prime}\right)^{-1}\widetilde{X}_{3iT}\Sigma_{00}^{-1}\alpha\left(\Sigma_{\beta\beta}\alpha^{\prime}\Sigma_{00}^{-1}\alpha\right)^{-1} \\
&\quad -\widetilde{X}_{1iT}^{\prime}\Sigma_{00}^{-1}\widetilde{X}_{4iT}^{\prime}\left(\int W_iW_i^{\prime}\right)^{-1}\widetilde{X}_{4iT}\Sigma_{00}^{-1}\alpha\left(\Sigma_{\beta\beta}\alpha^{\prime}\Sigma_{00}^{-1}\alpha\right)^{-1} \\
&\quad +\widetilde{X}_{3iT}^{\prime}\left(\int W_iW_i^{\prime}\right)^{-1}\widetilde{X}_{5iT}\alpha_{\perp}\left(\alpha_{\perp}^{\prime}\Sigma_{00}\alpha_{\perp}\right)^{-1}\alpha_{\perp}^{\prime}
\end{aligned}$$

and

$$\begin{aligned}
Y_{iT} &\equiv -\widetilde{Y}_{1iT}^{\prime}\Sigma_{00}^{-1}\widetilde{X}_{3iT}^{\prime}\left(\int W_iW_i^{\prime}\right)^{-1}\widetilde{X}_{3iT}\Sigma_{00}^{-1}\alpha\left(\Sigma_{\beta\beta}\alpha^{\prime}\Sigma_{00}^{-1}\alpha\right)^{-1} \\
&\quad -\widetilde{Y}_{1iT}^{\prime}\Sigma_{00}^{-1}\widetilde{X}_{4iT}^{\prime}\left(\int W_iW_i^{\prime}\right)^{-1}\widetilde{X}_{4iT}\Sigma_{00}^{-1}\alpha\left(\Sigma_{\beta\beta}\alpha^{\prime}\Sigma_{00}^{-1}\alpha\right)^{-1} \\
&\quad +\widetilde{X}_{3iT}^{\prime}\left(\int W_iW_i^{\prime}\right)^{-1}\widetilde{Y}_{5iT}\alpha_{\perp}\left(\alpha_{\perp}^{\prime}\Sigma_{00}\alpha_{\perp}\right)^{-1}\alpha_{\perp}^{\prime}.
\end{aligned}$$

Uniform integrability of the sequences $\|X_{iT}\|$ and $\|Y_{iT}\|$ may be proved along the same lines as in Larsson *et al* (1998). Details are omitted here. ■

The plan for the rest of the proof is to show that $\overline{Z}_{0T} \equiv N^{-1}\sum_{i=1}^{N} Z_{0iT}$, correctly normed, converges to the normal distribution as $N,T\to\infty$, whereas the remainder term $T^{-1/2}Z_{1iT}$, or rather $T^{-1/2}\overline{Z}_{1T} \equiv T^{-1/2}N^{-1}\sum_{i=1}^{N} Z_{1iT}$, converges to zero. The $o_P\left(T^{-1/2}\right)$ term may be considered as merged with the $T^{-1/2}Z_{1iT}$ term, hence no extra treatment of these terms are needed.

To show the convergence of $\overline{Z}_{0T}$, we will at first consider the normed quantity

$$Z_T^* \equiv \sqrt{N}\frac{\overline{Z}_{0T}-\mu_T}{\sigma_T},$$

where for the existence of the moments $\mu_T \equiv E\left(Z_{0iT}\right)$ and $\sigma_T^2 \equiv Var\left(Z_{0iT}\right)$, we refer to Larsson *et al* (1998). To this end, we utilize the Lindeberg type theorem, theorem 2 of Phillips and Moon (1999), which states that if for all $\delta > 0$,

$$\lim_{N,T\to\infty}\sum_{i=1}^{N} E\left[\xi_{i,N,T}^2 1\left\{\left|\xi_{i,N,T}\right| > \delta\right\}\right] = 0, \tag{9.10}$$

then as $N,T\to\infty$,

$$\sum_{i=1}^{N}\xi_{i,N,T} \xrightarrow{d} N\left(0,1\right).$$

In our case, we take

$$\xi_{i,N,T} \equiv N^{-1/2}\frac{Z_{0iT}-\mu_T}{\sigma_T},$$

and because the Z_{0iT} are *iid*, (9.10) simplifies into

$$\lim_{N,T\to\infty} E\left[\left(\frac{Z_{01T}-\mu_T}{\sigma_T}\right)^2 1\left\{\left|Z_{01T}-\mu_T\right| > \sqrt{N}\sigma_T\delta\right\}\right] = 0,$$

which holds if the Z_{01T} sequence is uniformly integrable. This may be proved in a similar manner as in Larsson *et al* (1998), but we omit these details here.

Next, we assess the convergence of $T^{-1/2}\overline{Z}_{1T}$. For this, we will apply theorem 1 (in simplified form) of Phillips and Moon (1999), which says the following: Let Y_{iT} be independent across i for all T and integrable. Assume that $\eta_{i,T} \xrightarrow{P} 0$ as $T\to\infty$ for all i.[3] Furthermore, assume that[4]

[3] Phillips and Moon have the more general condition $Y_{iT} \xrightarrow{d} Y_i$. With $Y_i = 0$ as in our case, assumption (iv) of Phillips and Moon is trivially satisfied, and hence omitted in our context.

[4] $1\left\{A\right\}$ is the indicator function of the event A.

(i) $\limsup_{N,T} N^{-1} \sum_{i=1}^{N} E \|\eta_{iT}\| < \infty$
(ii) $\limsup_{N,T} N^{-1} \sum_{i=1}^{N} \|E(\eta_{iT})\| = 0$
(iii) $\limsup_{N,T} N^{-1} \sum_{i=1}^{N} E \|\eta_{iT}\| 1\{\|\eta_{iT}\| > N\varepsilon\} = 0$ for all $\varepsilon > 0$
Then, $N^{-1} \sum_{i=1}^{N} \eta_{iT} \xrightarrow{P} 0$ as $N, T \to \infty$.

Observe that from the Cauchy-Schwarz inequality (cf Magnus and Neudecker (1988), p. 201), we have $\|C_i\| \leq \|\alpha'_\perp \beta_\perp\| \left\| (\alpha'_\perp \Gamma_i \beta_\perp)^{-1} \right\|$, so assumption 3 implies $\sup_i \|C_i\| < \infty$. Now, putting $\eta_{iT} \equiv C_i T^{-1/2} X_{iT}$, we have $\eta_{i,T} \xrightarrow{d} 0$ as $T \to \infty$ for all i, since X_{iT} is $O_P(1)$. To verify (i), the Cauchy-Schwarz inequality (cf Magnus and Neudecker (1988), p. 201) yields $E\|\eta_{iT}\| \leq \|C_i\| T^{-1/2} E\|X_{iT}\|$, and because $\sup_i \|C_i\| < \infty$, it is enough to show that $\limsup_{N,T} N^{-1} T^{-1/2} \sum_{i=1}^{N} E\|X_{iT}\| < \infty$. But this is seen because all $E\|X_{iT}\|$ exist, so for an arbitrary $\delta > 0$, for each N there is a T_0 such that for all $T > T_0$, $T^{-1/2} E\|X_{iT}\| < \delta$ for all $i \leq N$. Hence, $\sup_{T > T_0} N^{-1} T^{-1/2} \sum_{i=1}^{N} E\|X_{iT}\| < \delta$ for each N, and (i) follows. Moreover, because $\|E(\eta_{iT})\| \leq E\|\eta_{iT}\|$, and because δ may be chosen to be arbitrarily small, the same argument proves (ii). Finally, (iii) follows from (ii) simply by majorizing the indicator function with 1.

Because the corresponding story is similarly seen to be true for $\eta_{iT} = Y_{iT}$, it follows that $T^{-1/2} \overline{Z}_{1T} \xrightarrow{P} 0$ as $N, T \to \infty$.

