Characterization of Pareto Dominance

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The Pareto dominance relation is shown to be the unique nontrivial partial order on the set of finite-dimensional real vectors satisfying a number of intuitive properties.

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1. INTRODUCTION

The purpose of this note — inspired by and using a number of properties from Barbara and Jackson [2] — is to characterize the Pareto dominance relation. The set-up and interpretation is completely analogous to [2]. In particular, situations are considered where different actions are to be compared on the basis of their consequences, which are described by vectors of real numbers. Ranking actions then becomes equivalent with ranking vectors and a decision criterion can be seen as a binary relation on the set of real vectors (see [2, p. 35]).

If the coordinates of a vector $x \in \mathbb{R}^n$ measure positive attributes, like a firm's profit, utility of a decision maker, or the quantity of a certain good to a nonsatiable consumer, it Pareto dominates a vector $y \in \mathbb{R}^n$ if $x_i \geqq y_i$ for all coordinates i, with strict inequality for at least one coordinate. Conversely, if coordinates measure negative attributes (loss, disutility, quantities of 'bads', etc.), x Pareto dominates y if $x_i \leqq y_i$ for all coordinates i, with strict inequality for at least one coordinate. If an alternative x is not Pareto dominated in a given set of alternatives, it is Pareto optimal.

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Pareto dominance and Pareto optimality lie at the basis of numerous economic and game theoretic studies. Pareto optimality is in fact so fundamental, that it is often used as a characterizing property in axiomatizations of equilibria and solution concepts, rather than as the *subject* of an axiomatic study.

There are some exceptions in the related literature. Sen [6, p. 76] characterizes the collective choice rule that assigns to each tuple of preference relations the social order where an alternative a is weakly preferred to alternative b unless all agents weakly prefer b over a and at least one agent's preference is strict. His axiom P^* [6, p. 53], however, explicitly uses the Pareto dominance criterion. Campbell and Nagahisa [3] replace this property with a weaker topological condition. Another strand of literature, cf. [1], [4], [7], investigates which type of orders on a choice set can be obtained as a Pareto order through a suitable choice of preferences of the concerned agents.

The current note has a different starting point — that of Barbara and Jackson [2] and the classical Milnor [5] — by basing the comparison on vectors of real numbers. After settling matters of notation, a number of axioms is provided in Section 2. In Section 3, the Pareto dominance relation is shown to be the unique nontrivial binary relation on the set of finite-dimensional real vectors satisfying these properties. The axioms are shown to be logically independent in Section 4.

2. NOTATION AND AXIOMS

Let $\mathbb{N}$ be the set of positive integers. For $n \in \mathbb{N}$, $\mathbb{R}^n$ is the n-dimensional Euclidean space. Let $\mathcal{R} = \cup_{n \in \mathbb{N}} \mathbb{R}^n$ be the set of finite-dimensional vectors of real numbers. Let $x \in \mathcal{R}$. The dimension of x is denoted by $\dim(x)$. For $i \in \{1, \ldots, \dim(x)\}$, x_i is the i-th coordinate of x. If $\dim(x) \geqq 2$, then $x_{-i} \in \mathcal{R}$ is the $(\dim(x) - 1)$-dimensional vector obtained from x by deleting its i-th coordinate. A partial order on $\mathcal{R}$ is a binary relation $\succcurlyeq$ on $\mathcal{R}$ satisfying for all $x, y, z \in \mathcal{R}$: (i) reflexivity: $x \succcurlyeq x$, (ii) antisymmetry: $x \succcurlyeq y$ and $y \succcurlyeq x$ imply $x = y$, (iii) transitivity: $x \succcurlyeq y$ and $y \succcurlyeq z$ imply $x \succcurlyeq z$.

Equality, the Pareto order associated with positive attributes, and the Pareto order associated with negative attributes are three such partial orders on $\mathcal{R}$. They are, respectively, defined as follows:

$$\begin{aligned} x = y &\Leftrightarrow \dim(x) = \dim(y) \text{ and } x_i = y_i \text{ for all coordinates } i, \\ x \geqq y &\Leftrightarrow \dim(x) = \dim(y) \text{ and } x_i \geqq y_i \text{ for all coordinates } i, \\ x \leqq y &\Leftrightarrow \dim(x) = \dim(y) \text{ and } x_i \leqq y_i \text{ for all coordinates } i. \end{aligned}$$

Equality is a trivial partial order. It is the smallest possible partial order on $\mathcal{R}$: only identical elements are comparable, in accordance with the reflexivity condition.

