The market for melons: Cournot competition with unobservable qualities

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Abstract

Two firms produce different qualities at possibly different, constant marginal costs. They compete in quantities on a market where buyers only observe the average quality supplied. The model is a generalization of the standard Cournot duopoly, which corresponds to the special case where the two qualities are equal. When the quality differential is large, the firms' output levels are not always strategic substitutes. There can be no, or up to three pure-strategy equilibria. Yet, as long as the cost differential is not extreme, there always exists a stable duopolistic equilibrium. In that sense, strategic quantity-setting helps prevent market unraveling.

KEYWORDS: Cournot competition, quality, duopoly, asymmetric information, Nash equilibrium.

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1 Introduction

With summer comes the time of relishing those flavorful and refreshing melons which you find at your local marketplace. Well, they are not always flavorful, are they? As a matter of fact, it is not uncommon to get disappointed with a gourd which looked particularly enticing at the time of purchase. If only you remembered the brand name on the sticker (possibly) affixed to the last one you ate, which you particularly savored, but you don't! That would not be a problem if your local seller consistently supplied melons from the same source but sellers typically don't. They very often adjust their inventory in accordance with the availaibility, characteristics and prices of the different products sold on wholesale markets. Often, retailers sell different batches of a given fruit or vegetable at the same price, indepently of their origin; in supermarkets, cases are just stacked and only the curious and unhurried shoppers pay attention to the slight differences in the boxes' appearance, wondering whether they should act upon this information, and dig the pile or not.

This description makes clear that melons are experience goods, whose quality is not observable at the time of purchase. This quality is also subject to some variation at the retail level. Although we all have a friend who claims to know how to choose a good melon, it is clear that most of us face a lottery in doing so. In these circumstances, our willingness to pay might well depend upon the features of this lottery. We here make the assumption that the quantity demanded is linear in the average quality, a magnitude that can often be observed or infered by consumers. (Indeed, by word-of-mouth, one generally gets an idea of the worth of the season's harvest.)

Abstracting from production and retailing details, we study a game in which two producers of some variants of a given good have to decide about the quantity they will bring to the market on which their undistinguishable products are sold. Given these quantities and the corresponding average quality, which consumers observe or infer, the market is cleared by setting the price so as to equate demand with supply. In effect, the two producers compete *à la Cournot* with given but unobservable qualities when consumers have correct beliefs regarding the average quality in all circumstances. We attempt at characterizing the pure-strategy Nash equilibria of the game, which is a generalization of the standard Cournot game.

The existence of an unobserved difference in quality introduces an additional effect into the Cournot model. When a producer considers an increase in the quantity he or she brings to the market, he or she must anticipate not only that the market price will decrease along the current demand curve but also that the average quality will change, shifting the demand curve altogether. The high-quality producer thus has an incentive to produce more than the typical Cournot quantity, while the low-quality producer is led to produce less. If the marginal cost of production does not increase too quickly with quality, in equilibrium the high-quality producer produces more than the low-quality producer, even if she faces higher costs. This is the case, for instance, whenever quality is determined by an initial investment affecting the fixed cost but not the variable cost of production. In a sense, in this situation, there is favorable, or advantageous, selection. More generally, the strategic behavior of the producers mitigates adverse-selection phenomena of the type described by Akerlof (1970). In particular, it is easy to come up with examples where the only competitive equilibrium involves unraveling of the market for "lemons", whereas on our market for melons, for moderate cost differentials, high-quality products continue being supplied, sometimes on a high scale.

Because of this feature of quantity choice under asymmetric information, there are instances in which consumers would prefer to face two unequally able producers rather than two identical producers displaying the (unweighted) average level of ability. That is, assuming that melon producers can be ranked on a linear quality scale, consumers could prefer their local market being supplied by a first-class producer along with a third-class producer to having the certainty of buying a second-class melon, for in equilibrium the average quality will increase more than the price.

At the same time, the unobserved difference in quality have the potential to give rise to surprising outcomes. Large quality differentials can produce a non-monotonic best-response curve for the low-quality firm, and a discontinuous curve for the high-quality firm. That can lead to the non-existence of an equilibrium in pure strategies. Even in cases where the quality differential is small and firms' programs are well-behaved, up to three pure-strategy equilibria may co-exist, thus raising an equilibrium selection problem.