It remains to prove the convergence of $\Upsilon_{\overline{LR}}$. To this end, we have from the results above and lemma 2.1 that, writing $\mu \equiv E(Z_k)$, $\sigma^2 = Var(Z_k)$ and using Taylor expansion,

$$\begin{aligned}
\Upsilon_{\overline{LR}} &= \sqrt{N} \frac{\overline{LR}_{NT} - \mu}{\sigma} = \sqrt{N} \frac{\left\{ \overline{Z}_{0T} + T^{-1/2} \overline{Z}_{1iT} + o_P\left(T^{-1/2}\right) \right\} - \left\{ \mu_T + O\left(T^{-\kappa}\right) \right\}}{\left\{ \sigma_T + O\left(T^{-\kappa}\right) \right\}} \\
&= \sqrt{N} \left\{ \frac{\overline{Z}_{0T} - \mu_T}{\sigma_T} + O_P\left(T^{-\gamma}\right) \right\} \xrightarrow{d} N(0,1),
\end{aligned}$$

as $N, T \to \infty$, $\sqrt{N} T^{-\gamma} \to 0$, where $\gamma = \min(\kappa, 1/2)$. This completes the proof of the theorem. ■

Proof of lemma 2.3: We will at first show that for any fixed s and conditional on W_2, the covariance matrix of $V(s)$ is $\rho(s,s) I_r$. To this end, define the scalar

$$F(s,u) \equiv \int_0^s W_2(v)' dv \left(\int W_2 W_2' \right)^{-1} W_2(u),$$

and note that

$$V(s) = W_1(s) - \int_0^1 F(s,u) dW_1(u) = \int_0^1 \left\{ 1_{\{u<s\}} - F(s,u) \right\} dW_1(u), \quad (9.11)$$

where 1_A is the indicator function of the event A. Conditional on W_2, the distribution of $V(s)$ is clearly normal with expectation zero. Denoting expectation conditional on W_2 by E_2, we find for $s \leq t$

$$\begin{aligned}
& E_2\left\{V(s)V(t)'\right\} \\
= & \int_0^1 \int_0^1 \left\{1_{\{u<s\}} - F(s,u)\right\}\left\{1_{\{v<t\}} - F(t,v)\right\} E_2\left\{dW_1(u)\, dW_1(v)'\right\} \\
= & \int_0^1 \left\{1_{\{u<s\}} - F(s,u)\right\}\left\{1_{\{u<t\}} - F(t,u)\right\} du I_r \\
= & \left\{s - \int_0^s F(t,u)\, du - \int_0^t F(s,u)\, du + \int_0^1 F(s,u)\, F(t,u)\, du\right\} I_r. \quad (9.12)
\end{aligned}$$

Here,

$$\begin{aligned}
\int_0^t F(s,u)\, du &= \int_0^s W_2(v)'\, dv \left(\int W_2 W_2'\right)^{-1} \int_0^t W_2(u)\, du \\
&= A_{p-r}(s,t) \\
&= \int_0^s F(t,u)\, du
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^1 F(s,u)\, F(t,u)\, du \\
= & \int_0^1 \int_0^s W_2(v)'\, dv \left(\int W_2 W_2'\right)^{-1} W_2(u)\, W_2(u)' \\
& \left(\int W_2 W_2'\right)^{-1} \int_0^t W_2(x)\, dx du \\
= & A_{p-r}(s,t).
\end{aligned}$$

Hence, (9.12) implies

$$E_2\left\{V(s)\, V(t)'\right\} = \{s - A_{p-r}(s,t)\}\, I_r = \rho(s,t)\, I_r.$$

It immediately follows that $E\{\rho(s,s)\}$ is positive.

To see that $E\{\rho(s,s)\} \leq s$ we only need to observe that the quantity

$$\int_0^s W_2(t)'\, dt \left(\int W_2 W_2'\right)^{-1} \int_0^s W_2(t)\, dt$$

is positive with probability one. ■

Proof of lemma 2.4: Because $E\left\{A_{p-r}(s,s)\right\} = a_{p-r}(s,s)$ and

$$\rho(s,s) = s - A_{p-r}(s,s),$$

$$E(X) = \int_0^1 \operatorname{tr}\left[E\left\{\rho(s,s)\right\} I_r\right] ds = r\int_0^1 \left\{s - a_{p-r}(s,s)\right\} ds = r\left(\frac{1}{2} - b_{p-r}\right), \tag{9.13}$$

where

$$b_{p-r} = \int_0^1 a_{p-r}(s,s)\, ds.$$

Moreover,

$$\begin{aligned}
& b_{p-r} \\
= & \int_0^1 E\left\{\int_0^1 1_{\{v<s\}} W_2(v)'\, dv \left(\int W_2 W_2'\right)^{-1} \int_0^1 1_{\{u<s\}} W_2(u)\, du\right\} ds \\
= & E\left\{\int_0^1 W_2(v)' \int_0^1 \left(\int_0^1 1_{\{v<s\}} 1_{\{u<s\}} ds\right) \left(\int W_2 W_2'\right)^{-1} W_2(u)\, du dv\right\} \\
= & 2E\left\{\int_0^1 (1-v)\, W_2(v)' \left(\int W_2 W_2'\right)^{-1} \int_0^t W_2(u)\, du dt\right\}.
\end{aligned}$$

The conditional second moment of X is obtained from

$$\begin{aligned}
E_2\left(X^2\right) &= E_2\left\{\int_0^1 V(s)' V(s)\, ds \int_0^1 V(t)' V(t)\, dt\right\} \\
&= \int_0^1 \int_0^1 E_2\left\{V(s)' V(s) V(t)' V(t)\right\} ds dt \\
&= 2\int_0^1 \int_0^t E_2\left\{V(s)' V(s) V(t)' V(t)\right\} ds dt
\end{aligned} \tag{9.14}$$

Now, for $s \leq t$, let

$$\widetilde{V}(s,t) \equiv \begin{pmatrix} V(s) \\ V(t) \end{pmatrix},$$

which is $2r$-variate normal with expectation zero and covariance matrix $\Sigma \otimes I_r$, where

$$\Sigma \equiv \begin{pmatrix} \rho(s,s) & \rho(s,t) \\ \rho(s,t) & \rho(t,t) \end{pmatrix}.$$

To find the required moment, the following result, which is lemma 6.2 of Magnus (1978), is useful.

Lemma 9.3. *Let A and B be symmetric matrices, and let ε be $N_n(0, \Upsilon)$, Υ positive definite. Then,*

$$E(\varepsilon' A \varepsilon \varepsilon' B \varepsilon) = \operatorname{tr}(A\Upsilon)\operatorname{tr}(B\Upsilon) + 2\operatorname{tr}(A\Upsilon B\Upsilon).$$

Thus, taking $\varepsilon = \widetilde{V}(s,t)$, $\Upsilon = \Sigma \otimes I_r$ and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes I_r, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes I_r,$$

we find

$$E_2\left\{V(s)' V(s) V(t)' V(t)\right\} = r^2 \rho(s,s)\rho(t,t) + 2r\rho(s,t)^2.$$

Hence, by (9.14),

$$\begin{aligned}
& E\left(X^2\right) \\
= \; & 2r^2 \int_0^1 \int_0^t E\{\rho(s,s)\rho(t,t)\}\, dsdt + 4r \int_0^1 \int_0^t E\left\{\rho(s,t)^2\right\} dsdt \\
= \; & 2r^2 \int_0^1 \int_s^1 E\{\rho(s,s)\rho(t,t)\}\, dtds + 4r \int_0^1 \int_s^1 E\left\{\rho(s,t)^2\right\} dtds \\
= \; & 2r^2 \int_0^1 \int_s^1 E\left[\{s - A_{p-r}(s,s)\}\{t - A_{p-r}(t,t)\}\right] dtds \\
& + 4r \int_0^1 \int_s^1 E\left[\{s - A_{p-r}(s,t)\}^2\right] dtds \\
= \; & 2r^2 \left[\frac{1}{8} - \frac{1}{2}\int_0^1 \left(1 - s^2\right) a_{p-r}(s,s)\, ds - \int_0^1 s \int_s^1 a_{p-r}(t,t)\, dtds \right. \\
& \left. + \int_0^1 \int_s^1 E\{A_{p-r}(s,s) A_{p-r}(t,t)\}\, dtds \right] \\
& + 4r \left[\frac{1}{12} - 2\int_0^1 s \int_s^1 a_{p-r}(s,t)\, dtds \right. \\
& \left. + \int_0^1 \int_s^1 E\left\{A_{p-r}(s,t)^2\right\}\right] dtds.
\end{aligned}$$

