Appendix to "Endogenous market segmentation and the law of one price"

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In the following sections we calculate profits, prices and quantities for each strategy profile. The demand functions for the goods are:

	Home country	Foreign country
Home product	$p(x) = 1 - x - \gamma y$	$P(X) = A - X - \gamma Y$
Foreign product	$p(y) = 1 - y - \gamma x$	$P(Y) = A - Y - \gamma X$

As discussed in the paper we not only need to calculate profits for the four strategy profiles when all markets are open, (S,S),(S,N),(N,S) and (N,N) but also need to check whether firms would rather keep both markets open or close one of their markets. We model this as a two stage game, where the second stage is the segmentation-quantity game treated in the paper. In the first stage, a firm can choose whether to close the Home market. For simplicity, we assume that a closed market is also segmented (at zero cost). In analogy to the segment/not segment decision we use C to indicate when a firm has closed the Home market in the stage 1 game: $\Pi^F_{S,C}$ for instance is the profit in stage 1 for the Foreign firm if it closes the Home market, given that the Home firm operates on both markets and segments. In this extended (a first and a second period) game, we call a strategy profile an equilibrium if it is subgame perfect. Hence, we use backwards induction to find the equilibria of the extended game. Given the profits in the second stage, we analyze a stage one game of the following form:

		Foreign: Sell on both markets	Foreign: Close Home market
Home	Sell on both markets	1	2
	Close Home market	3	4

Each numbered square represents the equilibrium profits in the stage two game given the strategies in the first stage. Square 1 represents the main analysis in the paper.

1 Segment/not segment game when both markets are open (sq 1)

In the following section we calculate profits, prices and quantities for each of the strategy profiles (S,S),(S,N),(N,S) and (N,N) .

1.1 Both segments

Assume that both firms segment, then the Home firm's problem is

$$\Pi^H_{S,S} - K = \max_{x,X} xp(x) + XP(X) - tX - K,$$

and the Foreign firm's problem is

$$\Pi^F_{S,S} - K = \max_{y,Y} y\ p(y) + YP(Y) - ty - K.$$

1.1.1 Home *country* FOCs

The first-order conditions are

$$\frac{\partial}{\partial x}(xp(x) + XP(X) - tX - K) = 1 - 2x - \gamma y = 0,$$

which gives

$$x = \frac{1}{2} - \frac{1}{2}\gamma y,$$

and

$$\frac{\partial}{\partial y}(y\ p(y) + YP(Y) - ty - K) = 1 - 2y - \gamma x - t = 0,$$

which gives

$$y = \frac{1}{2} - \frac{1}{2}\gamma x - \frac{1}{2}t.$$

Combining the FOCs gives

$$\begin{aligned} y &= \frac{2(1-t) + \gamma(-1)}{4-\gamma^2}, \\ x &= \frac{2 + \gamma t + \gamma(-1)}{4-\gamma^2}. \end{aligned}$$

Thus, the Home firm's price in the Home country is $p(x) = \frac{2-\gamma+\gamma t}{(2-\gamma)(\gamma+2)}$, and the Foreign firm's price in the Home country is $p(y) = \frac{2-\gamma+2t-\gamma^2 t}{(2-\gamma)(\gamma+2)}$.

1.1.2 Foreign *country* FOCs

The first-order conditions are

$$\frac{\partial}{\partial X}\left(x\left(1-x-\gamma y\right)+X\left(A-X-\gamma Y\right)-tX-K\right)=A-2X-\gamma Y-t=0,$$

which gives

$$X=\frac{1}{2}A-\frac{1}{2}\gamma Y-\frac{1}{2}t,$$

and

$$\frac{\partial}{\partial Y}\left(y\left(1-y-\gamma x\right)+Y\left(A-Y-\gamma X\right)-ty-K\right)=A-2Y-\gamma X=0,$$

which gives

$$Y=\frac{1}{2}A-\frac{1}{2}\gamma X.$$

Combining the FOCs gives

$$\begin{aligned}X &= \frac{2A-\gamma A-2t}{4-\gamma^2},\\ Y &= \frac{2A-\gamma A+\gamma t}{4-\gamma^2}.\end{aligned}$$

Thus, the Home firm's price in the Foreign country is $P\left(X\right)=\frac{2A-\gamma A-\gamma^2 t+2t}{(2-\gamma)(\gamma+2)}$, and the Foreign firm's price in the Foreign Country is $P\left(Y\right)=\frac{2A-\gamma A+\gamma t}{(2-\gamma)(\gamma+2)}$.