Importantly, our setting requires that goods be undistinguishable to the eyes of the potential buyers. Thus, either the legal system does not support proprietary brands (as in the important case of counterfeiting), or the costs of establishing or maintaining such brands are prohibitive.

We do not want to claim that a model where goods are undistinguishable and producers do not set their price is general, although we believe that some concentrated agricultural or mineral product markets correspond to that description. We note that, even in environments where producers are straightforwardly identifiable, such as the markets for wines or spirits, where labelling or branding are common, high-quality producers very often express their fear that the market be flooded by low-quality variants taking advantage of the good "reputation" of the product and depressing its price. For example, it is arguably hard to confuse a bottle of Champagne from a *grande maison* with a bottle of sparkling wine produced in any other region by an unknown wine-grower. Yet, Champagne producers have always protested against the use of this name outside the historical region of production. This might well be an anti-competitive strategy but it is also likely that, because the purchase of sparkling wine is not repeated enough (or information acquisition, or processing, costs are high, or consumers have cognitive limitations), purchasers tend to bunch these distinguishable products into the same category. Indeed, concerns of this kind have led members of the World Trade Organization to grant a so-called higher level of protection to the place names used to identify the origin and quality, reputation or other characteristics of wines and spirits. Thus, we believe that there are many markets on which products are not absolutely undistinguishable but the asymmetric information problem we tackle here is present, to some extent, with the same qualitative consequences. Similarly, outside the realm of centralized markets, firms very likely have some pricing power. We take the quantity competition assumption to stand for a form of moderate competition where the law of one price does hold but firms cannot commit to serve any level of demand addressed to them, as opposed to the case of Bertrand competition.

There is of course a voluminous literature on Cournot competition, thoroughly surveyed, most recently, by Vives (1999). Our model is not the first not to guarantee the existence of a pure-strategy Nash equilibrium under quantity competition. Early examples were provided (in a more general context) by Roberts and Sonnenschein (1976). Even in the homogenous product case, some well-accepted demand or cost structures can lead to the non-quasiconcavity of firms' payoffs. In our model, the linear presence of the quality average term in the inverse demand function is sufficient to generate some non-convexity of profits. (We argue that this problem is not due to our functional form but is instead a general feature of the economic situation of interest.) The study of Cournot competition with differentiated products was marked by the seminal contributions of Singh and Vives (1984) and Vives

(1985) but to our knowledge, our analysis is the first to incorporate an element of asymmetric information between firms and consumers, in the spirit of Akerlov (1970)'s market for lemons.

We introduce the formal model in Section 2. Section 3 briefly recalls the standard Cournot model, which corresponds to the special case when qualities are identical. Section 4 deals with the general features of the case when qualities are dissimilar. Section 5 attemps at classifying the equilibrium outcomes on the basis of cost and quality heterogeneity. Section 6 develops some welfare considerations. Section 7 concludes.

2 Model

We first describe the model in its general form. After showing the unavoidable difficulties it leads to, we describe the special case on which we will focus.

Two firms indexed by $i \in \{L, H\}$ produce two variants of the same good, whose qualities are denoted x_L and x_H, respectively. We assume that $0 < x_L \leq x_H$. It is thus understood that our choice of subscript corresponds to the quality ranking of the variants. The cost of production depends upon quantity and quality and is given by a function $\mu : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$. Because the firms' qualities are exogenously fixed, we will slightly abuse notation by indexing μ with i. That is, $\mu_i(q_i) \equiv \mu(q_i, x_i)$.

Firms face a market (inverse) demand that is given by

$$P = P(Q, \bar{x}), \tag{1}$$

a function that strictly increases with $\bar{x}$, the average quality of the units brought to the market, and decreases with Q, the total quantity produced by the duopolists. We make the assumption that for any quadruple (x_L, x_H, q_L, q_H), consumers infer or observe the "true" average quality $\bar{x}$. This is so even when firms consider "deviations" from the prescribed equilibrium behavior.[1]

[1] One could be precise about the time and information structure of the interaction, seen as an extensive-form game. Because consumers cannot observe the variants' quality, their demand obviously depends upon their beliefs regarding this quality. Our assumption here is that in all circumstances, on and off the equilibrium path, consumers' (uniform) belief corresponds to the correct average. In effect, we prevent firms from taking advantage of some inertia in the prevailing beliefs: although not directly observable, no "deviation" can take place at unchanged beliefs. This is likely if production takes time and output decisions have to be made well ahead of sales, which prevents "instantaneous deviations".

