Patience and Ultimatum in Bargaining

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Abstract

This study investigates in a two-stage two-player model how the decision to make an ultimatum and how much to demand depends on the impatience of the agents and the pie uncertainty. First, players simultaneously decide on their ultimatums. If the ultimatum(s) are compatible then the player(s) receive his (their) demand(s) in the second period and the eventually remaining player becomes residual claimant. If no ultimatums are made then there is a Rubinstein-Ståhl bargaining. Relative impatience induces ultimatums but does not affect the demanded amount. In a discrete (continuous) setting there exist no equilibrium without an ultimatum (with mutual ultimatums).

1. Introduction

Bargaining is the standard tool to divide a surplus among the concerned economic agents (henceforth players). One important factor determining the outcome is the

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patience of the players. The amount a player receives is typically increasing in his patience. A relatively impatient player may therefore be tempted to make an ultimatum prior to the bargaining in order to avoid the unfavorable bargaining situation. By making an irrevocable ultimatum he may increase his share of the surplus. When the surplus is uncertain this increase is at the expense of a possible break down in negotiations if the surplus is not large enough to satisfy the demand. The decision to make an ultimatum and, if so, how much to demand should reflect this trade off. This study investigates how the decision to make an ultimatum and how much to demand depends on the impatience of the agents and the uncertainty of the surplus.

The model is a two-stage game. In the first stage the players simultaneously decides on their ultimatums. The size of the pie is unknown to the players but its distribution and the discount factors of the players are common knowledge. In the second stage the size of the pie is reviled and if no ultimatums have been made it is divided through a Rubinstein&Ståhl bargaining. If only one player has made an ultimatum and the pie exceeds the ultimatum then the player receives the demanded amount and the other player takes what is left. When two ultimatums have been made and the pie exceeds the sum of the ultimatums then both players receives their demanded amounts. Negotiations break down if the ultimatum(s) cannot be met and both players receive nothing. Two specific settings of this general model are investigated; one where the pie is discretely distributed over two outcomes and one where it is exponentially distributed. The main results are that the decision to make an ultimatum or not depend on the relative impatience of the players while the size of the ultimatum(s) does not depend on the impatience of the players. Moreover, if one player has made an ultimatum then it is never a strictly best response for the other player to make an ultimatum as well. The intuition is that ultimatums do not change the point in time at which payoffs are realized. The maximization problems are thus unaffected of the discount factors and the ultimatum that maximizes the expected payoff of an impatient player will also maximize that of a patient player. The future payoff from an ultimatum is thus the same for all types of player. The relative impatience of a player does affect the expected outcome of a bargaining and the relatively impatient player may therefore gain from making an ultimatum. If one player has made an ultimatum then the other player becomes residual claimant given that he does not make an ultimatum himself. In this situation making an ultimatum is weakly dominated by not making an ultimatum. In the discrete setting it is shown that ultimatums are always made in equilibrium and a condition for the existence of equilibria in which both players make ultimatums is derived. In the exponential setting there exist no equilibrium with mutual ultimatums and depending on the relative impatience of the players there exist equilibria in which no or one ultimatum is made. That there exist no equilibrium with mutual ultimatums is also shown to true in the general continuous case.

The two-stage model relates to the Nash demand game. Nash [7] considered a strategic model of bargaining that has come to play an important role in economic theory. Two players simultaneously make demands on a pie of known size. If the demands are compatible then the players receive the amount they demanded and otherwise disagreement occurs and the players receive nothing. The extension made in this study is to make the size of the pie uncertain and to make it possible for the players to refrain from making ultimatums and then to study whether it is in the interest of the players to do so or not. A similar approach is made by Güth and Ritzberger [3] but in their model players may differ in the concavity of their utility functions and there is no explicit bargaining procedure in the second stage. The pie is exponentially distributed. They show that commitment is a dominant strategy for favorable (low variance) distribution while the reversed is true for unfavorable distributions. There also exists a mixed equilibrium and equilibria in which one player commits. Young [10] show that players with high risk tolerance make higher demands than less risk-loving players.

Muthoo [5][6] studies partial commitment in a bargaining model where the size of the pie is known. The equilibrium ultimatum is increasing in the cost of revoking. Crawford [2] lets the cost of revoking be uncertain. Kambe [4] investigates imperfect commitment and reputation effects. Osborne and Rubinstein [8] and Bolt and Houba [1] investigate models where the players can undertake actions that affect the disagreement point. Schelling [9] discusses the role of ultimatums in his insightful book.