1.1.3 The profits

The profits to the firms are then

$$\begin{aligned}\Pi_{S,S}^{H}-K &= x\left(1-x-\gamma y\right)+X\left(A-X-\gamma Y\right)-tX-K\\ &= \frac{1}{2}t\frac{-2\left(A-1\right)-\gamma t-2t}{\left(\gamma+2\right)\left(2-\gamma\right)}+\\ &\quad+\left(\frac{1}{4}\frac{\left(2A-t\right)^2}{\left(\gamma+2\right)^2}+\frac{1}{2}\frac{t^2}{\left(\gamma-2\right)^2}+\frac{1}{4}\frac{\left(t-2\right)^2}{\left(\gamma+2\right)^2}+\frac{1}{2}\frac{t^2}{2-\gamma}-K\right),\end{aligned}$$

and

$$\begin{aligned}\Pi_{S,S}^{F}-K &= y\left(1-y-\gamma x\right)+Y\left(A-Y-\gamma X\right)-ty-K\\ &= \frac{1}{2}t\frac{2\left(A-1\right)-\gamma t-2t}{\left(\gamma+2\right)\left(2-\gamma\right)}+\\ &\quad+\left(\frac{1}{4}\frac{\left(2A-t\right)^2}{\left(\gamma+2\right)^2}+\frac{1}{2}\frac{t^2}{\left(\gamma-2\right)^2}+\frac{1}{4}\frac{\left(t-2\right)^2}{\left(\gamma+2\right)^2}+\frac{1}{2}\frac{t^2}{2-\gamma}-K\right).\end{aligned}$$

Further, we have

$$\Pi^F_{S,S} - \Pi^H_{S,S} = 2t\frac{A-1}{4-\gamma^2} > 0.$$

1.2 Home segments / Foreign doesn't

Let us now assume that the Home firm segments and the Foreign firm does not segment. The Home firm's problem is

$$\Pi^H_{S,N} - K = \max_{x,X} xp(x) + XP(X) - tX - K,$$

and the Foreign firm's problem is

$$\Pi^F_{S,N} = \max_{y,Y} y\ p(y) + Y\ P(Y) - ty,$$
$$\text{s.t. } |\ p(y) - P(Y)| \leq t.$$

Remember that the Home firm has paid its Segmentation fee K and does not need to worry about fulfilling the constraint $P(Y) - p(y) < t$ (although its products enter into this inequality). Thus, given our assumptions, the Home firm optimizes without restrictions and ignores the knowledge how the Foreign firm sets quantities. The Foreign firm optimizes under the restriction $P(Y) - p(y) = t$. This effectively decides one of the two quantities for the Foreign firm and the Foreign firm will thus only have one FOC to consider.

1.2.1 Home *firm* FOCs

The FOCs give two equations

$$\begin{aligned}
\frac{\partial}{\partial x}\left(x(1-x-\gamma y) + X(A-X-\gamma Y) - tX - K\right) &= 1 - 2x - \gamma y = 0, \\
\frac{\partial}{\partial X}\left(x(1-x-\gamma y) + X(A-X-\gamma Y) - tX - K\right) &= A - 2X - \gamma Y - t = 0.
\end{aligned}$$

1.2.2 Foreign *firm* FOCs

Given that the Foreign firm must uphold LOP on its market, the firm can only choose one of its quantities freely. The LOP constraint

$$t = P(Y) - p(y) = A - Y - \gamma X - 1 + y + \gamma x,$$

gives y as a function of Y,

$$y = -A + Y + \gamma X + 1 - \gamma x + t.$$

The Foreign firm's problem is then

$$\max_{Y}\ y\ p(y) + Y\ P(Y) - ty,$$

given that y is a function of Y. The FOC is then (given restriction on y)

$$\frac{\partial}{\partial Y}\left(y\left(1-y-\gamma x\right)+Y\left(A-Y-\gamma X\right)-ty\right)=3A-4Y-3\gamma X-3t-1+\gamma x=0.$$

1.2.3 Equilibrium

There are now four equations that have to be fulfilled. Setting up the problem

$$\begin{pmatrix} -2 & 0 & -\gamma & 0 \\ 0 & -2 & 0 & -\gamma \\ -\gamma & \gamma & -1 & 1 \\ \gamma & -3\gamma & 0 & -4 \end{pmatrix}\begin{pmatrix} x \\ X \\ y \\ Y \end{pmatrix}=\begin{pmatrix} -1 \\ t-A \\ A-1-t \\ 3t+1-3A \end{pmatrix},$$

where the first row is the Home firm's FOC w.r.t. x, the second row w.r.t. X. Further, the third row is the constraint implied by LOP for the Foreign firm and finally, the fourth row is the Foreign firm's FOC w.r.t. Y. The determinant of the matrix is $D=2\left(\gamma-2\right)\left(\gamma+2\right)\left(-2+\gamma^2\right)$, which is non-zero for $\gamma\in[0,1]$.