To avoid trivialities, we assume that

$$P(0, x_L) > c_L \quad \text{and} \quad P(0, x_H) > c_H.$$

These inequalities reflect the idea that, should one only of these goods be sold on the market, it could be profitably supplied.[2] We will refer to this assumption as the *profitable supply assumption*.

Firms compete *à la Cournot*, simultaneously deciding about the quantity $q_i \geq 0$ they will bring to the market and then letting a fictitious auctioneer set the price that equates market demand with market supply. Yet they are not price-takers: at the time they decide about their volume of production, they recognize that a change in q_i will affect the market price. We attempt at characterizing the (pure-strategy) Nash equilibria of this two-player, simultaneous-move game.

Each firm's profit is thus given by:

$$\pi_i(q_L, q_H) = q_i P\left(q_L + q_H, \frac{q_L x_L + q_H x_H}{q_L + q_H}\right) - \mu_i\left(q_i\right). \tag{2}$$

Using standard calculus notation, we have that

$$\frac{\partial \pi_i}{\partial q_i} = P(Q, \bar{x}) + q_i \frac{\partial P}{\partial Q} + \frac{\partial P}{\partial \bar{x}} \cdot \frac{q_L q_H \left(x_i - x_{-i}\right)}{\left(q_L + q_H\right)^2} - \frac{\partial \mu_i}{\partial q_i}; \tag{3}$$

$$\begin{aligned} \frac{\partial^2 \pi_i}{\left(\partial q_i\right)^2} &= 2\frac{\partial P}{\partial Q} + q_i \frac{\partial^2 P}{\partial q_i \partial Q} + \frac{\partial^2 P}{\partial q_i \partial \bar{x}} \frac{q_L q_H \left(x_i - x_{-i}\right)}{\left(q_L + q_H\right)^2} \\ &\quad + \frac{\partial P}{\partial \bar{x}} \cdot \frac{q_{-i}\left(x_i - x_{-i}\right)}{\left(q_L + q_H\right)^2} \cdot \left(1 - \frac{2q_i}{q_L + q_H}\right) - \frac{\partial^2 \mu_i}{\left(\partial q_i\right)^2}; \end{aligned} \tag{4}$$

$$\begin{aligned} \frac{\partial^2 \pi_i}{\partial q_{-i} \partial q_i} &= \frac{\partial P}{\partial Q} + q_i \frac{\partial^2 P}{\partial q_{-i} \partial Q} + q_i \frac{\partial P}{\partial \bar{x}} \frac{\left(q_H - q_L\right)\left(x_H - x_L\right)}{\left(q_L + q_H\right)^3} \\ &\quad + \frac{\partial P}{\partial \bar{x}} \frac{q_i \left(x_{-i} - x_i\right)}{\left(q_L + q_H\right)^2} + \frac{\partial^2 P}{\partial q_{-i} \partial \bar{x}} \cdot \frac{q_L q_H \left(x_i - x_{-i}\right)}{\left(q_L + q_H\right)^2}. \end{aligned} \tag{5}$$

These expressions make clear that the presence of the average-quality term considerably complicates the duopoly problem.

[2] These inequalities guarantee that the most enthusiastic consumer's willingness to pay will cover the cost of the "first" unit.

Assume for instance that the inverse demand curve intersects both axes and all its derivatives are finite. In that case,

$$\lim_{q_H \to 0} \frac{\partial^2 \pi_H}{(\partial q_H)^2} = 2\frac{\partial P}{\partial Q} - \frac{\partial^2 \mu_H}{(\partial q_H)^2} + \frac{\partial P}{\partial \bar{x}} \cdot \frac{q_L (x_H - x_L)}{(q_L + q_H)^2}, \tag{6}$$

which is positive for sufficiently low q_L. That means that firm H's program is not always quasi-concave. Hence, the best-response correspondence may fail to be a continuous function and we cannot appeal to the usual fixed-point theorems to prove the existence of (at least) one pure-strategy equilibrium. The reason for the convexity of the profit function in this quantity range will soon become very clear.