The general model is presented in Section 2, the discrete case in Section 3 and the exponential in Section 4. The discussion is in Section 5 and Section 6 contains a few examples. Proofs are in the Appendix.

2. The Model

The model consists of a two-stage game with the following structure.

1. Nature chooses the size of the pie. Without being informed about natures move player 1 and 2 simultaneously decide whether to make an ultimatum

or not. If a player decides to make an ultimatum he also decides upon how much to demand. The players may differ in their patience and their types is assumed to be common knowledge.

2. The size of the pie and the ultimatums become common knowledge. If no ultimatum has been made the players bargain over the pie a' la Rubinstein&Ståhl. If only one ultimatum has been made and the size of the pie exceeds the ultimatum then the player who made the ultimatum receives the demanded amount and the other player gets the remaining part. If they both make ultimatums that can be simultaneously satisfied then each player receives the demanded amount. If the ultimatum or the sum of the ultimatums exceeds the size of the pie then the bargaining breakes down and the players get zero.

In the next two sections the two stages are described in detail. The second stage is presented first and thereafter the first stage.

2.1. The Second Stage

The second stage begins with the players being informed about the size of the pie, denoted X, and the ultimatum made by the other player. Let $\underline{x}_1 \geq 0$ and $\underline{x}_2 \geq 0$ denote the ultimatum of player 1 and 2, respectively. If $\underline{x}_1 = 0$ then player 1 made no ultimatum in the first stage and if he did then his demand is described by $\underline{x}_1 > 0$. Let x_1 (x_2) denote the amount of money received by player 1 (2). His objective is to maximize the amount of money and his discount factor is δ_1, i.e. her utility from x_1 one period from now is $\delta_1 x_1$. When no ultimatums has been made ($\underline{x}_1 = \underline{x}_2 = 0$) then the two players bargain a' la Rubinstein-Stahl over the pie. There is no reason to give the first move to one of the players so a fair coin is tossed before the bargaining starts. Each player has with probability 1/2 the privilege of making the first move and player 1 thus expects to get

$$\begin{aligned} x_1 &= \left(\frac{1-\delta_2}{1-\delta_1\delta_2} + \left(1 - \frac{1-\delta_1}{1-\delta_1\delta_2}\right)\right)\frac{X}{2} \\ &= \left(\frac{1}{2} + \frac{\delta_1 - \delta_2}{2\left(1-\delta_1\delta_2\right)}\right)X \\ &= b\left(\delta\right)X \end{aligned} \tag{1}$$

where $\delta = (\delta_1, \delta_2)$. Hence, when no ultimatums has been made in the first stage then player 2's expected share of the pie is $1 - b(\delta)$. Notice that if the players have identical discount factors then each of them expects to receive half of the pie. A player who is alone of making an ultimatum gets what he demanded if the pie exceeds the ultimatum. Otherwise both players get zero. When both players make ultimatums then they receive their demands if the pie exceeds the sum of the demands and zero otherwise. The expected payoff for player 1[1] is summarized below.

$$x_1 = \begin{cases} b(\delta)X & if \quad \underline{x}_1 = \underline{x}_2 = 0 \\ X - \underline{x}_2 & if \quad \underline{x}_1 = 0 \text{ and } 0 < \underline{x}_2 < X \\ \underline{x}_1 & if \quad \underline{x}_1 > 0 \text{ and } \underline{x}_1 + \underline{x}_2 \leq X \\ 0 & if \quad \underline{x}_1 + \underline{x}_2 > X \end{cases}$$

If one player makes an ultimatum it will not be in the interest of the other player to make one since, if he does not, he will be the residual claimant. Making an ultimatum cannot increase his expected payoff. The assumption that a player only receives his demand if the pie exceeds the sum of the demands may therefore seem restrictive. However, notice that the alternative assumption, that the players divide remaining part of the pie according to some rule, does not change this. The assumption above is made for the reason of simplicity even though it may lead to inefficiencies.