The solution to this linear equation system is

$$\begin{aligned} Y &= \frac{-A\gamma^2+\gamma^2 t-1+3A-3t}{\left(\gamma+2\right)\left(2-\gamma^2\right)}, \\ y &= \frac{3-\gamma^2-A+t}{\left(\gamma+2\right)\left(2-\gamma^2\right)}, \\ X &= \frac{1}{2}\frac{-A\left(2\gamma^2+\gamma-4\right)+\gamma+\gamma t+2\gamma^2 t-4t}{\left(\gamma+2\right)\left(2-\gamma^2\right)}, \\ x &= \frac{1}{2}\frac{-2\gamma^2-\gamma+4+\gamma A-\gamma t}{\left(\gamma+2\right)\left(2-\gamma^2\right)}. \end{aligned}$$

Price differences are then

$$\begin{aligned} P\left(X\right)-p\left(x\right) &= t+\frac{\left(A-1-t\right)\left(1-\gamma\right)}{2-\gamma^2}, \\ P\left(Y\right)-p\left(y\right) &= t. \end{aligned}$$

1.2.4 Profits

The Home firm's profit is

$$\begin{aligned} \Pi^H_{S,N}-K &= x\left(1-x-\gamma y\right)+X\left(A-X-\gamma Y\right)-tX-K \\ &= \frac{1}{2}\frac{\left(A-t+1\right)^2}{\left(\gamma+2\right)^2}+\frac{1}{2}\left(A-1-t\right)^2\frac{\left(\gamma-1\right)^2}{\left(2-\gamma^2\right)^2}-K. \end{aligned}$$

The Foreign firm's profit is

$$\begin{aligned}\Pi^F_{S,N} &= y(1-y-\gamma x)+Y(A-Y-\gamma X)-ty \\ &= \frac{1}{2}\frac{(A-t+1)^2}{(\gamma+2)^2}+t(A-1-t)\frac{2-\gamma}{2-\gamma^2}.\end{aligned}$$

1.3 Foreign segments / Home doesn't

Let us now assume that the Foreign firm segments and the Home firm does not segment. The Home firm's problem is

$$\Pi^H_{N,S} = \max_{x,X} xp(x)+XP(X)-tX,$$
$$\text{s.t. } |p(x)-P(X)| \le t,$$

and the Foreign firm's problem is

$$\Pi^F_{N,S}-K = \max_{y,Y} y\,p(y)+Y\,P(Y)-ty-K.$$

Similar to the previous case, the Foreign firm optimizes without restrictions, while the Home firm optimizes under the restriction $P(X)-p(x)=t$. This effectively decides one of the two quantities for the Home firm and the Home firm will thus only have one FOC to consider.

1.3.1 Home *firm's* FOCs

Given that the Home firm must uphold LOP on its market, the firm can only choose one of its quantities freely. The LOP constraint

$$t = P(X)-p(x) = A-X-\gamma Y-1+x+\gamma y,$$

gives x as a function of X,

$$x = -A+X+\gamma Y+1-\gamma y+t.$$

The FOC is then (given that x is a function of X)

$$\frac{\partial}{\partial X}(x(1-x-\gamma y)+X(A-X-\gamma Y)-tX) = 3A-4X-3\gamma Y-3t-1+\gamma y=0.$$

1.3.2 Foreign *firm's* FOCs

The problem is

$$\max_{y,Y} y\,p(y)+Y\,P(Y)-ty-K.$$

The FOCs are

$$\begin{aligned}\frac{\partial}{\partial Y}(y(1-y-\gamma x)+Y(A-Y-\gamma X)-ty-K) &= A-2Y-\gamma X=0, \\ \frac{\partial}{\partial y}(y(1-y-\gamma x)+Y(A-Y-\gamma X)-ty-K) &= 1-2y-\gamma x-t=0.\end{aligned}$$

1.3.3 Equilibrium

There are now four equations that have to be fulfilled. Setting up the problem

$$\begin{pmatrix} -1 & 1 & -\gamma & \gamma \\ 0 & -4 & \gamma & -3\gamma \\ 0 & -\gamma & 0 & -2 \\ -\gamma & 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ X \\ y \\ Y \end{pmatrix} = \begin{pmatrix} A-1-t \\ -3A+3t+1 \\ -A \\ t-1 \end{pmatrix},$$

where the first row is the LOP constraint for the Home firm and the second row is the FOC w.r.t. X for the Home firm. Further, the third row is the FOC w.r.t. Y and finally, the fourth row is the FOC w.r.t. y for the Foreign firm. The determinant of the matrix is $D = -2(\gamma-2)(\gamma+2)\left(-2+\gamma^2\right)$, which is non-zero for $\gamma \in [0,1]$.

