Gain, Loss, and Asset Pricing: It is Much Easier. A note*

I.R. Longarela
Department of Finance
Stockholm School of Economics†

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Abstract

Bernardo and Ledoit (2000) develop a very appealing framework to compute pricing bounds based on what they call gain-loss ratio. Their method has many advantages and very interesting properties and so far one important drawback: the complexity of the numerical computation of the pricing bounds. In this note we provide a simple procedure for their computation which only entails solving a linear optimization program.

We will follow as closely as possible Bernardo and Ledoit's notation and we will concentrate on the finite-state framework. Thus, consider a two-period economy with S future states of nature which occur with strictly positive probabilities p_j, $j = 1, \ldots, S$. Let Z be the space of portfolio payoffs which is spanned by a set of N payoffs $\widetilde{z}^1, \ldots, \widetilde{z}^N$. Every $\widetilde{z} \in Z$ is a random variable $\widetilde{z} = [z_1, \ldots, z_S]$.

Asset prices are given by a linear function π defined on Z, that is, the portfolio with payoff $\widetilde{z} \in Z$ has price $\pi(\widetilde{z})$. We assume absence of arbitrage and hence, there exists at least one random variable $\widetilde{m} > 0$ such that $E(\widetilde{m}\widetilde{z}) =$

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†Address: Box 6501, S-113 83 Stockholm, telephone: +46-8-7369167, fax: +46-8-312327, e-mail: finir@hhs.se, web: http://www.hhs.se/personal/rodriguez.

$\pi(z)\ \forall\, \widetilde{z} \in Z$. Define the set $M = \left\{ \widetilde{m} : \widetilde{m} \in \Re^S_{++} \text{ and } E(\widetilde{m}\widetilde{z}) = \pi(\widetilde{z})\ \ \forall \widetilde{z} \in Z \right\}$, that is, the set of admissible *stochastic discount factors*. Each $\widetilde{m} \in M$ has an associated vector of state prices given by $\mu_j = p_j m_j$, $j = 1, \ldots, S$, where m_j represents the value of $\widetilde{m}$ at state of nature j.

Also, we will assume that there is a riskless asset with return r_F and we will define the set of excess payoffs $X = \{\widetilde{z} - (1 + r_F)\,\pi(\widetilde{z}) : \widetilde{z} \in Z\}$.

Consider a strictly positive benchmark stochastic discount factor $\widetilde{m}^* = [m_1, \ldots, m_S]$. This random variable correctly prices the assets in Z if and only if $E(\widetilde{m}^*\widetilde{x}) = 0\ \forall \widetilde{x} \in X$ or equivalently, if and only if for all $\widetilde{x} \in X$

$$\frac{E^*(\widetilde{x}^+)}{E^*(\widetilde{x}^-)} = 1$$

where $E^*(\cdot)$ is the expectation under the risk-neutral probabilities $\mu_j^* = p_j m_j^*$, $j = 1, \ldots, S$ and $\widetilde{x} = \widetilde{x}^+ - \widetilde{x}^-$ is the decomposition of a payoff into its positive part $\widetilde{x}^+ = \max(\widetilde{x}, 0)$ and its negative part $\widetilde{x}^- = \max(-\widetilde{x}, 0)$. Bernardo and Ledoit call $E^*(\widetilde{x}^+)$ the gain, $E^*(\widetilde{x}^-)$ the loss and $E^*(\widetilde{x}^+)/E(\widetilde{x}^-)$ the gain-loss ratio.

For each $\widetilde{m} \in M$ define the value

$$L_{\widetilde{m}} \equiv \frac{\max\limits_{j=1,\ldots,S} \left(m_j/m_j^*\right)}{\min\limits_{j=1,\ldots,S} \left(m_j/m_j^*\right)}$$

and consider a payoff $\widetilde{z}^* \notin Z$. Bernardo and Ledoit define pricing bounds on $\widetilde{z}^*$ as the solution to the programs

$$\min_{\substack{\widetilde{m} \in M \\ L_{\widetilde{m}} \le \overline{L}}} E(\widetilde{m}\widetilde{z}^*) \tag{1}$$

and

$$\max_{\substack{\widetilde{m} \in M \\ L_{\widetilde{m}} \le \overline{L}}} E(\widetilde{m}\widetilde{z}^*) \tag{2}$$

where $\overline{L}$ is a ceiling to be set by the user which must satisfy $\overline{L} \ge \min_{\widetilde{m} \in M} L_{\widetilde{m}}$. The economic intuition behind each value $L_{\widetilde{m}}$ can be easily deduced from Theorem 1 in Bernardo and Ledoit (2000). $L_{\widetilde{m}}$ gives the maximum gain-loss ratio for excess payoffs in the span of all contingent claims under the extension of π to all $z \in \Re^S$ implied by $\widetilde{m}$.

These authors devote three pages to give details on a numerical procedure to compute the bounds on a call price where the space of basis assets is the underlying stock and a bond. This procedure gets even less tractable if we want to include more assets in the set of basis assets. The following result provides the key to our simple numerical recipe.