2.2. The First Stage

In the beginning of the first stage nature decides on X, the size of the pie, which is distributed over a subset of $\mathbb{R}_+$ in accordance to the cumulative distribution F. Let f denote the corresponding probability density function. Without being informed about natures move the players simultaneously decide on what ultimatums to make. The distribution F and their types δ_1 and δ_2 are common knowledge. Player 1 holds a belief $\underline{x}_2^e$ over player 2's ultimatum and vice versa. Here, the focus is on pure strategies and the beliefs $\underline{x}_i^e$ are thus assumed to numbers, not distributions over numbers. Player 1 aims at maximizing his expected utility given his beliefs. His ultimatum will consequently be one of those solving his maximization problem, i.e.

$$\underline{x}_1 \in \arg\max_{y \geq 0} \delta_1 E\left[x_1 \mid y, \underline{x}_2^e\right].$$

[1] One should possibly consider the case where the two parties bargain over the remains when $\underline{x}_1, \underline{x}_2 > 0$ and $\underline{x}_1 + \underline{x}_2 < X$, i.e. $x_1 = \underline{x}_1 + b_1(\delta_1, \delta_2)(X - \underline{x}_1 - \underline{x}_2)$. Otherwise is it by construction not advantageous to make an ultimatum when one's opponent has made one.

Notice that the solution to the maximization problem does not depend on the discount factor of any of the players.

Proposition 1. *Any ultimatum maximizing the expected payoff of any of the players is independent of the discount factors.*

Proof. See Appendix.

The intuition behind Proposition 1 is that an ultimatum does not change the point in time where payoffs are realized. If an ultimatum maximizes the payoff of an impatience player then it also maximizes that of a more patient player. In equilibrium both players anticipate the ultimatum of the other player correctly (rational expectations). An equilibrium is then a pair of ultimatums simultaneously solving the two players maximization problems given $\underline{x}_i^e = \underline{x}_i$.

In this general setting it is difficult to show the existence of a pure equilibrium and to derive equilibrium properties. The standard theorems does not apply; the strategy spaces may not be compact, the expected payoffs may not be continuous at $\underline{x}_i = 0$, the pie could be discretely distributed resulting in a game without a pure equilibrium (rock-paper-scissors game). In order to simplify and to make results more intuitive two special cases will be studied, each represented by a special distribution F. The first case is a discrete distribution over only two outcomes and the second case is the exponential distribution.

3. The Discrete Case

The pie is small and its value is low ($X = x^L \geq 0$) with probability p. With probability $(1-p)$ it is large and its value is high ($X = x^H > x^L$). If no one of the players makes an ultimatum then the expected payoff of player 1 is

$$b(\delta)\left(px^L + (1-p)\,x^H\right) \tag{2}$$

Suppose player 1 but not player 2 makes an ultimatum $\underline{x}_1 > 0$. If his ultimatum does not exceed x^L then his expected payoff is

$$\underline{x}_1 \tag{3}$$

and if the ultimatum exceeds x^L then

$$(1-p)\,\underline{x}_1 \tag{4}$$

The expressions 3 and 4 are maximized at $\underline{x}_1 = x^L$ and $\underline{x}_1 = x^H$, respectively. Hence, if $p \geq \overline{p} \equiv \left(x^H - x^L\right)/x^H$ then $\underline{x}_1 = x^L$ is the best non-zero ultimatum and if $p \leq \overline{p}$ then $x_1 = x^H$ is the best non-zero ultimatum. Comparing 3 and 4 with 2 gives that $\underline{x}_1 = x^L$ is his best reply to $\underline{x}_2 = 0$ if $b(\delta) \leq x^L/E[X]$ and $p \geq \overline{p}$. Similarly is $\underline{x}_1 = x^H$ a best reply against $\underline{x}_2 = 0$ if $b(\delta) \leq x^H(1-p)/E[X]$ and $p \leq \overline{p}$. If the inequalities hold with equality or are violated then $\underline{x}_1 = 0$ is a best reply. Consider a situation where both players make an ultimatum. If player 2's demand exceeds x^L then player 1's best alternatives are to demand $x^H - \underline{x}_2$ or nothing at all. Both alternatives yield the same expected payoff, $(1-p)\left(x^H - \underline{x}_2\right)$. In the case where player 2 demands less than x^L the best alternative for player 1 is to demand nothing and thereby become residual claimant. The alternative, to demand $x^L - \underline{x}_2$ or $x^H - \underline{x}_2$, are both dominated by $\underline{x}_1 = 0$. The unique best reply is thus to demand nothing. To summarize, player 1's best reply correspondence is