Hence, since by (9.13),

$$\{E(X)\}^2 = r^2 \left(\frac{1}{2} - b_{p-r}\right)^2 = \frac{r^2}{4} - r^2 b_{p-r} + r^2 b_{p-r}^2,$$

where

$$b_{p-r} = \int_0^1 a_{p-r}(s,s)\, ds,$$

and because

$$\int_0^1 s \int_s^1 a_{p-r}(t)\, dtds = \int_0^1 a_{p-r}(t) \int_0^t sdsdt = \frac{1}{2}\int_0^1 t^2 a_{p-r}(t)\, dt,$$

we get

$$\begin{aligned} & Var(X) \\ = \; & 2r^2 \left[\int_0^1 \int_s^1 E\{A_{p-r}(s,s)\, A_{p-r}(t,t)\}\, dtds - \frac{1}{2} b_{p-r}^2 \right] \\ & +4r \left[\frac{1}{12} - 2\int_0^1 s \int_s^1 a_{p-r}(s,t)\, dtds \right. \\ & \left. + \int_0^1 \int_s^1 E\left\{ A_{p-r}(s,t)^2 \right\} \right] dtds. \end{aligned}$$

This completes the proof of lemma 2.4. ■

Proof of lemma 2.5: To obtain the requried moments numerically for $p-r=1$, we will need the joint Laplace transform of

$$\left\{ \int_0^s W_x dx, \int_0^t W_y dy, \int_0^1 W_z^2 dz \right\},$$

where W is a standard one-dimensional Wiener process.

We will utilize the fact that (cf Chan and Wei (1988))

$$\begin{aligned} & \left(T^{-3/2} \sum_{t=1}^{[Ts]} S_{t-1}, T^{-3/2} \sum_{t=1}^{[Tt]} S_{t-1}, T^{-2} \sum_{t=1}^{T} S_{t-1}^2 \right) \\ & \xrightarrow{w} \left(\int_0^s W_x dx, \int_0^t W_y dy, \int_0^1 W_z^2 dz \right) \end{aligned}$$

as $T \to \infty$, where $S_t = \sum_{i=1}^t \varepsilon_i$ with ε_i as independent unit normals, and where $[Ts]$ is the integer part of Ts. It follows that

$$A_T(s,t) \equiv T^{-1} \frac{\sum_{i=1}^{[Ts]} S_{i-1} \sum_{j=1}^{[Tt]} S_{j-1}}{\sum_{t=1}^{T} S_{t-1}^2} \xrightarrow{w} \frac{\int_0^s W_x dx \int_0^t W_y dy}{\int_0^1 W_z^2 dz} = A_1(s,t),$$

as $T \to \infty$. Now, define the Laplace transform

$$\begin{aligned} & \varphi(r,u,v) \\ \equiv \; & E\left\{ \exp\left(-rT^{-3/2} \sum_{i=1}^{[Ts]} S_{i-1} - uT^{-3/2} \sum_{j=1}^{[Tt]} S_{j-1} - vT^{-2} \sum_{t=1}^{T} S_{t-1}^2 \right) \right\}. \end{aligned}$$

Noting the identity

$$\int_0^{\infty} v^{n-1} \exp(-av)\, dv = (n-1)!a^{-n},$$

we obtain the useful formulae

$$E\{A_T(s,s)\} = \int_0^{\infty} \left.\frac{\partial^2}{\partial r^2}\varphi(r,u,v)\right|_{r=0,u=0} dv, \tag{9.15}$$

$$E\{A_T(s,t)\} = \int_0^{\infty} \left.\frac{\partial^2}{\partial r \partial u}\varphi(r,u,v)\right|_{r=0,u=0} dv, \tag{9.16}$$

$$\begin{aligned} E\left\{A_T(s,t)^2\right\} &= E\{A_T(s,s)\, A_T(t,t)\} \\ &= \int_0^{\infty} v \left.\frac{\partial^4}{\partial r^2 \partial u^2}\varphi(r,u,v)\right|_{r=0,u=0} dv, \end{aligned} \tag{9.17}$$

and then, from uniform integrability[5],

$$\lim_{T\to\infty} E\{A_T(s,t)\} = E\{A_1(s,t)\} = a_1(s,t), \tag{9.18}$$

etcetera. Now, introducing the integration variable $\underline{x} = (x_1, ..., x_T)'$, we find

$$\begin{aligned} &\varphi(r,u,v) \\ = &\int (2\pi)^{-T/2} \exp\left\{-rT^{-3/2}\sum_{i=1}^{[Ts]-1} x_i - uT^{-3/2}\sum_{i=1}^{[Tt]-1} x_i \right. \\ &\left. -vT^{-2}\sum_{i=1}^{T-1} x_i^2 - \frac{1}{2}\sum_{i=1}^{T}(x_i - x_{i-1})^2\, d\underline{x}\right\} \\ = &\int (2\pi)^{-T/2} \exp\left(-\frac{1}{2}\underline{x}'P\underline{x} - \underline{q}'\underline{x}\right) d\underline{x} \\ = &\det P^{-1/2} \exp\left(\frac{1}{2}\underline{q}'P^{-1}\underline{q}\right), \end{aligned}$$

where the $T \times T$ matrix

$$P \equiv \begin{pmatrix} \alpha & -1 & 0 & \cdots & 0 \\ -1 & \alpha & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & & 0 \\ \vdots & \ddots & & \alpha & -1 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix}, \quad \alpha \equiv 2\left(1 + vT^{-2}\right),$$

[5] We do not prove uniform integrability here, but a similar proof may be found in Larsson (1997).

and $\underline{q} \equiv rT^{-3/2}\underline{e}_s + uT^{-3/2}\underline{e}_t$, where $\underline{e}_s$ is a T-dimensional vector with ones in entries $1, ..., [Ts]-1$ and zeroes in the other entries, and similarly for $\underline{e}_t$. The differentiation w.r.t. r, u is handled via the Taylor expansion

$$\begin{aligned}
& \exp\left(\frac{1}{2}\underline{q}'P^{-1}\underline{q}\right) \\
= \ & 1 + \frac{1}{2}\underline{q}'P^{-1}\underline{q} + \frac{1}{8}\left(\underline{q}'P^{-1}\underline{q}\right)^2 + ... \\
= \ & 1 + \frac{1}{2}\left(r^2T^{-3}\underline{e}_s'P^{-1}\underline{e}_s + 2ruT^{-3}\underline{e}_s'P^{-1}\underline{e}_t + u^2T^{-3}\underline{e}_t'P^{-1}\underline{e}_t\right) \\
& + \frac{1}{8}\left\{2r^2u^2T^{-6}\left(\underline{e}_s'P^{-1}\underline{e}_s\right)\left(\underline{e}_t'P^{-1}\underline{e}_t\right) + 4r^2u^2T^{-6}\left(\underline{e}_s'P^{-1}\underline{e}_t\right)^2 + ...\right\} \\
& + ...,
\end{aligned}$$

from which we conclude that

$$\left.\frac{\partial^2}{\partial r^2}\varphi(r,u,v)\right|_{r=0,u=0} = T^{-3}\underline{e}_s'P^{-1}\underline{e}_s \det P^{-1/2}, \tag{9.19}$$

$$\left.\frac{\partial^2}{\partial r \partial u}\varphi(r,u,v)\right|_{r=0,u=0} = T^{-3}\underline{e}_s'P^{-1}\underline{e}_t \det P^{-1/2}, \tag{9.20}$$

$$\begin{aligned}
\left.\frac{\partial^4}{\partial r^2 \partial u^2}\varphi(r,u,v)\right|_{r=0,u=0} = \ & T^{-6}\left\{\left(\underline{e}_s'P^{-1}\underline{e}_s\right)\left(\underline{e}_t'P^{-1}\underline{e}_t\right)\right. \\
& \left. + 2\left(\underline{e}_s'P^{-1}\underline{e}_t\right)^2\right\} \det P^{-1/2}.
\end{aligned} \tag{9.21}$$