A partial order $\succcurlyeq$ coincides with a partial order $\gg$ on a subset $S \subseteq \mathcal{R}$ if for all $x, y \in S$: $x \succcurlyeq y$ if and only if $x \gg y$.

A function $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing if $x > y$ implies $f(x) > f(y)$.

Let us proceed to the axioms. Axioms (A0), (A2), and (A3) are taken from Barbara and Jackson [2], to which the reader is referred for further discussion. The axioms require, for each $x, y \in \mathcal{R}$, the following:

(A0) $x \succcurlyeq y \Rightarrow \dim(x) = \dim(y)$.

This axiom states that only vectors of the same dimension are to be compared. It is consistent with the interpretation that the different coordinates reflect different attributes.

(A1) Partial order: $\succcurlyeq$ is a partial order.

Analogously to axiom (1) in Barbara and Jackson [2], this axiom specifies the type of binary relation this note aims to characterize.

(A2) Independence of duplicated states: if $\dim(x) = \dim(y) \geqq 2$ and there are coordinates i, j with $i \neq j, x_i = x_j$, and $y_i = y_j$, then

$$x \succcurlyeq y \Leftrightarrow x_{-i} \succcurlyeq y_{-i}.$$

According to this axiom, the order does not depend on the number of attributes giving rise to the same evaluation.

(A3) Independence of identical consequences: if $\dim(x) = \dim(y) \geqq 2$ and $x_i = y_i$ for some coordinate i, then

$$x \succcurlyeq y \Leftrightarrow x_{-i} \succcurlyeq y_{-i}.$$

This axiom states that the order depends only on coordinates with different evaluations.

(A4) Ordinality: if $\dim(x) = \dim(y) = n$ and $f_i : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function for each $i \in \{1, \ldots, n\}$, then

$$x \succcurlyeq y \Leftrightarrow (f_1(x_1), \ldots, f_n(x_n)) \succcurlyeq (f_1(y_1), \ldots, f_n(y_n)).$$

Ordinality states that the evaluations are based on qualitative, rather than quantitative, differences: it is the direction of the difference that matters, not its size. This axiom reflects the ordinal, rather than the cardinal, character of utilities as a means to represent preferences.

Remark 1. Strictly increasing functions mapping the smaller of two numbers to zero and the larger to one are important tools in the proof of Theorem 1.

3. CHARACTERIZATION

The characterization result indicates that the Pareto orders $\geqq$ and $\leqq$ are the only nontrivial partial orders on $\mathcal{R}$ satisfying (A0) to (A4).

THEOREM 1. *There are only three binary relations on $\mathcal{R}$ satisfying (A0) to (A4). They are $=$, $\geqq$, and $\leqq$.*

Proof. The binary relations $=$, $\geqq$, and $\leqq$ satisfy (A0) to (A4). Let $\succcurlyeq$ be a binary relation on $\mathcal{R}$ that also satisfies the properties. We prove by induction on $n \in \mathbb{N}$ the statement

$P(n)$: $\succcurlyeq$ coincides with one of the relations $=$, $\geqq$, or $\leqq$ on $\cup_{k=1}^{n} \mathbb{R}^k$.

For $n = 1$, $\succcurlyeq$ either coincides with $=$ on $\mathbb{R}$, or there exist $x, y \in \mathbb{R}$ with $x \neq y$ and $x \succcurlyeq y$. If $x > y$, we show that $a \succcurlyeq b$ for all $a, b \in \mathbb{R}$ with $a > b$, i.e., $\succcurlyeq$ coincides with $\geqq$ on $\mathbb{R}$. Let $a, b \in \mathbb{R}, a > b$. Consider the strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ given, for each $z \in \mathbb{R}$, by

$$f(z) = \frac{a-b}{x-y}(z-y) + b.$$

Ordinality (A4) implies that $a = f(x) \succcurlyeq f(y) = b$. Similarly, if $x < y$, it follows that $\succcurlyeq$ coincides with $\leqq$ on $\mathbb{R}$. This proves $P(1)$.

Let $n \in \mathbb{N}, n \geqq 2$, and assume $P(k)$ is true for all $k \in \mathbb{N}$ with $k < n$. By induction, the binary relation $\succcurlyeq$ coincides with one of the orders $=$, $\geqq$, or $\leqq$ on $\cup_{k=1}^{n-1} \mathbb{R}^k$. To show that $\succcurlyeq$ coincides with the same order on $\cup_{k=1}^{n} \mathbb{R}^k$, it suffices by (A0), (A1), and reflexivity of partial orders to consider vectors $x, y \in \mathbb{R}^n$ with $x \neq y$.