$$\underline{x}_1^D(\underline{x}_2) = \begin{cases} \{x^L\} & if \quad \underline{x}_2 = 0,\ p > \overline{p} \text{ and } b(\delta) < x^L/E[X] \\ \{0, x^L\} & if \quad \underline{x}_2 = 0,\ p > \overline{p} \text{ and } b(\delta) = x^L/E[X] \\ \{x^L, x^H\} & if \quad \underline{x}_2 = 0,\ p = \overline{p} \text{ and } b(\delta) < x^L/E[X] \\ \{0, x^L, x^H\} & if \quad \underline{x}_2 = 0,\ p = \overline{p} \text{ and } b(\delta) = x^L/E[X] \\ \{x^H\} & if \quad \underline{x}_2 = 0,\ p < \overline{p} \text{ and } b(\delta) < x^H(1-p)/E[X] \\ \{0, x^H - \underline{x}_2\} & if \quad \underline{x}_2 \geq x^L \\ \{0\} & otherwise \end{cases} \tag{5}$$

Player 2' best-reply correspondence is defined analogously. The set of equilibria is then the set of fixed points to the combined best-reply correspondence $\underline{x}_1^D \times \underline{x}_2^D$. Let $M = \left\{(x_1, x_2) \mid x_1 + x_2 = x^H, x_1 \geq x^L, x_2 \geq x^L\right\}$ be the possibly empty set of non-zero ultimatums summing up to x^H.

Proposition 2. *In the discrete case the set of Nash equilibria is*

$$M \cup \begin{cases} \left\{\left(x^H, 0\right)\right\} & if \quad b(\delta) < \frac{E[X]-(1-p)x^H}{E[X]} \\ \left\{\left(x^H, 0\right), \left(0, x^H\right)\right\} & if \quad \frac{E[X]-(1-p)x^H}{E[X]} \leq b(\delta) \leq \frac{(1-p)x^H}{E[X]} \\ \left\{\left(0, x^H\right)\right\} & if \quad b(\delta) > \frac{(1-p)x^H}{E[X]} \end{cases}$$

if $p < \overline{p}$,

$$M \cup \begin{cases} \left\{\left(x^L,0\right),\left(x^H,0\right)\right\} & if \quad b\left(\delta\right) < \frac{E[X]-(1-p)x^H}{E[X]} \\ \left\{\left(x^L,0\right),\left(x^H,0\right),\left(0,x^L\right),\left(0,x^H\right)\right\} & if \quad \frac{E[X]-(1-p)x^H}{E[X]} \leq b\left(\delta\right) \leq \frac{(1-p)x^H}{E[X]} \\ \left\{\left(0,x^L\right),\left(0,x^H\right)\right\} & if \quad b\left(\delta\right) > \frac{(1-p)x^H}{E[X]} \end{cases}$$

if $p = \overline{p}$, *and*

$$M \cup \begin{cases} \left\{\left(x^L,0\right)\right\} & if \quad b\left(\delta\right) < \frac{E[X]-x^L}{E[X]} \\ \left\{\left(x^L,0\right),\left(0,x^L\right)\right\} & if \quad \frac{E[X]-x^L}{E[X]} \leq b\left(\delta\right) \leq \frac{x^L}{E[X]} \\ \left\{\left(0,x^L\right)\right\} & if \quad b\left(\delta\right) > \frac{x^L}{E[X]} \end{cases}$$

if $p > \overline{p}$.

Proof. See Appendix.

Corollary 3. *In the discrete setting:*
(i) An ultimatum is always made in equilibrium, i.e. $\underline{x}_1 = \underline{x}_2 = 0$ *is not a Nash equilibrium.*
(ii) There always exist an equilibrium where only one of the players makes an ultimatum. The ultimatum is not always made by the most impatient player.
(iii) If $x^H \geq 2x^L$ *then there always exist an equilibrium in which both players make ultimatums.*
(iv) If both players make ultimatums in equilibrium then the ultimatums are pie efficient with respect to the high outcome, i.e. $\underline{x}_1 + \underline{x}_2 = x^H$.

Proof. Follows from Proposition 2. ■

4. The Exponential Case

Let the size of the pie be exponentially distributed, $f(X) = \lambda e^{-\lambda X}$ and $F(X) = 1 - e^{-\lambda X}$ for all $x > 0$ and $\lambda > 0$. The expected value of the pie is $1/\lambda$ and the variance is $1/\lambda^2$.