Now, it follows as in Larsson (1998b) that, putting $v \equiv x^2/2$,

$$\det P = \cosh x + O\left(T^{-1}\right), \tag{9.22}$$

and moreover, denoting the elements of P^{-1} by p_{ij} and $y \equiv i/T$ and $z \equiv j/T$, for $i \leq j$ (because of the symmetry, $p_{ji} = p_{ij}$ for all i, j)

$$T^{-1}p_{ij} = f(x,y,z) + O\left(T^{-1}\right), \quad f(x,y,z) \equiv \frac{\sinh(xy)\cosh\{x(1-z)\}}{x\cosh x},$$

It follows that for $s \leq t$, adapting the integral approximation technique of Larsson (1998b),

$$\lim_{T\to\infty} T^{-3}\underline{e}_s'P^{-1}\underline{e}_t$$

$$
\begin{aligned}
&= \lim_{T\to\infty} T^{-3}\sum_{i=1}^{[Ts]}\sum_{j=1}^{[Tt]} p_{ij} = \lim_{T\to\infty} T^{-3}\sum_{i=1}^{[Ts]}\left(\sum_{j=1}^{i-1} p_{ji} + \sum_{j=i}^{[Tt]} p_{ij}\right) \\
&= \int_0^s \left(\int_0^y f(x,z,y)\,dz + \int_y^t f(x,y,z)\,dz\right) dy \\
&= \frac{1}{x\cosh x}\int_0^s \Big[\cosh\{x(1-y)\}\int_0^y \sinh(xz)\,dz \\
&\quad + \sinh(xy)\int_y^t \cosh\{x(1-z)\}\,dz\Big]\,dy \\
&= \frac{1}{x\cosh x}\int_0^s \Bigg[\cosh\{x(1-y)\}\,\frac{\cosh(xy)-1}{x} \\
&\quad + \sinh(xy)\,\frac{\sinh\{x(1-y)\}-\sinh\{x(1-t)\}}{x}\Bigg]\,dy \\
&= \frac{1}{x^2\cosh x}\int_0^s \left[\cosh x - \cosh\{x(1-y)\} - \sinh\{x(1-t)\}\sinh(xy)\right] dy \\
&= \frac{1}{x^2\cosh x}\Bigg[s\cosh x - \frac{\sinh x - \sinh\{x(1-s)\}}{x} \\
&\quad - \sinh\{x(1-t)\}\,\frac{\cosh(xs)-1}{x}\Bigg] \\
&\equiv g(s,t,x). \qquad (9.23)
\end{aligned}
$$

Now, because $dv = xdx$, we find via (9.15), (9.18) (9.19) and (9.22) that

$$
\begin{aligned}
a_1(s,s) &= \lim_{T\to\infty}\int_0^\infty T^{-3}\underline{e}_s' P^{-1}\underline{e}_s \det P^{-1/2} x dx \\
&= \int_0^\infty x(\cosh x)^{-1/2} g(s,s,x)\,dx, \qquad (9.24)
\end{aligned}
$$

implying, simplifying and interchanging the ordering of integrations,

$$
\begin{aligned}
& b_1 \\
&= \int_0^1 a_1(s,s)\,ds = \int_0^\infty x(\cosh x)^{-1/2}\int_0^1 g(s,s,x)\,dsdx \\
&= \int_0^\infty \frac{1}{x(\cosh x)^{3/2}}\left\{\frac{1}{2}\cosh x - \frac{3}{2}\frac{\sinh x}{x} + \frac{2}{x^2}(\cosh x - 1)\right\} dx. \qquad (9.25)
\end{aligned}
$$

Moreover, to find $Var(X)$, we at first note that from uniform integrability, (9.15)-(9.22) and $v = x^2/2$, $dv = xdx$,

$$
a_1(s,t) = \int_0^\infty x(\cosh x)^{-1/2} g(s,t,x)dx,
$$

$$
\begin{aligned}
E\left\{A_1(s,t)^2\right\} &= E\left\{A_1(s,s)\,A_1(t,t)\right\} \\
&= \frac{1}{2}\int_0^\infty x^3(\cosh x)^{-1/2}\left\{g(s,s,x)\,g(t,t,x)+2g(s,t,x)^2\right\}dx,
\end{aligned}
$$

which may be plugged in into the variance formula of lemma 2.4 to yield

$$
\begin{aligned}
Var(X) &= 2r^2\left(\frac{1}{2}c_1-\frac{1}{2}b_1^2\right)+4r\left(\frac{1}{12}-2c_2+\frac{1}{2}c_1\right) \\
&= \frac{1}{3}r+\left(r^2+2r\right)c_1-r^2b_1^2-8rc_2, \qquad (9.26)
\end{aligned}
$$

where b_1 is as above and

$$
\begin{aligned}
c_1 &\equiv \int_0^1\int_s^1\int_0^\infty x^3(\cosh x)^{-1/2}\left\{g(s,s,x)\,g(t,t,x)+2g(s,t,x)^2\right\}dxdtds, \\
c_2 &\equiv \int_0^1 s\int_s^1\int_0^\infty x(\cosh x)^{-1/2}\,g(s,t,x)dxdtds.
\end{aligned}
$$

Here,

$$
c_1=\int_0^\infty x^3(\cosh x)^{-1/2}\,k_1(x)\,dx, \quad k_1(x)\equiv h_1(x)+2h_{12}(x),
$$

where

$$
\begin{aligned}
& h_{11}(x) \\
\equiv\ & \int_0^1 g(s,s,x)\int_s^1 g(t,t,x)\,dtds \\
=\ & \frac{1}{x^4\cosh^2 x}\int_0^1\left[s\cosh x-\frac{\sinh x-\sinh\{x(1-s)\}}{x}\right. \\
& \left.-\sinh\{x(1-s)\}\frac{\cosh(xs)-1}{x}\right] \\
& \int_s^1\left[t\cosh x-\frac{\sinh x-\sinh\{x(1-t)\}}{x}-\sinh\{x(1-t)\}\frac{\cosh(xt)-1}{x}\right]dtds \\
=\ & \frac{1}{x^4\cosh^2 x}\int_0^1\left[s\cosh x-\frac{\sinh x}{x}+2\frac{\sinh\{x(1-s)\}}{x}\right. \\
& \left.-\sinh\{x(1-s)\}\frac{\cosh(xs)}{x}\right] \\
& \left[\frac{1-s^2}{2}\cosh x-(1-s)\frac{\sinh x}{x}+2\frac{\cosh\{x(1-s)\}-1}{x^2}\right.
\end{aligned}
$$

$$-\frac{1-s}{2}\frac{\sinh x}{x}+\frac{1}{4}\frac{\cosh x}{x^2}-\frac{1}{4}\frac{\cosh\{x(2s-1)\}}{x^2}\Bigg]$$

$$= \frac{1}{16x^8\cosh^2 x}\Big\{48-x^2+x^4-16\left(4+x^2\right)\cosh x+48x\sinh x$$
$$+\left(16+17x^2+x^4\right)\cosh(2x)-6x\left(4+x^2\right)\sinh(2x)\Big\}$$

and

$$h_{12}(x)$$
$$\equiv \int_0^1\int_s^1 g(s,t,x)^2\,dtds$$
$$= \frac{1}{x^4\cosh^2 x}\int_0^1\int_s^1\Bigg[s\cosh x-\frac{\sinh x-\sinh\{x(1-s)\}}{x}$$
$$-\sinh\{x(1-t)\}\frac{\cosh(xs)-1}{x}\Bigg]^2 dtds$$
$$= \frac{1}{48x^8\cosh^2 x}\Big\{-48-54x^2+2x^4+144x\sinh x+48\left(4-x^2\right)\cosh x$$
$$-144\cosh(2x)+75x\sinh(2x)-8x^3\sinh(2x)+2x^4\cosh(2x)\Big\}.$$