The solution to this linear equation system is

$$\begin{aligned} x &= \frac{3-\gamma^2-A+2\gamma t+t+\gamma^2 t}{(\gamma+2)(2-\gamma^2)}, \\ X &= \frac{3A-2\gamma t-\gamma^2 A-1-3t}{(\gamma+2)(2-\gamma^2)}, \\ y &= \frac{1}{2}\frac{4-\gamma+\gamma A-3\gamma t-2\gamma^2-4t}{(\gamma+2)(2-\gamma^2)}, \\ Y &= \frac{1}{2}\frac{4A-2\gamma^2 A-\gamma A+2\gamma^2 t+\gamma+3\gamma t}{(\gamma+2)(2-\gamma^2)}. \end{aligned}$$

Price differences are then

$$\begin{aligned} P(Y)-p(y) &= t\frac{\gamma^2+\gamma-1}{2-\gamma^2}+\frac{(A-1)(1-\gamma)}{2-\gamma^2}, \\ P(X)-p(x) &= t. \end{aligned}$$

1.3.4 Profits

The Home firm's profit is

$$\begin{aligned} \Pi^H_{N,S} &= x(1-x-\gamma y)+X(A-X-\gamma Y)-tX \\ &= \frac{1}{2}\frac{(t-A-1)^2}{(\gamma+2)^2}. \end{aligned}$$

The Foreign firm's profit is

$$\begin{aligned} \Pi^F_{N,S}-K &= y(1-y-\gamma x)+Y(A-Y-\gamma X)-ty-K \\ &= \frac{1}{2}\frac{(t-A-1)^2}{(\gamma+2)^2}+\frac{1}{2}\frac{((A-1)(1-\gamma)+\gamma t+t)^2}{(2-\gamma^2)^2}-K. \end{aligned}$$

1.4 No Segmentation

Let us now assume that no firm segments. The Home firm's problem is

$$\Pi^{H}_{N,N} = \max_{x,X} xp(x) + XP(X) - tX,$$
$$\text{s.t. } |p(x) - P(X)| \leq t.$$

and the Foreign firm's problem is

$$\Pi^{F}_{N,N} = \max_{y,Y} yp(y) + Y\ P(Y) - ty,$$
$$\text{s.t. } |p(y) - P(Y)| \leq t.$$

1.4.1 FOCs

Here we can take the relevant constraints and first-order conditions from the above sections where firms segment. From the section on the case Home segments / Foreign doesn't, we have the constraint

$$y = -A + Y + \gamma X + 1 - \gamma x + t,$$

and FOC is then

$$\frac{\partial}{\partial Y}\left(y(1-y-\gamma x) + Y(A - Y - \gamma X) - ty\right) = 3A - 4Y - 3\gamma X - 3t - 1 + \gamma x = 0.$$

Further, from the section on the case Foreign segments / Home doesn't, we have the constraint

$$x = -A + X + \gamma Y + 1 - \gamma y + t,$$

and the FOC is then

$$\frac{\partial}{\partial X}\left(x(1-x-\gamma y) + X(A - X - \gamma Y) - tX\right) = 3A - 4X - 3\gamma Y - 3t - 1 + \gamma y = 0.$$

1.4.2 Equilibrium

There are now four equations that have to be fulfilled,

$$\begin{aligned}
0 &= -\gamma x + \gamma X - y + Y - A + 1 + t : \quad y - \text{constraint} \\
0 &= \gamma x - 3\gamma X + 0 * y - 4Y + 3A - 3t - 1 : \ Y - \text{FOC} \\
0 &= -x + X - \gamma y + \gamma Y - A + 1 + t : \quad x - \text{constraint} \\
0 &= 0 * x - 4X + \gamma y - 3\gamma Y + 3A - 3t - 1 : \ X - \text{FOC}
\end{aligned}$$

As a linear system

$$\begin{pmatrix} -\gamma & \gamma & -1 & 1 \\ \gamma & -3\gamma & 0 & -4 \\ -1 & 1 & -\gamma & \gamma \\ 0 & -4 & \gamma & -3\gamma \end{pmatrix} \begin{pmatrix} x \\ X \\ y \\ Y \end{pmatrix} = \begin{pmatrix} A - 1 - t \\ -3A + 3t + 1 \\ A - 1 - t \\ -3A + 3t + 1 \end{pmatrix}.$$

The determinant of the matrix is $D = 4(\gamma - 1)(\gamma - 2)(\gamma + 2)(\gamma + 1)$, which is non-zero for $\gamma \in [0, 1)$. For $\gamma = 1$, there is a problem, since there is only one single LOP constraint and the firms then both optimize under the same constraint.

The solution to this linear equation system for $\gamma \neq 1$ is

$$\begin{aligned} Y &= \frac{1}{2}\frac{2\gamma A - 2\gamma t - 3t - 1 + 3A}{(\gamma + 2)(\gamma + 1)}, \\ x &= \frac{1}{2}\frac{2\gamma + 3 - A + t}{(\gamma + 2)(\gamma + 1)}, \\ y &= \frac{1}{2}\frac{2\gamma + 3 - A + t}{(\gamma + 2)(\gamma + 1)}, \\ X &= \frac{1}{2}\frac{2\gamma A - 2\gamma t - 3t - 1 + 3A}{(\gamma + 2)(\gamma + 1)}. \end{aligned}$$

Price differences are then

$$\begin{aligned} P(Y) - p(y) &= t, \\ P(X) - p(x) &= t, \end{aligned}$$

as we would expect.