Suppose player 1 wants to make a non-zero ultimatum and expects player 2 to make the ultimatum $\underline{x}_2^e \geq 0$. The best non-zero ultimatum is the solution to

$$\max_{\underline{x}_1 > 0} \delta_1 \underline{x}_1 e^{-\lambda\left(\underline{x}_1 + \underline{x}_2^e\right)}. \tag{6}$$

It turns out that the best non-zero ultimatum is constant with respect to δ_1, δ_2, and $\underline{x}_2^e$.

Lemma 4. *In the exponential setting the best non-zero ultimatum is* $1/\lambda$.

Proof. See Appendix.

The expected payoff for player 1 when no ultimatum has been made is $b_1(\delta)/\lambda$ and when making the ultimatum $1/\lambda$ it is $1/\lambda e$. When player 2 made the ultimatum $\underline{x}_2 > 0$ player 1's expected payoff from not making an ultimatum is $\underline{x}_2/e^{\lambda \underline{x}_2}$. Making the ultimatum $1/\lambda$ yields $1/\lambda e^{\lambda(\underline{x}_2+1/\lambda)}$. Comparing the payoffs above gives the best-reply correspondence for player 1

$$\underline{x}_1^C(\underline{x}_2) = \begin{cases} 1/\lambda & if \quad x_2 = 0 \text{ and } b(\delta) < 1/e \text{ or } x_2 \in (0, 1/\lambda e) \\ \{0, 1/\lambda\} & if \quad x_2 = 0 \text{ and } b(\delta) < 1/e \text{ or } x_2 = 1/\lambda e \\ 0 & otherwise \end{cases}$$

The best-reply correspondence for player 2 is defined analogously. Using the best-reply correspondences gives the set of equilibria.

Proposition 5. *In the continuous case with exponential distribution the set of equilibria is*

$$\begin{cases} \{(1/\lambda, 0)\} & if \quad b(\delta) < 1/e \\ \{(0,0), (1/\lambda, 0)\} & if \quad b(\delta) = 1/e \\ \{(0,0)\} & if \quad b(\delta) \in (1/e, (1-e)/e) \\ \{(0,0), (0, 1/\lambda)\} & if \quad b(\delta) = (e-1)/e \\ \{(0, 1/\lambda)\} & if \quad b(\delta) > (e-1)/e. \end{cases}$$

Proof. Follows from the best-reply correspondences. ■

Corollary 6. *There exists no equilibrium in which both players make an ultimatum. If an ultimatum is made in equilibrium then it is made by the most impatience player.*

Proof. Follows from 5. ■

4.1. The General Continuous Case

Consider the general continuous case where F is defined over $\mathbb{R}_+$. One important result obtained in the exponential setting carries over to this more general case. There exists no equilibrium in which both players make ultimatums.

Proposition 7. *If* $\underline{x}_i > 0$ *for* $i = 1, 2$ *then* $(\underline{x}_1, \underline{x}_2)$ *is not an equilibrium in the general continuous setting.*

Proof. See Appendix.

The intuition is that refraining from making an ultimatum increases the chances of receiving a payoff and results in a higher payoff in those cases a positive demand would have been satisfied. Making no ultimatum makes the player a residual claimant.

5. Discussion

In a simple two-stage two-player game it has been shown that only the choice whether to make an ultimatum or not depend on the relative impatience of the player. If an ultimatum is made then its magnitude is independent of the relative impatience of the demanding player. The set of equilibria for two specific setting was derived showing how sensitive the equilibrium set is of the assumed distribution.

Ultimatums are here in absolute terms and not in relative terms (e.g. 60% of X). The interpretation of this is that the size of the pie is unobservable to outside observers while the received amount is. A player that makes a relative demand can consequently not show that his demand has been met. This undermines the commitment effect of an ultimatum, e.g. a union leader who cannot show whether he has been tough or soft in wage negotiations is not credibly committed to an earlier demand. An ultimatum made in absolute terms enables him to show this and thereby to gain commitment. Ultimatums are assumed to be irrevocable and, hence, commitment to be complete. Complete commitment does not require irrevocable ultimatums, only that the cost of revoking is sufficiently high. The union leader may revoke from his ultimatum but if this is likely to cost him his leadership then he will never choose to do so.