Hence, via simplifications,

$$k_1(x)$$
$$= \frac{1}{48x^8\cosh^2 x}\Big\{48-111x^2+7x^4-48\left(-4+3x^2\right)\cosh x+432x\sinh x$$
$$(-240+51x^2+7x^4)\cosh(2x)+2x\left(39-17x^2\right)\sinh(2x)\Big\}.$$

Moreover,

$$c_2=\int_0^\infty x(\cosh x)^{-1/2}h_2(x)\,dx,$$

where

$$h_2(x)$$
$$\equiv \int_0^1 s\int_s^1 g(s,t,x)dtds$$
$$= \frac{1}{x^2\cosh x}\int_0^1 s\int_s^1\Bigg[s\cosh x-\frac{\sinh x-\sinh\{x(1-s)\}}{x}$$
$$-\sinh\{x(1-t)\}\frac{\cosh(xs)-1}{x}\Bigg]dtds$$

$$= \frac{1}{12x^6 \cosh x}\left\{24 - 6x^2 + \left(-24 - 3x^2 + x^4\right)\cosh x + \left(21x - 2x^3\right)\sinh x\right\}.$$

Hence, via (9.26),

$$\begin{aligned} & Var\left(X\right) \\ = \; & \frac{1}{3}r + \left(r^2 + 2r\right)\int_0^\infty x^3 \left(\cosh x\right)^{-1/2} k_1\left(x\right) dx - r^2 b_1^2 \\ & -8r\int_0^\infty x\left(\cosh x\right)^{-1/2} h_2\left(x\right) dx \\ = \; & d_{12}r^2 + d_{11}r, \end{aligned}$$

where

$$\begin{aligned} d_{12} &= \int_0^\infty x^3 \left(\cosh x\right)^{-1/2} k_1\left(x\right) dx - b_1^2, \\ d_{11} &= \frac{1}{3} + 2\int_0^\infty x\left(\cosh x\right)^{-1/2} k_2\left(x\right) dx, \end{aligned}$$

with

$$k_2\left(x\right) \equiv x^2 k_1\left(x\right) - 4h_2\left(x\right).$$

Further, we have the simplification

$$\begin{aligned} & k_2\left(x\right) \\ = \; & \frac{1}{48x^6\cosh^2 x}\left\{240 - 87x^2 - x^4 - 48\left(4 + x^2\right)\cosh x + 432x\sinh x \right. \\ & \left. - \left(48 - 75x^2 + x^4\right)\cosh\left(2x\right) - 18x\left(5 + x^2\right)\sinh\left(2x\right)\right\}, \end{aligned}$$

and the proof is completed. ■

Proof of theorem 2.7: This proof is basically a combination of arguments as in the proof of theorem 2.2 and of the proof of theorem 7 in Harris (1997). With notation as in section 2.2, we have as in the proof of theorem 7, Harris (1997) (we suppress individual-specific indeces i)

$$\widehat{c} = T^{-2}\sum_{t=1}^{T} \widehat{S}_t' \widehat{\Omega}_{zz}^{*-1} \widehat{S}_t$$

where

$$\widehat{S}_t' \widehat{\Omega}_{zz}^{*-1} \widehat{S}_t = \widehat{S}_t' \left(\beta'\widehat{\beta}^*\right)^{-1} \beta'\beta \left\{\beta'\beta\left(\widehat{\beta}^{*\prime}\beta\right)^{-1} \widehat{\Omega}_{zz}^{*} \left(\beta'\widehat{\beta}^*\right)^{-1} \beta'\beta\right\}^{-1} \beta'\beta\left(\widehat{\beta}^{*\prime}\beta\right)^{-1} \widehat{S}_t$$

with, because $\widehat{\beta}^* = \beta + O_P(T^{-1})$ and $\widehat{\Omega}^*_{zz} = \Omega^*_{zz} + O_P(T^{-1})$, where Ω^*_{zz} is the covariance matrix of $z_t^* \equiv \beta' y_t^*$,

$$\beta'\beta\left(\widehat{\beta}^{*\prime}\beta\right)^{-1}\widehat{\Omega}^*_{zz}\left(\beta'\widehat{\beta}^*\right)^{-1}\beta'\beta = \Omega^*_{zz} + O_P\left(T^{-1}\right).$$

Hence, because $\widehat{S}_t$ is $O_P\left(T^{1/2}\right)$,

$$T^{-1}\widehat{S}_t'\widehat{\Omega}^{*-1}_{zz}\widehat{S}_t = T^{-1}\widehat{S}_t'\left(\beta'\widehat{\beta}^*\right)^{-1}\beta'\beta\Omega^{*-1}_{zz}\beta'\beta\left(\widehat{\beta}^{*\prime}\beta\right)^{-1}\widehat{S}_t + O_P\left(T^{-1}\right). \tag{9.27}$$

Moreover, we have the identity

$$\begin{aligned} & \beta'\beta\left(\widehat{\beta}^{*\prime}\beta\right)^{-1}T^{-1/2}\widehat{S}_{[Ts]} \\ = \; & T^{-1/2}\sum_{t=1}^{[Ts]}\beta' y_t^* + T\left\{\widehat{\beta}^*\left(\beta'\widehat{\beta}^*\right)^{-1}\beta'\beta - \beta\right\}' T^{-3/2}\sum_{t=1}^{[Ts]} y_t^*. \end{aligned} \tag{9.28}$$

Here, from the definition of y_t^* and the consistency of $\widehat{\beta}$, $\widehat{\Omega}_{zw}$ and $\widehat{\Omega}_{ww}$,

$$\begin{aligned} z_t^* &= \beta' y_t^* = \beta' y_t - \Omega_{zw}\Omega_{ww}^{-1}\beta_\perp'\Delta y_t + O_P\left(T^{-1}\right) \\ &= z_t - \Omega_{zw}\Omega_{ww}^{-1}w_t + O_P\left(T^{-1}\right), \end{aligned}$$

where Ω_{zw} and Ω_{ww} are suitably defined blocks of the covariance matrix for $\zeta_t = (z_t', w_t')'$. Hence, because $w_t^* = w_t + O_P(T^{-1})$, we have

$$\zeta_t^* \equiv \begin{pmatrix} z_t^* \\ w_t^* \end{pmatrix} = M\zeta_t + O_P\left(T^{-1}\right), \quad M \equiv \begin{pmatrix} I_r & -\Omega_{zw}\Omega_{ww}^{-1} \\ 0 & I_{p-r} \end{pmatrix}. \tag{9.29}$$

Now, as in theorem 2.1 of Johansen (1995) (cf also Larsson (1999)), we have the representation

$$\zeta_t = C_t(L)\varepsilon_t + a_t,$$

where $C_t(L) \equiv \sum_{j=1}^{t-1} C_j L^j$, with L as the lag operator and where a_t depends on initial values of the ζ_t process. It follows that we have the representation (cf Johansen (1995), chap. 2)

$$\zeta_t = C_t(1)\varepsilon_t + b_t, \quad b_t \equiv C_t^{(1)}(L)\Delta\varepsilon_t + a_t, \tag{9.30}$$

where $C_t(1) = \sum_{j=1}^{t-1} C_j$. Furthermore, as in Larsson (1999)[6] we find

$$D \equiv \lim_{t\to\infty} C_t(1) = \begin{pmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{pmatrix}\begin{pmatrix} \overline{\alpha}' \\ \overline{\alpha}_\perp' \end{pmatrix}, \tag{9.31}$$