Besides, observe that, since the very meaning of quality justifies $\partial P/\partial \bar{x} > 0$, it follows that whatever the assumption one is willing to make about the relationship between the marginal willingness to pay for quality and quantity (i.e. about the sign of $\partial^2 P/\partial q_{-i}\partial \bar{x}$), terms in $(x_i - x_{-i})$ and $(x_{-i} - x_i)$ both appear in one of the two cross-derivatives. That implies that the best-response of one of the two firms might well be non-monotonic, in which case we cannot appeal to the standard existence theorems based on super-, or sub-modularity. Again, the reason for this non-monotonicity will become clear in the special case to which we will restrict our attention.

These issues arise independently on the assumptions made about the curvature of the inverse demand curve or the cost function, as is usually the case in oligopoly theory. Instead, they are rooted in the economics of the problem at hand.

We therefore choose to concentrate on a special case. We take the simplest one, in which the inverse demand curve is linear in both arguments. Thus, the inverse demand curve is given by:

$$P = a + \bar{x} - bQ \tag{7}$$

where a is a positive demand-shifting parameter and b, a strictly positive demand-rotating parameter.

We also assume that the marginal cost of production only depends upon quality. That is, technology exhibits constant returns to scale and firms produce at unit costs c_L and c_H, respectively.

3 Standard Cournot duopoly

Setting $x_L = x_H = x$ gives the usual Cournot duopoly model as a special case. Indeed, both firms face a demand of the form

$$P = a + x - bQ. \tag{8}$$

Still, they can have different unit costs. In order to avoid any confusion, we choose the firm indices so that $c_L < c_H$.

Firm i's profit is given by

$$\pi_i = q_i \left[a + x - b\left(q_i + q_{-i}\right) - c_i\right], \tag{9}$$

where q_{-i} stands for the quantity produced by firm i's rival. For any q_{-i}, the profit function, being quadratic and strictly concave in q_i, admits a unique maximizer on $\mathbb{R}_+$, which, if interior, is characterized by the first-order condition:

$$a + x - 2bq_i - bq_{-i} - c_i = 0 \tag{10}$$

Disallowing negative quantities, one gets the best-response functions:

$$q_i = \max\left\{0, \frac{a + x - bq_{-i} - c_i}{2b}\right\}; i \in \{L, H\} \tag{11}$$

Because $dq_i/dq_{-i} < 0$ at the interior solution, the two choice variables are seen to be strategic substitutes.

Solving for the intersection of the two interior best-response curves, one gets:

$$\begin{aligned}
q_L^* &= \frac{a + x + (c_H - 2c_L)}{3b} \\
q_H^* &= \frac{a + x + (c_L - 2c_H)}{3b} \\
Q^* &= \frac{2(a + x) - (c_L + c_H)}{3b} \\
P^* &= \frac{a + x + c_L + c_H}{3}
\end{aligned}$$

The producer facing the highest production cost ends up producing less than its more efficient competitor in equilibrium.

One has to check that the values of q_L^* and q_H^* indeed lead both firms to remain active on the market, a situation to which we will refer as a *duopolistic equilibrium*. It is easily verified that if $c_H - c_L \geq a + x - c_H$, then firm H wants to withdraw from the market, for its margin turns negative.

There could then be an equilibrium in which the low-cost firm serves the market and the high-cost producer decides to withdraw. We call such a situation, in which only one firm remains active in equilibrium, a *monopolistic equilibrium*. At this equilibrium, firm L produces the monopoly quantity, q_L^M, and firm H's margin is negative at all quantity levels. From the best-response characterization, one gets the condition under which such an equilibrium exists:

$$bq_L^M \geq a + x - c_H, \tag{12}$$

which is verified if and only if

$$c_H - c_L \geq a + x - c_H. \tag{13}$$

So if the cost differential is large enough in comparison to firm H's margin on the first unit sold, then a monopolistic equilibrium exists and is unique. Conversely, if the cost differential is low enough, then the duopolistic equilibrium is the only equilibrium. If firms have the same cost function, then the duopolistic equilibrium is unique and symmetric.

In view of the subsequent analysis, we find it convenient to express these results by reference to a (symmetric) cost heterogeneity parameter, δ. So for any fixed c_L and c_H, let c be their arithmetic average and set $\delta = (c_H - c_L)/2$. Then, $c_L = c - \delta$ and $c_H = c + \delta$. We thus summarize this section with the following claim.