1.4.3 The profits

The Home firm's profit is

$$\begin{aligned} \Pi^H_{N,N} &= x(1 - x - \gamma y) + X(A - X - \gamma Y) - tX \\ &= \frac{1}{2}\frac{(A - t + 1)^2}{(\gamma + 2)^2}, \end{aligned}$$

and the Foreign firm's profit is

$$\begin{aligned} \Pi^F_{N,N} &= y(1 - y - \gamma x) + Y(A - Y - \gamma X) - ty \\ &= \frac{1}{2}\frac{(A - t + 1)^2}{(\gamma + 2)^2} + t\frac{A - 1 - t}{\gamma + 1}. \end{aligned}$$

Then, $\Pi^F_{N,N} - \Pi^H_{N,N} = t\frac{(A-1-t)}{\gamma+1} > 0$.

1.5 Derivation of differences in profits between strategies

Given the profits derived in the previous sections we have

$$\begin{aligned} & \Pi^H_{S,S}(A, t, K) - \Pi^H_{N,S}(A, t) \\ = & \frac{1}{4}t\frac{A - 1}{-2 + \gamma} + \frac{1}{2}\frac{t^2}{(-2 + \gamma)^2} - \frac{1}{4}t\frac{A - 1}{\gamma + 2} + \frac{1}{2}\frac{A^2 + 1 - 2A}{(\gamma + 2)^2}, \end{aligned}$$

$$\begin{aligned}&\Pi^H_{S,N}(A,t,K)-\Pi^H_{N,N}(A,t)\\ =\;&\frac{1}{2}\frac{(\gamma-1)^2}{(\gamma^2-2)^2}A^2-(\gamma-1)^2\frac{t+1}{(\gamma^2-2)^2}A+\frac{1}{2}(t+1)^2\frac{(\gamma-1)^2}{(\gamma^2-2)^2}\end{aligned}$$

$$\begin{aligned}&\Pi^F_{S,S}(A,t,K)-\Pi^F_{S,N}(A,t)\\ =\;&-\frac{1}{4}t\frac{A-1}{-2+\gamma}+\frac{1}{2}\frac{t^2}{(-2+\gamma)^2}+\frac{1}{4}t\frac{A-1}{\gamma+2}+\frac{1}{2}\frac{A^2+1-2A}{(\gamma+2)^2}-\\ &-t\frac{-2A+2+2t+\gamma A-\gamma-\gamma t}{\gamma^2-2}\end{aligned}$$

$$\begin{aligned}&\Pi^F_{N,S}(A,t,K)-\Pi^F_{N,N}(A,t)\\ =\;&\frac{1}{2}\frac{(A-1-t)^2}{\gamma^2-2}-t\frac{A-1-t}{\gamma+1}-\\ &-\frac{1}{2}\frac{-2(t-1+A)(t+1-A)\gamma-3(A-1)^2-t(-2A+2+3t)}{(\gamma^2-2)^2}\end{aligned}$$

2 Market closedown by Foreign (sq2)

All the above was under the assumption that both firms were selling on both markets. This is not neccesarily optimal, in particular the Foreign firm might choose to ignore the Home (poor) market and achieve higher profits by setting an optimal price to the Foreign market only. Demand is then

	Home country	Foreign country
Home product	$p(x)=1-x$	$P(X)=A-X-\gamma Y$
Foreign product		$P(Y)=A-Y-\gamma X$

The payoffs to Foreign of this strategy depend on what Home does, so we have to examine both the case where Home segments and the case where it does not.

		Foreign
		Close
Home	Segment	$\Pi^H_{S,C}-K,\ \Pi^F_{S,C}$
	Not Segment	$\Pi^H_{N,C},\ \Pi^F_{N,C}$

2.1 Foreign close, home segment

Assume that the Foreign firm does not sell to the Home market, then the Home firm's problem is

$$\Pi^H_{S,C}=\max_{x,X}\ xp(x)+XP(X)-tX-K,$$

and the Foreign firm's problem is

$$\Pi^F_{S,C}=\max_{Y}\ YP(Y)$$

2.1.1 Home *country* FOCs

The first-order condition is

$$\frac{\partial}{\partial x}\left(xp\left(x\right)+XP\left(X\right)-tX-K\right)=1-2x=0,$$

which gives

$$x=\frac{1}{2},$$

Thus, the Home firm's price in the Home country is $p\left(x\right)=1/2$.