Case A: $x \ngeqq y$ and $x \nleqq y$.
To show: x and y are also incomparable according to $\succcurlyeq$. This is done in two steps.

Step 1: We prove that the vectors $(0,1)$ and $(1,0)$ in $\mathbb{R}^2$ are incomparable according to $\succcurlyeq$. Suppose $(1,0) \succcurlyeq (0,1)$. We derive a contradiction[2]. Let $a = (1,0,1)$ and $b = (0,1,0)$. Since $a_1 = a_3$ and $b_1 = b_3$, independence of duplicated states (A2) together with $a_{-3} = (1,0) \succcurlyeq (0,1) = b_{-3}$ implies that $x \succcurlyeq y$. Independence of duplicated states (A2) together with $a \succcurlyeq b$ then implies that $a_{-1} = (0,1) \succcurlyeq (1,0) = b_{-1}$. Hence $(1,0) \succcurlyeq (0,1)$ and $(0,1) \succcurlyeq (1,0)$, which together with antisymmetry of the partial order $\succcurlyeq$ (see (A1)) implies that $(1,0) = (0,1)$, a contradiction.

Step 2: We proceed to show that x and y are incomparable according to $\succcurlyeq$. Since $x \ngeqq y$, there is an $i \in \{1, \ldots, n\}$ with $x_i < y_i$. Since $y \ngeqq x$, there is a $j \in \{1, \ldots, n\}$ with $x_j > y_j$. Define $v, w \in \mathbb{R}^n$ by taking for each

[2] The case $(0,1) \succcurlyeq (1,0)$ is similar and therefore omitted.

$k \in \{1, \ldots, n\}$:

$$\begin{array}{ll} v_k = 0, w_k = 1 & \text{if } x_k < y_k, \\ v_k = 0, w_k = 0 & \text{if } x_k = y_k, \\ v_k = 1, w_k = 0 & \text{if } x_k > y_k. \end{array}$$

Ordinality (A4) and Remark 1 imply that $\succcurlyeq$ orders x and y the same way as v and w. By definition, $v_1 = 0, w_1 = 1, v_2 = 1, w_2 = 0$. For all other coordinates $k \in \{1, \ldots, n\} \setminus \{i, j\}$, either $v_k = w_k$, in which case coordinate k can be eliminated by independence of identical consequences (A3), or $\{v_k, w_k\} = \{0, 1\}$, in which case coordinate k can be eliminated by independence of duplicated states, since it is a duplication of either coordinate i or j. Thus eliminating all but the i-th and j-th coordinate, the vectors v and w, and hence x and y, are incomparable according to $\succcurlyeq$, since $(1, 0)$ and $(0, 1)$ are incomparable by Step 1.

Case B: $x \geqq y$ or $y \geqq x$.
Assume without loss of generality that $x \geqq y$. To show:

(a) If $\succcurlyeq$ coincides with $=$ on $\cup_{k=1}^{n-1} \mathbb{R}^k$, then x and y are incomparable;

(b) If $\succcurlyeq$ coincides with $\geqq$ on $\cup_{k=1}^{n-1} \mathbb{R}^k$, then $x \succcurlyeq y$;

(c) If $\succcurlyeq$ coincides with $\leqq$ on $\cup_{k=1}^{n-1} \mathbb{R}^k$, then $y \succcurlyeq x$.

By independence of identical consequences (A3), we may assume that $x_k \neq y_k$ for all coordinates k. Hence $x_k > y_k$ for all $k \in \{1, \ldots, n\}$. By ordinality (A4) and Remark 1, $\succcurlyeq$ orders x and y the same way as $(1, \ldots, 1) \in \mathbb{R}^n$ and $(0, \ldots, 0) \in \mathbb{R}^n$. By independence of duplicated states (A2), $\succcurlyeq$ orders $(1, \ldots, 1)$ and $(0, \ldots, 0)$, and hence x and y, the same way as the real numbers 1 and 0. Apply the induction step:

(a) If $\succcurlyeq$ coincides with $=$ on $\cup_{k=1}^{n-1} \mathbb{R}^k$, then 1 and 0 and consequently x and y are incomparable;

(b) If $\succcurlyeq$ coincides with $\geqq$ on $\cup_{k=1}^{n-1} \mathbb{R}^k$, then $1 \succcurlyeq 0$. Consequently $x \succcurlyeq y$;

(c) If $\succcurlyeq$ coincides with $\leqq$ on $\cup_{k=1}^{n-1} \mathbb{R}^k$, then $0 \succcurlyeq 1$. Consequently $y \succcurlyeq x$.