[6] See also Johansen (1995), p. 51, where the inverse of this matrix is given as $\widetilde{A}(1)$.

where

$$\begin{aligned}
C_{11} &\equiv -I_r, \\
C_{12} &\equiv \overline{\alpha}'\Gamma\overline{\beta}_\perp\left(\overline{\alpha}_\perp'\Gamma\overline{\beta}_\perp\right)^{-1}, \\
C_{22} &\equiv \left(\overline{\alpha}_\perp'\Gamma\overline{\beta}_\perp\right)^{-1},
\end{aligned}$$

and the lag polynomial $C_t^{(1)}(L)$ is as in Larsson (1999) and $\Delta\varepsilon_t \equiv \varepsilon_t - \varepsilon_{t-1}$. Hence, (9.29) yields

$$\begin{aligned}
\zeta_t^* &= M\zeta_t + O_P\left(T^{-1}\right) \\
&= M\left(D\varepsilon_t + b_t\right) + O_P\left(T^{-1}\right). \qquad (9.32)
\end{aligned}$$

Thus, because b_t does not affect the asymptotic behaviour of ζ_t^* (cf Larsson (1999)), $(MD)^{-1}\zeta_t^*$ is asymptotically standard p-variate normal, and we may define the standard p-variate Brownian motion $W(s)$ such that

$$\sum_{t=1}^{[Ts]}\zeta_t^* \xrightarrow{d} MDW(s).$$

Consequently, (9.32) yields

$$T^{-1/2}\sum_{t=1}^{[Ts]}\zeta_t^* = M\left\{DW(s) + T^{-1/2}R(s)\right\} + O_P\left(T^{-1}\right),$$

where $R(s)$ is $O_P(1)$ (cf Larsson (1999)). Hence,

$$T^{-1/2}\sum_{t=1}^{[Ts]}z_t^* = \begin{pmatrix} I_r & 0 \end{pmatrix} M\left\{DW(s) + T^{-1/2}R(s)\right\} + O_P\left(T^{-1}\right). \qquad (9.33)$$

Note the simplification

$$\begin{pmatrix} 0 & I_{p-r} \end{pmatrix} MD = C_{22}\overline{\alpha}_\perp'. \qquad (9.34)$$

Moreover, because from (9.30) and (9.31),

$$\begin{pmatrix} \Omega_{zz} & \Omega_{zw} \\ \Omega_{wz} & \Omega_{ww} \end{pmatrix} = \Omega_{\zeta\zeta} = \begin{pmatrix} -I_r & C_{12} \\ 0 & C_{22} \end{pmatrix} \begin{pmatrix} \overline{\alpha}' \\ \overline{\alpha}_\perp' \end{pmatrix} \begin{pmatrix} \overline{\alpha} & \overline{\alpha}_\perp \end{pmatrix} \begin{pmatrix} -I_r & 0 \\ C_{12}' & C_{22}' \end{pmatrix},$$

implying

$$\Omega_{zw} = C_{12}\overline{\alpha}'_{\perp}\overline{\alpha}_{\perp}C'_{22},$$
$$\Omega_{ww} = C_{22}\overline{\alpha}'_{\perp}\overline{\alpha}_{\perp}C'_{22},$$

and hence

$$C_{12} - \Omega_{zw}\Omega_{ww}^{-1}C_{22} = 0,$$

we have

$$\begin{aligned} \begin{pmatrix} I_r & 0 \end{pmatrix} MD &= \begin{pmatrix} I_r & -\Omega_{zw}\Omega_{ww}^{-1} \end{pmatrix} \begin{pmatrix} -I_r & C_{12} \\ 0 & C_{22} \end{pmatrix} \begin{pmatrix} \overline{\alpha}' \\ \overline{\alpha}'_{\perp} \end{pmatrix} \\ &= \begin{pmatrix} -I_r & 0 \end{pmatrix} \begin{pmatrix} \overline{\alpha}' \\ \overline{\alpha}'_{\perp} \end{pmatrix} = -\overline{\alpha}'. \end{aligned} \tag{9.35}$$

Furthermore, from the proof of theorem 2 of Harris (1997), we get

$$T\left\{\widehat{\beta}^{*}\left(\beta'\widehat{\beta}^{*}\right)^{-1}\beta'\beta - \beta\right\} = -\beta_{\perp}\left(T^{-1}\beta'_{\perp}S^{*}_{yy}\beta_{\perp}\right)^{-1}\beta'_{\perp}S^{*}_{yy}\beta + O_P\left(T^{-1}\right), \tag{9.36}$$

where via (9.34) and convergence theorems as in Billingsley (1968)[7]

$$\begin{aligned} & T^{-1}\beta'_{\perp}S^{*}_{yy}\beta_{\perp} \\ = \; & T^{-2}\sum_{t=1}^{T}\left(\sum_{u=1}^{[Tt]} w_u\right)\left(\sum_{v=1}^{[Tt]} w'_v\right) + O_P\left(T^{-1}\right) \\ = \; & \begin{pmatrix} 0 & I_{p-r} \end{pmatrix} M \int \left(DW + T^{-1/2}R\right)\left(DW + T^{-1/2}R\right)' M' \begin{pmatrix} 0 \\ I_{p-r} \end{pmatrix} \\ & + O_P\left(T^{-1}\right) \\ = \; & C_{22}\overline{\alpha}'_{\perp}\left(\int WW'\right)\overline{\alpha}_{\perp}C'_{22} + T^{-1/2}\left(R_1 + R'_1\right) + O_P\left(T^{-1}\right), \end{aligned} \tag{9.37}$$

with

$$R_1 \equiv C_{22}\overline{\alpha}'_{\perp}\left(\int WR'\right) M' \begin{pmatrix} 0 \\ I_{p-r} \end{pmatrix}, \tag{9.38}$$

[7] Observe the short-hand notation, where for example, $\int WW'$ means $\int_0^1 W(t)\,W(t)'\,dt$.

and

$$
\begin{aligned}
& \beta_{\perp}' S_{yy}^{*} \beta \\
= \; & T^{-2} \sum_{t=1}^{T} \left(\sum_{u=1}^{[Tt]} w_u \right) z_t^{*} + O_P\left(T^{-1}\right) \\
= \; & \left(\begin{array}{cc} 0 & I_{p-r} \end{array} \right) M \int \left(DW + T^{-1/2} R \right) d \left(DW + T^{-1/2} R \right)' M' \left(\begin{array}{c} I_r \\ 0 \end{array} \right) \\
& + O_P\left(T^{-1}\right) \\
= \; & -C_{22} \overline{\alpha}_{\perp}' \int W dW' \overline{\alpha} + T^{-1/2} \left(R_{21} + R_{22} \right) + O_P\left(T^{-1}\right), \qquad (9.39)
\end{aligned}
$$

where

$$
\begin{aligned}
R_{21} & \equiv C_{22} \overline{\alpha}_{\perp}' \left(\int W dR' \right) M' \left(\begin{array}{c} I_r \\ 0 \end{array} \right), \qquad (9.40) \\
R_{22} & \equiv - \left(\begin{array}{cc} 0 & I_{p-r} \end{array} \right) M \left(\int R dW' \right) \overline{\alpha}. \qquad (9.41)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
T^{-3/2} \beta_{\perp}' \sum_{t=1}^{[Ts]} y_t^{*} & = \left(\begin{array}{cc} 0 & I_{p-r} \end{array} \right) M \int_0^s \left(DW + T^{-1/2} R \right) + O_P\left(T^{-1}\right) \\
& = C_{22} \overline{\alpha}_{\perp}' \int_0^s W + T^{-1/2} R_3(s) + O_P\left(T^{-1}\right), \qquad (9.42)
\end{aligned}
$$

where

$$
R_3(s) \equiv \left(\begin{array}{cc} 0 & I_{p-r} \end{array} \right) M \int_0^s R. \qquad (9.43)
$$