2.1.2 Foreign *country* FOCs

The first-order conditions are

$$\frac{\partial}{\partial X}\left(x\left(1-x\right)+X\left(A-X-\gamma Y\right)-tX-K\right)=A-2X-\gamma Y-t=0,$$

which gives

$$X=\frac{1}{2}A-\frac{1}{2}\gamma Y-\frac{1}{2}t,$$

and

$$\frac{\partial}{\partial Y}\left(Y\left(A-Y-\gamma X\right)-ty-K\right)=A-2Y-\gamma X=0,$$

which gives

$$Y=\frac{1}{2}A-\frac{1}{2}\gamma X.$$

Combining the FOCs gives

$$\begin{aligned} X &= \frac{2A-\gamma A-2t}{4-\gamma^2}, \\ Y &= \frac{2A-\gamma A+\gamma t}{4-\gamma^2}. \end{aligned}$$

Thus, the Home firm's price in the Foreign country is $P\left(X\right)=\frac{2A-\gamma A-\gamma^2 t+2t}{(2-\gamma)(\gamma+2)}$, and the Foreign firm's price in the Foreign Country is $P\left(Y\right)=\frac{2A-\gamma A+\gamma t}{(2-\gamma)(\gamma+2)}$.

2.1.3 The profits

The profits to the firms are then

$$\begin{aligned} \Pi^H_{S,C}-K &= x\left(1-x\right)+X\left(A-X-\gamma Y\right)-tX-K \\ &= \frac{1}{4}+\frac{2A-\gamma A-2t}{4-\gamma^2}\left(A-\frac{2A-\gamma A-2t}{4-\gamma^2}-\gamma\frac{2A-\gamma A+\gamma t}{4-\gamma^2}\right) \\ &\quad -t\frac{2A-\gamma A-2t}{4-\gamma^2}-K \end{aligned}$$

and

$$
\begin{aligned}
\Pi_{S,C}^{F} &= Y\left(A - Y - \gamma X\right) \\
&= \frac{2A - \gamma A + \gamma t}{4 - \gamma^2}\left(A - \frac{2A - \gamma A + \gamma t}{4 - \gamma^2} - \gamma\frac{2A - \gamma A - 2t}{4 - \gamma^2}\right).
\end{aligned}
$$

2.2 Foreign close, Home integrate

Let us now examine the case where the Foreign firm closes the Home market and that the Home firm does not segment. The Home firm's problem is

$$
\begin{gathered}
\Pi_{N,C}^{H} = \max_{x,X} xp\left(x\right) + XP\left(X\right) - tX, \\
\text{s.t. } \left| p\left(x\right) - P\left(X\right) \right| \leq t,
\end{gathered}
$$

and the Foreign firm's problem is

$$
\Pi_{N,C}^{F} = \max_{Y} Y \; P\left(Y\right).
$$

Similar to the previous case, the Foreign firm optimizes without restrictions, while the Home firm optimizes under the restriction $P\left(X\right) - p\left(x\right) = t$. This effectively decides one of the two quantities for the Home firm and the Home firm will thus only have one FOC to consider.

2.2.1 Home *firm's* FOCs

Given that the Home firm must uphold LOP on its market, the firm can only choose one of its quantities freely. The LOP constraint

$$
t = P\left(X\right) - p\left(x\right) = A - X - \gamma Y - 1 + x = t,
$$

gives x as a function of X,

$$
x = -A + X + \gamma Y + 1 + t.
$$

The FOC is then (given that x is a function of X)

$$
\frac{\partial}{\partial X}\left(x\left(1 - x\right) + X\left(A - X - \gamma Y\right) - tX\right) = 3A - 4X - 3\gamma Y - 3t - 1 = 0.
$$

2.2.2 Foreign *firm's* FOCs

The problem is

$$
\max_{Y} Y \; P\left(Y\right).
$$

The FOC is

$$
\frac{\partial}{\partial Y}\left(Y\left(A - Y - \gamma X\right)\right) = A - 2Y - \gamma X = 0,
$$

2.2.3 Equilibrium

We now have a linear equation system with the two FOCs and the LOP constraint to solve for three quantities

$$\begin{aligned} A - 2Y - \gamma X &= 0 &(1)\\ 3A - 4X - 3\gamma Y - 3t - 1 &= 0 &(2)\\ A - X - \gamma Y - 1 + x &= t &(3) \end{aligned}$$

which yields

$$\begin{aligned} X &= \frac{-6A + 3\gamma A + 6t + 2}{-8 + 3\gamma^2} &(4)\\ Y &= \frac{-4A + 3\gamma A - 3\gamma t - \gamma}{-8 + 3\gamma^2} &(5)\\ x &= -\frac{-2A + \gamma A + 2t + 6 - 2\gamma^2}{-8 + 3\gamma^2} &(6) \end{aligned}$$

2.2.4 Profits

The Home firm's profit is

$$\begin{aligned} \Pi^H_{N,C} &= x(1-x) + X(A - X - \gamma Y) - tX \\ &= 2\frac{\left(-2 - 2A + \gamma A + \gamma^2 + 2t\right)^2}{\left(-8 + 3\gamma^2\right)^2}. \end{aligned}$$