Since x and y were arbitrary elements of $\mathbb{R}^n$, this finishes the proof. ∎

As mentioned before, the equality $=$ is a trivial partial order in which only identical vectors are comparable. Hence, the theorem indicates that the Pareto dominance relations $\leqq$ and $\geqq$ are the unique nontrivial partial orders on $\mathcal{R}$ satisfying the properties (A0) till (A4). Their difference is a benchmark property indicating whether attributes measure positive or negative qualities of the alternatives: setting $1 \succcurlyeq 0$ ('something is better than nothing') would single out the $\geqq$-relation:

$$\forall x, y \in \mathcal{R} : x \succcurlyeq y \Leftrightarrow \dim(x) = \dim(y) \text{ and } x \geqq y,$$

while the opposite would select the $\leqq$-relation.

4. LOGICAL INDEPENDENCE

In this section it is shown that none of the axioms used in Theorem 1 is implied by the others:

PROPOSITION 1. *Axioms (A0) till (A4) are logically independent.*

This result is proven by five examples of binary relations on $\mathcal{R}$, each of which violates exactly one of the axioms. In each of the examples it is straightforward to check that certain axioms are indeed satisfied. This part is left to the reader.

EXAMPLE 1. Define the binary relation $\succcurlyeq$ on $\mathcal{R}$ by taking for each $x, y \in \mathcal{R}$:

$$x \succcurlyeq y \Leftrightarrow \dim(x) \geqq \dim(y) \text{ and } x_i = y_i \text{ for all } i = 1, \ldots, \dim(y).$$

This binary relation satisfies all axioms, except (A0): $(1,0) \succcurlyeq 1$.

EXAMPLE 2. Define the binary relation $\succcurlyeq$ on $\mathcal{R}$ by taking for each $x, y \in \mathcal{R}$:

$$x \succcurlyeq y \quad \Leftrightarrow \quad x = y \text{ or } (\dim(x) = \dim(y) \text{ and } x_i > y_i \text{ for some } i \in \{1, \ldots, \dim(x)\}).$$

This binary relation satisfies all axioms, except (A1): $(0,1) \succcurlyeq (3,0)$ and $(3,0) \succcurlyeq (2,2)$, but $(0,1) \not\succcurlyeq (2,2)$, so $\succcurlyeq$ is not transitive. (It is not antisymmetric either). Hence $\succcurlyeq$ is not a partial order.

EXAMPLE 3 (THE LEXICOGRAPHIC ORDER). Define the binary relation $\succcurlyeq$ on $\mathcal{R}$ by taking for each $x, y \in \mathcal{R}$:

$$x \succcurlyeq y \quad \Leftrightarrow \quad x = y \text{ or } (\dim(x) = \dim(y) \text{ and } \exists k \in \{1, \ldots, \dim(x)\} : x_i = y_i \text{ for all } i < k \text{ and } x_k > y_k).$$

This binary relation satisfies all axioms, except independence of duplicated states (A2): $(1,0,1) \succcurlyeq (0,1,0)$, yet after deleting the first coordinate: $(0,1) \not\succcurlyeq (1,0)$.

EXAMPLE 4. Define the binary relation $\succcurlyeq$ on $\mathcal{R}$ by taking for each $x, y \in \mathcal{R}$:

$$x \succcurlyeq y \quad \Leftrightarrow \quad x = y \text{ or } (\dim(x) = \dim(y) \text{ and } x_i > y_i \text{ for all } i = 1, \ldots, \dim(x)).$$

This binary relation satisfies all axioms, except independence of identical consequences (A3): $(1,1)$ and $(1,0)$ are incomparable according to $\succcurlyeq$, yet after deleting the first coordinate: $1 \succcurlyeq 0$.

EXAMPLE 5. Define the binary relation $\succcurlyeq$ on $\mathcal{R}$ by taking for each $x, y \in \mathcal{R}$:

$$x \succcurlyeq y \quad \Leftrightarrow \quad x = y \text{ or } (\dim(x) = \dim(y) \text{ and } x_i \geqq y_i + 2 \text{ for all } i = 1, \ldots, \dim(x) \text{ with } x_i \neq y_i).$$

This binary relation satisfies all axioms, except ordinality (A4): $(2,2) \succcurlyeq (0,0)$, yet after dividing each coordinate by two: $(1,1) \not\succcurlyeq (0,0)$.

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