Now, the invertibility of C_{22} and (9.35)-(9.43) yield

$$
\begin{aligned}
& T \left\{ \widehat{\beta}^{*} \left(\beta' \widehat{\beta}^{*} \right)^{-1} \beta' \beta - \beta \right\}' T^{-3/2} \sum_{t=1}^{[Ts]} y_t^{*} \\
= \; & -\overline{\alpha}' \int dW W' \overline{\alpha}_{\perp} C_{22}' \left\{ C_{22} \overline{\alpha}_{\perp}' \left(\int W W' \right) \overline{\alpha}_{\perp} C_{22}' \right\}^{-1} C_{22} \overline{\alpha}_{\perp}' \int_0^s W \\
& + T^{-1/2} R_4(s) + O_P\left(T^{-1}\right) \\
= \; & -\overline{\alpha}' \int dW W' \overline{\alpha}_{\perp} \left\{ \overline{\alpha}_{\perp}' \left(\int W W' \right) \overline{\alpha}_{\perp} \right\}^{-1} \overline{\alpha}_{\perp}' \int_0^s W \\
& + T^{-1/2} R_4(s) + O_P\left(T^{-1}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
& R_4(s) \\
\equiv\ & -\left(R_{21}'+R_{22}'\right)\left\{C_{22}\overline{\alpha}_\perp'\left(\int WW'\right)\overline{\alpha}_\perp C_{22}'\right\}^{-1}C_{22}\overline{\alpha}_\perp'\int_0^s W \\
& -\overline{\alpha}'\int dWW'\overline{\alpha}_\perp C_{22}'\left\{C_{22}\overline{\alpha}_\perp'\left(\int WW'\right)\overline{\alpha}_\perp C_{22}'\right\}^{-1}\left(R_1+R_1'\right) \\
& \times\left\{C_{22}\overline{\alpha}_\perp'\left(\int WW'\right)\overline{\alpha}_\perp C_{22}'\right\}^{-1}C_{22}\overline{\alpha}_\perp'\int_0^s W \\
& -\overline{\alpha}'\int dWW'\overline{\alpha}_\perp C_{22}'\left\{C_{22}\overline{\alpha}_\perp'\left(\int WW'\right)\overline{\alpha}_\perp C_{22}'\right\}^{-1}R_3(s) \\
=\ & -\left(R_{21}'+R_{22}'\right)C_{22}'^{-1}\left\{\overline{\alpha}_\perp'\left(\int WW'\right)\overline{\alpha}_\perp\right\}^{-1}\overline{\alpha}_\perp'\int_0^s W \\
& -\overline{\alpha}'\int dWW'\overline{\alpha}_\perp\left\{\overline{\alpha}_\perp'\left(\int WW'\right)\overline{\alpha}_\perp\right\}^{-1}C_{22}^{-1}\left(R_1+R_1'\right) \\
& \times C_{22}'^{-1}\left\{\overline{\alpha}_\perp'\left(\int WW'\right)\overline{\alpha}_\perp\right\}^{-1}\overline{\alpha}_\perp'\int_0^s W \\
& -\overline{\alpha}'\int dWW'\overline{\alpha}_\perp\left\{\overline{\alpha}_\perp'\left(\int WW'\right)\overline{\alpha}_\perp\right\}^{-1}C_{22}^{-1}R_3(s).
\end{aligned}
$$

Consequently, via (9.28), (9.33) and (9.35),

$$
\begin{aligned}
& \beta'\beta\left(\widehat{\beta}^{*\prime}\beta\right)^{-1}T^{-1/2}\widehat{S}_{[Ts]} \\
=\ & -\overline{\alpha}'\left\{W(s)-\int dWW'\overline{\alpha}_\perp\left\{\overline{\alpha}_\perp'\left(\int WW'\right)\overline{\alpha}_\perp\right\}^{-1}\overline{\alpha}_\perp'\int_0^s W\right\} \\
& +T^{-1/2}R_5(s),
\end{aligned}
$$

where

$$R_5(s)\equiv R_4(s)+R(s). \tag{9.44}$$

Now, because from (9.33) and (9.35),

$$\Omega_{zz}^*=\begin{pmatrix} I_r & 0\end{pmatrix}MDD'M'\begin{pmatrix} I_r \\ 0\end{pmatrix}=\overline{\alpha}'\overline{\alpha} \tag{9.45}$$

is the covariance function of the Brownian motion $-\overline{\alpha}'W(s)$, which is r-dimensional, and because $\overline{\alpha}_\perp'W(s)$ is a $(p-r)$-dimensional Brownian motion with covariance function $\overline{\alpha}_{\perp}'\overline{\alpha}_{\perp}$, we may define the independent standard Brownian motions

$$\begin{aligned} W_1(s) &\equiv -\left(\overline{\alpha}'\overline{\alpha}\right)^{-1/2}\overline{\alpha}'W(s), \\ W_2(s) &\equiv \left(\overline{\alpha}_{\perp}'\overline{\alpha}_{\perp}\right)^{-1/2}\overline{\alpha}_{\perp}'W(s), \end{aligned}$$

which are r- and $(p-r)$-dimensional, respectively. It follows that

$$\widehat{S}_{[Ts]}'\widehat{\Omega}_{zz}^{*-1}\widehat{S}_{[Ts]} = V(s)'V(s) + T^{-1/2}\left\{R_6(s) + R_6(s)'\right\},$$

where

$$V(s) = W_1(s) - \int dW_1W_2'\left(\int W_2W_2'\right)^{-1}\int_0^s W_2$$

and

$$\begin{aligned} & R_6(s) \\ \equiv\; & R_5(s)'\,\Omega_{zz}^{*-1}\overline{\alpha}'\left\{W(s) - \int dWW'\overline{\alpha}_{\perp}\left\{\overline{\alpha}_{\perp}'\left(\int WW'\right)\overline{\alpha}_{\perp}\right\}^{-1}\overline{\alpha}_{\perp}'\int_0^s W\right\} \\ =\; & R_5(s)'\left(\overline{\alpha}'\overline{\alpha}\right)^{-1/2}V(s). \qquad (9.46) \end{aligned}$$

Hence,

$$T^{-2}\sum_{t=1}^{T}\widehat{S}_t'\widehat{\Omega}_{zz}^{*-1}\widehat{S}_t = T^{-2}\sum_{t=1}^{T}\widehat{S}_{[Ts]}'\widehat{\Omega}_{zz}^{*-1}\widehat{S}_{[Ts]} = \int V'V + T^{-1/2}\int\left(R_6 + R_6'\right). \quad (9.47)$$

Moreover,

$$\begin{aligned} & \left(\overline{\alpha}'\overline{\alpha}\right)^{-1/2}R_4(s) \\ =\; & -\left(\overline{\alpha}'\overline{\alpha}\right)^{-1/2}\left(R_{21}' + R_{22}'\right)C_{22}'^{-1}\left(\int W_2W'\overline{\alpha}_{\perp}\right)^{-1}\int_0^s W_2 \\ & -\int dW_1W_2'\left(\overline{\alpha}_{\perp}'\int WW_2'\right)^{-1}C_{22}^{-1}\left(R_1 + R_1'\right)C_{22}'^{-1}\left(\int W_2W'\overline{\alpha}_{\perp}\right)^{-1}\int_0^s W_2 \\ & -\int dW_1W_2'\left(\overline{\alpha}_{\perp}'\int WW_2'\right)^{-1}C_{22}^{-1}R_3(s) \qquad (9.48) \end{aligned}$$

where, from (9.40), (9.41) and $\begin{pmatrix} 0 & I_{p-r} \end{pmatrix}M = \begin{pmatrix} 0 & I_{p-r} \end{pmatrix}$,