The Foreign firm's profit is

$$\begin{aligned} \Pi^F_{N,C} &= Y(A - Y - \gamma X) \\ &= \frac{\left(-4A + 3\gamma A - 3\gamma t - \gamma\right)^2}{\left(-8 + 3\gamma^2\right)^2}. \end{aligned}$$

3 Market closedown by Home (sq3)

Now instead examine the case where the Home firm closes its own market and only sells on the Foreign market. Demand is then

	Home country	Foreign country
Home product		$P(X) = A - X - \gamma Y$
Foreign product	$p(y) = 1 - y$	$P(Y) = A - Y - \gamma X$

The payoffs to Home of this strategy depend on what Foreign does, so we have to examine both the case where Foreign segments and that where it does not.

		Foreign	
		Segment	Not Segment
Home	Close	$\Pi^{\bar{H}}_{C,S}$, $\Pi^F_{C,S} - K$	$\Pi^{\bar{H}}_{C,S}$, $\Pi^F_{C,N}$

3.1 Home close, Foreign integrate

Let us now assume that the Home firm sells only on the foreign Market, while Foreign firm sells on both markets and integrates. The Home firm's problem is

$$\Pi^{H}_{C,N} = \max_{X} XP(X) - tX,$$

and the Foreign firm's problem is

$$\Pi^{F}_{C,N} = \max_{y,Y} y\ p(y) + Y\ P(Y) - ty,$$
$$\text{s.t. } |\ p(y) - P(Y)| \leq t.$$

The Foreign firm optimizes under the restriction $P(Y) - p(y) = t$. This effectively decides one of the two quantities for the Foreign firm and the Foreign firm will thus only have one FOC to consider.

3.1.1 Home *firm* FOCs

The FOC gives one equation

$$\frac{\partial}{\partial X}(X(A - X - \gamma Y) - tX) = A - 2X - \gamma Y - t = 0.$$

3.1.2 Foreign *firm* FOCs

Given that the Foreign firm must uphold LOP on its market, the firm can only choose one of its quantities freely. The LOP constraint

$$t = P(Y) - p(y) = A - Y - \gamma X - 1 + y,$$

gives y as a function of Y,

$$y = -A + Y + \gamma X + 1 + t.$$

The Foreign firm's problem is then

$$\max_{Y}\ y\ p(y) + Y\ P(Y) - ty,$$

given that y is a function of Y. The FOC is then (given restriction on y)

$$\frac{\partial}{\partial Y}(\,y(1-y) + Y(A - Y - \gamma X) - ty) = 3A - 4Y - 3\gamma X - 3t - 1 = 0.$$

3.1.3 Equilibrium

There are now three equations that have to be fulfilled (FOCs and LOP).

$$
\begin{aligned}
3A - 4Y - 3\gamma X - 3t - 1 &= 0 &(7)\\
A - 2X - \gamma Y - t &= 0 &(8)\\
y &= -A + Y + \gamma X + 1 + t &(9)
\end{aligned}
$$

The solution to this linear equation system is

$$
\begin{aligned}
Y &= \frac{-6A + 3\gamma A - 3\gamma t + 6t + 2}{-8 + 3\gamma^2} &(10)\\
X &= \frac{-4A + 3\gamma A - 3\gamma t - \gamma + 4t}{-8 + 3\gamma^2} &(11)\\
y &= -\frac{-2A + \gamma A - \gamma t + 2t + 6 - 2\gamma^2}{-8 + 3\gamma^2} &(12)
\end{aligned}
$$

3.1.4 Profits

The Home firm's profit is

$$
\begin{aligned}
\Pi^H_{C,N} &= X\left(A - X - \gamma Y\right) - tX\\
&= \frac{\left(-4A + 3\gamma A - 3\gamma t - \gamma + 4t\right)^2}{\left(-8 + 3\gamma^2\right)^2}.
\end{aligned}
$$

The Foreign firm's profit is

$$
\begin{aligned}
\Pi^F_{C,N} &= y\left(1 - y\right) + Y\left(A - Y - \gamma X\right) - ty\\
&= -\frac{-2A + \gamma A - \gamma t + 2t + 6 - 2\gamma^2}{-8 + 3\gamma^2}\left(1 + \frac{-2A + \gamma A - \gamma t + 2t + 6 - 2\gamma^2}{-8 + 3\gamma^2}\right) +\\
&\quad \frac{-6A + 3\gamma A - 3\gamma t + 6t + 2}{-8 + 3\gamma^2}\left(A - \frac{-6A + 3\gamma A - 3\gamma t + 6t + 2}{-8 + 3\gamma^2} - \gamma\frac{-4A + 3\gamma A - 3\gamma t - \gamma + 4t}{-8 + 3\gamma^2}\right.\\
&\quad + t\frac{-2A + \gamma A - \gamma t + 2t + 6 - 2\gamma^2}{-8 + 3\gamma^2}
\end{aligned}
$$