6. Examples

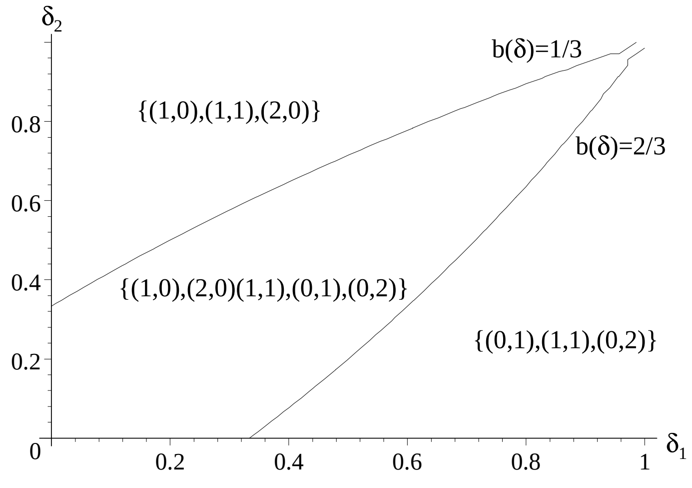
Reconsider the bivariate case and let $p = 1/2$, $x^L = 1$ and $x^H = 2$. Then $E[X] = 3/2$, $p = \overline{p}$ and $M = 1$. The set of equilibria is

$$\{(1,1)\} \cup \begin{cases} \{(1,0),(2,0)\} & if \quad b(\delta) < \frac{1}{3} \\ \{(1,0),(2,0),(0,1),(0,2)\} & if \quad \frac{1}{3} \leq b(\delta) \leq \frac{2}{3} \\ \{(0,1),(0,2)\} & if \quad b(\delta) > \frac{2}{3} \end{cases}$$

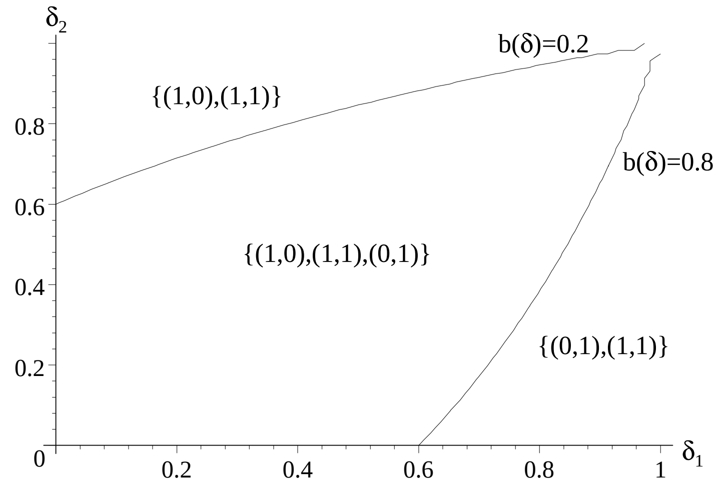
and it is illustrated in Figure **??**. For comparison, the case of $p = 3/4$ is illustrated in Figure **??**

In the exponential case, let $\lambda = 1$. Then $E[X] = \sigma^2 = 1$. The set of equilibria is

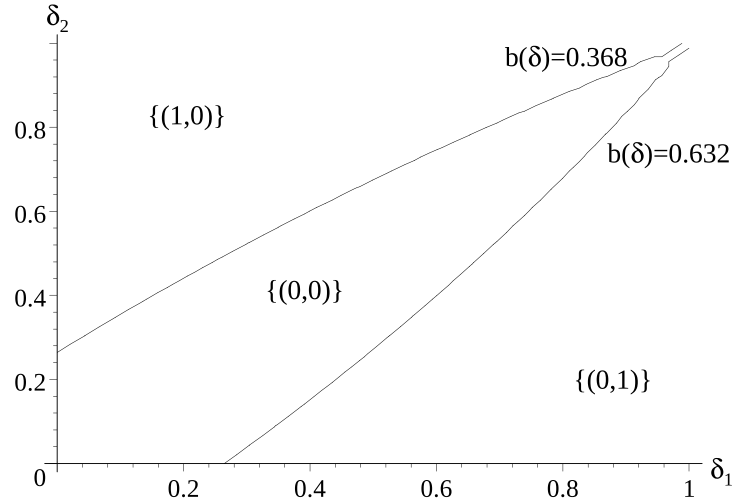
$$\begin{cases} \{(1,0)\} & if \quad b(\delta) < 0.368 \\ \{(0,0),(1,0)\} & if \quad b(\delta) = 0.368 \\ \{(0,0)\} & if \quad 0.368 < b(\delta) < 0.632 \\ \{(0,0),(0,1)\} & if \quad b(\delta) = 0.632 \\ \{(0,1)\} & if \quad b(\delta) > 0.632 \end{cases}$$



The set of equilibria for the discrete example.