$$\begin{aligned} & \left(\overline{\alpha}'\overline{\alpha}\right)^{-1/2}\left(R_{21}' + R_{22}'\right)C_{22}'^{-1}\left(\int W_2W'\overline{\alpha}_{\perp}\right)^{-1} \\ =\; & \left(\overline{\alpha}'\overline{\alpha}\right)^{-1/2}\begin{pmatrix} I_r & 0 \end{pmatrix}M\left(\int dRW_2'\right)\left(\int W_2W_2'\right)^{-1} \\ & -\left(\int dW_1R'\right)\begin{pmatrix} 0 \\ I_{p-r} \end{pmatrix}C_{22}'^{-1}\left(\overline{\alpha}_{\perp}'\overline{\alpha}_{\perp}\right)^{-1/2}\left(\int W_2W_2'\right)^{-1}, \end{aligned}$$

from (9.38),

$$
\begin{aligned}
& \left(\overline{\alpha}_{\perp}' \int W W_2'\right)^{-1} C_{22}^{-1} R_1 C_{22}'^{-1} \left(\int W_2 W' \overline{\alpha}_{\perp}\right)^{-1} \\
= & \left(\int W_2 W_2'\right)^{-1} \left(\int W_2 R'\right) \begin{pmatrix} 0 \\ I_{p-r} \end{pmatrix} C_{22}'^{-1} \left(\overline{\alpha}_{\perp}' \overline{\alpha}_{\perp}\right)^{-1/2} \left(\int W_2 W_2'\right)^{-1}
\end{aligned}
$$

and via (9.43),

$$
\begin{aligned}
& \left(\overline{\alpha}_{\perp}' \int W W_2'\right)^{-1} C_{22}^{-1} R_3(s) \\
= & \left(\int W_2 W_2'\right)^{-1} \left(\overline{\alpha}_{\perp}' \overline{\alpha}_{\perp}\right)^{-1/2} C_{22}^{-1} \begin{pmatrix} 0 & I_{p-r} \end{pmatrix} \int_0^s R.
\end{aligned}
$$

Hence, defining

$$
\begin{aligned}
F_1 & \equiv \left(\overline{\alpha}' \overline{\alpha}\right)^{-1/2} \begin{pmatrix} I_r & 0 \end{pmatrix} M, \\
F_2 & \equiv \left(\overline{\alpha}_{\perp}' \overline{\alpha}_{\perp}\right)^{-1/2} C_{22}^{-1} \begin{pmatrix} 0 & I_{p-r} \end{pmatrix},
\end{aligned}
$$

we have via (9.44) and (9.48) that

$$
\begin{aligned}
& \left(\overline{\alpha}' \overline{\alpha}\right)^{-1/2} R_5(s) \\
= & \left(\overline{\alpha}' \overline{\alpha}\right)^{-1/2} R(s) \\
& - \left(F_1 \int dR W_2' + \int dW_1 R' F_2'\right) \left(\int W_2 W_2'\right)^{-1} \int_0^s W_2 \\
& - \int dW_1 W_2' \left(\int W_2 W_2'\right)^{-1} \left(\int W_2 R' F_2' + F_2 \int R W_2'\right) \left(\int W_2 W_2'\right)^{-1} \int_0^s W_2 \\
& - \int dW_1 W_2' \left(\int W_2 W_2'\right)^{-1} F_2 \int_0^s R.
\end{aligned}
$$

Note the simplifications

$$
\begin{aligned}
F_1 & = \left(\overline{\alpha}' \overline{\alpha}\right)^{-1/2} \begin{pmatrix} I_r & -\Omega_{zw} \Omega_{ww}^{-1} \end{pmatrix} \\
& = \left(\overline{\alpha}' \overline{\alpha}\right)^{-1/2} \begin{pmatrix} I_r & -C_{12} C_{22}^{-1} \end{pmatrix} \\
& = \left(\overline{\alpha}' \overline{\alpha}\right)^{-1/2} \begin{pmatrix} I_r & -\overline{\alpha}' \Gamma \overline{\beta}_{\perp} \end{pmatrix} \\
& = \left(\alpha' \alpha\right)^{1/2} \begin{pmatrix} I_r & -\left(\alpha' \alpha\right)^{-1} \alpha' \Gamma \beta_{\perp} \left(\beta_{\perp}' \beta_{\perp}\right)^{-1} \end{pmatrix}
\end{aligned}
$$

and

$$\begin{aligned} F_2 &= \left(\overline{\alpha}_\perp'\overline{\alpha}_\perp\right)^{-1/2}\overline{\alpha}_\perp'\Gamma\overline{\beta}_\perp\begin{pmatrix} 0 & I_{p-r}\end{pmatrix} \\ &= \left(\alpha_\perp'\alpha_\perp\right)^{1/2}\overline{\alpha}_\perp'\Gamma\overline{\beta}_\perp\begin{pmatrix} 0 & I_{p-r}\end{pmatrix} \\ &= \left(\alpha_\perp'\alpha_\perp\right)^{-1/2}\alpha_\perp'\Gamma\beta_\perp\left(\beta_\perp'\beta_\perp\right)^{-1}\begin{pmatrix} 0 & I_{p-r}\end{pmatrix}. \end{aligned}$$

Hence, with $\Gamma = \Gamma_i$, it is seen as in the proof of theorem 2.2 that the result follows if the suprema of the sequences $\{\alpha'\Gamma_i\beta_\perp\}_i$ and $\{\alpha_\perp'\Gamma_i\beta_\perp\}_i$ are finite, which is assumption 4. ■

Proof of lemma 3.1: Because PC_r is a function of $\overline{c}_k(r)$ and LR_r is a function of $\overline{LR}_{NT}(H(r)|H(p))$, it is enough to show that $\overline{c}_k(r)$ and $\overline{LR}_{NT}(H(r)|H(p))$ are asymptotically independent. Moreover, these quantities in turn are averages of the $\widehat{c}$ and $LR_{NT}(H(r)|H(p))$quantities over the panel, which by assumption has independent individuals. Hence, it is enough to show asymptotic independence *within* panels. Now, for a given individual, we have as $T \to \infty$ that

$$\begin{aligned} \widehat{c} &\overset{w}{\to} X = \int_0^1 V(s)' V(s)\, ds, \\ LR_{NT}(H(r)|H(p)) &\overset{w}{\to} Z \equiv \operatorname{tr}\left\{\int dW_2W_2'\left(\int W_2W_2'\right)^{-1}\int W_2dW_2'\right\}, \end{aligned}$$

where

$$V(s) = W_1(s) - \int dW_1W_2'\left(\int W_2W_2'\right)^{-1}\int_0^s W_2(r)\, dr,$$

with the r and $p-r$ dimensional independent Brownian motions W_1 and W_2. We need to show that X and Z are independent. To see this, it is sufficient to prove that $V(s)$ and Z are independent for all s belonging to the unit interval. Now, observe that for s arbitrary (in the unit interval), $W_1(s)$, $\int dW_1W_2'$ and $\int_0^s W_2(r)\, dr$ are independent of $\int W_2W_2'$. Regarding $W_1(s)$, this is so because $W_1(s)$ and $\int W_2W_2'$ are uncorrelated, $W_1(s)$ is normally distributed and $\int W_2W_2'$ may be arbitrarily well approximated by a function of a finite number of normal random variables (see Chan and Wei (1988)). The same arguments go through to show that $\int dW_1W_2'$ and $\int_0^s W_2(r)\, dr$ are independent of $\int W_2W_2'$. Furthermore, it is similarly seen that $W_1(s)$, $\int dW_1W_2'$ and $\int_0^s W_2(r)\, dr$ are all independent of $\int W_2dW_2'$, and this independence holds also conditionally on $X \equiv \int W_2W_2'$. Hence, denoting the marginal densities by f_V, f_Z and f_X, the simultaneous densities by $f_{V,Z}$ etcetera and the conditional densities by e.g. $f_{V|X}$, we find

$$f_{V,Z} = \int f_{V,Z|X}\, dF_X = \int f_{V|X} f_{Z|X}\, dF_X = f_V \int f_{Z|X}\, dF_X = f_V f_Z.$$

Consequently, $V(s)$ and Z are independent for all s in the unit interval, as was to be shown. ■