3.2 Home close, Foreign segment

The Home firm's problem is

$$
\Pi^H_{C,S} = \max_X XP\left(X\right) - tX,
$$

and the Foreign firm's problem is

$$
\Pi^F_{C,S} - K = \max_{y,Y} y\ p\left(y\right) + YP\left(Y\right) - ty - K.
$$

3.2.1 Home *country* FOC

The first-order conditions is

$$\frac{\partial}{\partial y}\left(y\ p\left(y\right)+YP\left(Y\right)-ty-K\right)=1-2y-t=0,$$

$$y=\frac{1}{2}-\frac{1}{2}t.$$

3.2.2 Foreign *country* FOCs

The first-order conditions are

$$\frac{\partial}{\partial X}\left(X\left(A-X-\gamma Y\right)-tX\right)=A-2X-\gamma Y-t=0,$$

which gives

$$X=\frac{1}{2}A-\frac{1}{2}\gamma Y-\frac{1}{2}t,$$

and

$$\frac{\partial}{\partial Y}\left(y\left(1-y-\gamma x\right)+Y\left(A-Y-\gamma X\right)-ty-K\right)=A-2Y-\gamma X=0,$$

which gives

$$Y=\frac{1}{2}A-\frac{1}{2}\gamma X.$$

Combining the FOCs gives

$$\begin{aligned} X &= \frac{-2A+\gamma A+2t}{-4+\gamma^2} \\ Y &= \frac{-2A+\gamma A-\gamma t}{-4+\gamma^2} \end{aligned}$$

Thus, the Home firm's price in the Foreign country is $P\left(X\right)=\frac{-2A+\gamma A-2t-2\gamma t}{-4+\gamma^2}$, and the Foreign firm's price in the Foreign Country is $P\left(Y\right)=\frac{-2A+\gamma A-\gamma t}{-4+\gamma^2}$.

3.2.3 The profits

The profits to the firms are then

$$\begin{aligned} \Pi^H_{C,S} &= X\left(A-X-\gamma Y\right)-tX \\ &= \frac{\left(-2A+\gamma A+2t\right)^2}{\left(-4+\gamma^2\right)^2} \end{aligned}$$

and

$$\Pi_{C,S}^{F}-K = \frac{-2A+\gamma A-\gamma t}{-4+\gamma^2}\left(A-\frac{-2A+\gamma A-\gamma t}{-4+\gamma^2}-\gamma\frac{-2A+\gamma A+2t}{-4+\gamma^2}\right)$$
$$+\left(\frac{1}{2}-\frac{1}{2}t\right)\left(\frac{1}{2}+\frac{1}{2}t\right)-t\left(\frac{1}{2}-\frac{1}{2}t\right)-K$$

4 Duopoly in Foreign only (sq 4)

Now examine the case where both firms close the Home market and only sell on the Foreign market. Then the Home firm's problem is

$$\Pi_{C,C}^{H}=\max_{X} XP(X)-tX,$$

and the Foreign firm's problem is

$$\Pi_{C,C}^{F}=\max_{Y} YP(Y).$$

4.0.4 Foreign *country* FOCs

The first-order conditions are

$$\frac{\partial}{\partial X}\left(X\left(A-X-\gamma Y\right)-tX\right)=A-2X-\gamma Y-t=0,$$

which gives

$$X=\frac{1}{2}A-\frac{1}{2}\gamma Y-\frac{1}{2}t,$$

and

$$\frac{\partial}{\partial Y}\left(Y\left(A-Y-\gamma X\right)\right)=A-2Y-\gamma X=0,$$

which gives

$$Y=\frac{1}{2}A-\frac{1}{2}\gamma X.$$

Combining the FOCs gives

$$X = \frac{2A-\gamma A-2t}{4-\gamma^2},$$
$$Y = \frac{2A-\gamma A+\gamma t}{4-\gamma^2}.$$

Thus, the Home firm's price in the Foreign country is $P(X)=\frac{2A-\gamma A-\gamma^2 t+2t}{(2-\gamma)(\gamma+2)}$, and the Foreign firm's price in the Foreign Country is $P(Y)=\frac{2A-\gamma A+\gamma t}{(2-\gamma)(\gamma+2)}$.

4.0.5 The profits

The profits to the firms are then

$$\begin{aligned}\Pi^H_{C,C} &= X\left(A - X - \gamma Y\right) - tX \\ &= \frac{\left(-2A + \gamma A + 2t\right)^2}{\left(-4 + \gamma^2\right)^2}\end{aligned}$$

and

$$\begin{aligned}\Pi^F_{C,C} &= Y\left(A - Y - \gamma X\right) \\ &= \frac{\left(-2A + \gamma A - \gamma t\right)^2}{\left(-4 + \gamma^2\right)^2}\end{aligned}$$