The discrete case when $p = 3/4$.



The set of equilibria in the exponential example. At the bounderies between two sets the set of equilibria is the union of the two smaller sets.

7. Appendix

Proof 1. Maximizing $\delta_1 E[x_1 \mid \underline{x}_1, \underline{x}_2^e]$ over $\underline{x}_1$ is equivalent to maximizing $E[x_1 \mid \underline{x}_1, \underline{x}_2^e]$ over $\underline{x}_1$. ■

Proof 2. The outline is as follows: First it is shown that $(0,0)$ cannot be an equilibrium. Thereafter is the set of equilibria where only one player makes a non-zero ultimatum derived. Finally is the set of equilibria where both players make non-zero ultimatums derived.

(i) Here it is shown that $\underline{x}_1 = \underline{x}_2 = 0$ is not an equilibrium. First, suppose $x^L = 0$. Then $\underline{x}_1 = x^H$ is always the best non-zero ultimatum against $\underline{x}_2 = 0$. For $\underline{x}_1 = \underline{x}_2 = 0$ to be an equilibrium $\underline{x}_i = 0$ must yield a higher expected payoff than $\underline{x}_i = x^H$ for $i = 1, 2$, i.e.

$$b(\delta)(1-p)x^H \geq (1-p)x^H$$

and

$$(1-b(\delta))(1-p)x^H \geq (1-p)x^H.$$

Equilibrium thus requires $b(\delta) \geq 1$ and $b(\delta) \leq 0$. Hence, $\underline{x}_1 = \underline{x}_2 = 0$ cannot be an equilibrium. Next, suppose $x^L > 0$ and $\underline{x}_2 = 0$. Then $\underline{x}_1 = x^H$ is the best non-zero ultimatum if $p \geq \overline{p} \equiv \left(x^H - x^L\right)/x^H$ and $\underline{x}_1 = x^H$ if $p \leq \overline{p}$. Suppose $p \geq \overline{p}$. $\underline{x}_1 = \underline{x}_2 = 0$ to be an equilibrium requires

$$\frac{x^L}{E[X]} \leq b(\delta) \leq \frac{(1-p)\left(x^H - x^L\right)}{E[X]}.$$

In equilibrium $x^L \leq (1-p)\left(x^H - x^L\right)$ and this is most easily satisfied for $p = \overline{p}$ which gives

$$x^L \leq \frac{x^L}{x^H}\left(x^H - x^L\right) \Longleftrightarrow 0 \leq -x^L.$$

Since $x^L > 0$ by assumption this is not true and $\underline{x}_1 = \underline{x}_2 = 0$ is not an equilibrium. Next, suppose $p \leq \overline{p}$. Then $\underline{x}_1 = \underline{x}_2 = 0$ in equilibrium requires

$$\frac{(1-p)x^H}{E[X]} \leq b(\delta) \leq \frac{px^L}{E[X]}.$$

Hence, $(1-p)x^H \leq px^L$ which then must hold for $p = \overline{p}$ which is the most favorable case

$$\frac{x^L}{x^H}x^H \leq \frac{x^H - x^L}{x^H}x^L \Longleftrightarrow 0 \leq -x^L x^L.$$

This inequality cannot hold since $x^L > 0$ by assumption and $\underline{x}_1 = \underline{x}_2 = 0$ is not an equilibrium.
Suppose $\underline{x} = x^L$ and $\underline{x}_2 = 0$. $\underline{x}_2 = 0$ is a best reply to $\underline{x}_1 = x^L$. Equilibrium thus requires that $p \geq \overline{p}$ and

$$b(\delta) \leq \frac{x^L}{E[X]} \tag{7}$$

which makes $\underline{x}_1 = x^L$ a best reply against $\underline{x}_1 = 0$. Analogously is $\left(0, x^L\right)$ an equilibrium if

$$1 - b(\delta) \leq \frac{x^L}{E[X]} \Longleftrightarrow b(\delta) \geq \frac{E[X] - x^L}{E[X]}. \tag{8}$$