Proposition 1 *$L_{\widetilde{m}} \leq \overline{L}$ if and only if there exist two constants θ_1^* and θ_2^* such that $\frac{\theta_2^*}{\theta_1^*} = \overline{L}$ and*

$$\theta_1^* \leq \frac{\mu_j}{\mu_j^*} \leq \theta_2^*, \qquad j = 1, \ldots, S. \tag{3}$$

Proof. Note that since the true probabilities cancel (3) is equivalent to

$$\theta_1^* \leq \frac{m_j}{m_j^*} \leq \theta_2^*, \qquad j = 1, \ldots, S$$

which can be rewritten as

$$\theta_1^* \leq \min_{j=1,\ldots,S} \frac{m_j}{m_j^*} \qquad \text{and} \qquad \theta_2^* \geq \max_{j=1,\ldots,S} \frac{m_j}{m_j^*}. \tag{4}$$

Finally, (4) together with $\theta_2^*/\theta_1^* = \overline{L}$ gives a necessary and sufficient condition for

$$L_{\widetilde{m}} \leq \overline{L}.$$

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Thus, the above result allows us to transform a nonlinear restriction $L_m \leq \overline{L}$ into a linear one and hence, the computation of the bounds can be done by just solving

$$\min_{\mu_1,\ldots,\mu_S,\theta_1,\theta_2} \sum_{j=1}^{S} \mu_j z_j^* \tag{5}$$

$$\text{s.t.} \begin{cases} \sum_{j=1}^{S} \mu_j z_j^i = \pi\left(\widetilde{z}^i\right), & i = 1, \ldots, N \\ \theta_1 \leq \frac{\mu_j}{\mu_j^*} \leq \theta_2, & j = 1, \ldots, S \\ \theta_2 = \theta_1 \overline{L} \\ \theta_1 \end{cases}$$

and

$$\max_{\mu_1,\cdots,\mu_S,\theta_1,\theta_2} \sum_{j=1}^{S} \mu_j z_j^* \tag{6}$$

$$\text{s.t.} \begin{cases} \sum_{j=1}^{S} \mu_j z_j^i = \pi\left(\widetilde{z}^i\right), \qquad i = 1, \ldots, N \\ \theta_1 \leq \frac{\mu_j}{\mu_j^*} \leq \theta_2, \qquad j = 1, \ldots, S \\ \theta_2 = \theta_1 \overline{L} \\ \theta_1 \geq 0 \end{cases} .$$

That's it! The simplex method takes care of everything. The first set of constraints guarantees that $[\mu_1, \ldots, \mu_S]$ is a vector of state prices. Since $\overline{L}$ and $\widetilde{m}^*$ are strictly positive, for $\overline{L} < \infty$ we have that $\theta_1 \geq 0$ implies that $\mu_j p_j > 0$, $j = 1, \ldots, S$ and hence, the associated $\widetilde{m}$ is an element of M.

We have replicated the computation of the values in Figure 1 of Bernardo and Ledoit (2000). The approximation of the continuous support of the stock payoff was made by using 125 equally spaced states of nature. Its accuracy can be measured by comparing the true Black-Scholes prices with the values obtained for $\overline{L} = 1$. The maximum and mean difference in relative terms over the whole set of initial stock prices (80-110) was .0614% and .0204%, respectively. The corresponding Fortran routine is available upon request.[1]

Finally, the proposition below proves that from the dual of the above two linear programs we obtain Bernardo and Ledoit's dual expression of the bounds in (1) and (2).

Proposition 2 *The dual expressions of (5) and (6) are equivalent to*

$$\max_{\substack{\widetilde{z} \in Z \\ \frac{E^*\left[(\widetilde{z}^* - \widetilde{z})^+\right]}{E^*\left[(\widetilde{z}^* - \widetilde{z})^-\right]} \geq \overline{L}}} \pi\left(\widetilde{z}\right)$$

and

$$\min_{\substack{\widetilde{z} \in Z \\ \frac{E^*\left[(\widetilde{z}^* - \widetilde{z})^+\right]}{E^*\left[(\widetilde{z}^* - \widetilde{z})^-\right]} \geq \overline{L}}} \pi\left(\widetilde{z}\right),$$

respectively.

[1] Its execution for all $\overline{L} = 1, 2, \ldots, 10$ takes ten minutes in total in a Pentium III 500 with the Visual Fortran 5.0 compiler.

Proof. We will only prove the first equivalence. The dual of (5) is given by (see Luenberger (1973) for details)

$$\max_{c,d,v,w,\lambda_1,\lambda_2} \sum_{i=1}^{N} (c_i - d_i)\, \pi\left(\widetilde{z}^i\right)$$

$$\text{s.t} \begin{cases} \sum_{i=1}^{N} (c_i - d_i)\, z_j^i + v_j - w_j = z_j^*, \qquad j = 1, \ldots, S \\ -\sum_{j=1}^{s} \mu_j^* v_j - \lambda_1 \overline{L} + \lambda_2 \overline{L} = 0 \\ \sum_{j=1}^{s} \mu_j^* w_j + \lambda_1 - \lambda_2 \leq 0 \\ c_1, \ldots, c_n, d_1, \ldots, d_N \geq 0 \\ v_1, \ldots, v_S, w_1, \ldots, w_S, \lambda_1, \lambda_2 \geq 0 \end{cases},$$

which can be rewritten as

$$\max_{\widetilde{z} \in Z, \widetilde{y}, \lambda} \pi\left(\widetilde{z}\right)$$

$$\text{s.t.} \begin{cases} \widetilde{z} + \widetilde{y} = \widetilde{z}^* \\ E^*\left(\widetilde{y}^+\right) - \lambda \overline{L} = 0 \\ E^*\left(\widetilde{y}^-\right) - \lambda \leq 0 \end{cases}.$$

Finally, it is easy to see that the above three constraints are equivalent to

$$\frac{E^*\left[\left(\widetilde{z}^* - \widetilde{z}\right)^+\right]}{E^*\left[\left(\widetilde{z}^* - \widetilde{z}\right)^-\right]} \geq \overline{L}.$$

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References

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