By assumption is $p \geq \overline{p}$ which makes $2x_L \geq E[X]$. This implies that either 7 or 8, or both, holds. Hence, either $\left(x^L, 0\right)$, $\left(0, x^L\right)$ or both are equilibria.
What if $p \leq \overline{p}$? Then $\underline{x}_2 = 0$ is a best reply against $\underline{x}_1 = x^H$ and $\underline{x}_1 = x^H$ is a best reply against $\underline{x}_2 = 0$ if

$$b(\delta) \leq \frac{(1-p)\, x^H}{E[X]}. \tag{9}$$

Similarly, $\left(0, x^H\right)$ is an equilibrium if

$$1 - b(\delta) \leq \frac{(1-p)\, x^H}{E[X]} \Longleftrightarrow b(\delta) \geq \frac{E[X] - (1-p)\, x^H}{E[X]}. \tag{10}$$

Since $p \leq \overline{p}$ is $2(1-p)\, x^H \geq E[X]$ and either 9 or 10, or both, holds. Hence, $\left(x^H, 0\right)$, $\left(0, x^H\right)$, or both are equilibria. Notice that 7 coincides with 9 at $p = \overline{p}$ just as 8 and 10.
From the best-reply correspondence it follows that if $\underline{x}_1, \underline{x}_2 > 0$ in equilibrium, then $\underline{x}_1, \underline{x}_2 \geq x^L$. Otherwise one of the players would have an incentive to deviate by not making an ultimatum. Hence, if $x^H < 2x^L$ then there exist no such equilibrium. Suppose $x^H \geq 2x^L$, $\underline{x}_1 \geq x^L, \underline{x}_2 \geq x^L$ and $\underline{x}_1 + \underline{x}_2 = x^H$. Then $\underline{x}_2 = x^H - \underline{x}_1$ is a best reply against $\underline{x}_1$ and vice versa. Hence, the set of equilibria in which both players make non-zero ultimatums is $\left\{(x_1, x_2) \mid x_1 + x_2 = x^H, x_1 \geq x^L, x_2 \geq x^L\right\}$.

Proof 4. The FONC to 6 is

$$\frac{\partial}{\partial \underline{x}_1} E\left[x_1 \mid \underline{x}_1, \underline{x}_2^e\right] = \delta_1 e^{-\lambda\left(\underline{x}_1 + \underline{x}_2^e\right)} \left(1 - \underline{x}_1 \lambda\right) = 0$$

which has the unique solution $\underline{x}_1^* = 1/\lambda$. The SONC evaluated at $\underline{x}_1^*$ is

$$\frac{\partial}{\partial \underline{x}_1} \big|_{\underline{x}_1 = \underline{x}_1^*} E\left[x_1 \mid \overset{*}{\underline{x}_1}, \underline{x}_2^e\right] = -\lambda e^{-\lambda\left(\underline{x}_1^* + \underline{x}_2^e\right)} < 0$$

showing that $\underline{x}_1^*$ gives the global maximum. Furthermore, $\underline{x}_1^*$ does not depend on δ_1, δ_2 or $\underline{x}_2^e$. ■

Proof 7 Suppose $(\underline{x}_1^*, \underline{x}_2)$ is an equilibrium where $\underline{x}_1^*, \underline{x}_2 > 0$. Then

$$E\left[x_1 \mid \underline{x}_1^*, \underline{x}_2\right] \geq E\left[x_1 \mid 0, \underline{x}_2\right].$$

Rewriting gives

$$\begin{aligned} \underline{x}_1^* \int_{\underline{x}_1^* + \underline{x}_2}^{\infty} f(t)\, dt &> \int_{\underline{x}_2}^{\infty} (t - \underline{x}_2) f(t)\, dt \\ &= \int_{\underline{x}_2}^{\underline{x}_1^* + \underline{x}_2} (t - \underline{x}_2) f(t)\, dt + \int_{\underline{x}_1^* + \underline{x}_2}^{\infty} (t - \underline{x}_2) f(t)\, dt \end{aligned}$$

and simplifying

$$0 > \int_{\underline{x}_2}^{\underline{x}_1^* + \underline{x}_2} (t - \underline{x}_2) f(t)\, dt + \int_{\underline{x}_1^* + \underline{x}_2}^{\infty} (t - \underline{x}_1^* - \underline{x}_2) f(t)\, dt.$$

Both terms on the right-hand side is positive and $(\underline{x}_1^*, \underline{x}_2)$ cannot be an equilibrium. ■

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