Claim 1 *In the case when the two firms produce a homogenous product ($x_L = x_H = x$), the game admits a unique Nash equilibrium. Given a linear inverse demand curve and a cost average, c, the nature of the equilibrium depends upon the level of cost heterogeneity. (i) If $\delta \geq (a + x - c)/3$, then firm L produces $q_L^M = (a + x - c + \delta)/(2b)$ and firm H withdraws from the market. (ii) If $\delta < (a + x - c)/3$, then both firms are active in equilibrium* *and*

$$\begin{aligned} q_L^* &= \frac{a+x-c+3\delta}{3b} \\ q_H^* &= \frac{a+x-c-3\delta}{3b} \\ Q^* &= \frac{2(a+x-c)}{3b} \\ P^* &= \frac{a+x+2c}{3}. \end{aligned}$$

Observe that, at unchanged average, a small increase in the cost differential does not impact the aggregate variables. It only distorts the market shares in favor of the efficient firm.

4 Unobservable differences in quality

Suppose now that the two firms do no longer produce a homogenous product, and $x_L < x_H$.

Each firm's profit is given by[3]

$$\pi_i(q_L, q_H) = q_i \left[a + \frac{q_L x_L + q_H x_H}{q_L + q_H} - b\,(q_L + q_H) - c_i \right]. \tag{14}$$

It is a matter of computation to derive the following expressions, which we display here for reference:

$$\frac{\partial \pi_i}{\partial q_i} = a + \bar{x} - bQ - c_i - bq_i + \frac{q_L q_H\,(x_i - x_{-i})}{(q_L + q_H)^2} \tag{15}$$

$$\frac{\partial^2 \pi_i}{(\partial q_i)^2} = 2\frac{(q_{-i})^2\,(x_i - x_{-i})}{(q_L + q_H)^3} - 2b \tag{16}$$

$$\frac{\partial^2 \pi_i}{\partial q_{-i} \partial q_i} = 2\frac{q_L q_H}{(q_L + q_H)^3}\,(x_{-i} - x_i) - b \tag{17}$$

[3]In all rigor, given the usual meaning attached to a demand curve, we should write

$$\pi_i = q_i \left[\max \left\{ a + \frac{q_L x_L + q_H x_H}{q_L + q_H} - b\,(q_L + q_H)\,, 0 \right\} - c_i \right].$$

Firms will now recognize that they can affect the market price in two distinct manners: by producing more, they depress the willingness to pay of the marginal consumer along the current demand curve, but by doing so they also change the average quality, which shifts the demand curve altogether. This additional effect is captured by the last term in equation (15), which did not appear in the standard Cournot model.

We first attempt at giving a general description of the nature of the problem that firms face and its consequence on the existence and features of equilibria. In the next section, we try to make these results precise for various combinations of cost and quality heterogeneity levels.

4.1 Existence of interior solutions to the firms' problems

Observe that $\pi_L(0, q_H) = \pi_H(q_L, 0) = 0$ and $\lim_{q_i \to +\infty} \pi_i = -\infty$. With $x_L < x_H$, π_L is strictly concave in q_L for any $q_H \geq 0$. Since π_L is then single-peaked on $\mathbb{R}_+$, firm L's output decision is characterized by the first-order condition if and only if, given firm H's quality and quantity, firm L's margin on the "first" unit sold is positive (which amounts to saying that $\partial\pi_L(0, q_H)/\partial q_L \geq 0$). In turn, this condition is met when firm H does not depress the price too much by "flooding the market".

Remark 2 *The unique solution to firm L's problem is characterized by the first-order condition if and only if*

$$bq_H \leq a + x_H - c_L. \tag{18}$$

Denote ρ_H the supremum of the quantities q_H that elicit a strictly positive response from firm L.

By contrast, π_H may not be strictly concave in q_H everywhere (although it eventually turns so). In particular, there can be a convex section for low values of q_H if the quality differential is big relatively to the demand curve slope. So, either π_H, being continuous and eventually negative, achieves an interior global maximum for a finite $q_H > 0$, or there is a corner solution at $q_H = 0$, or both. The usual first-order condition is necessary in the first case but in general not sufficient. It takes a little work to establish the precise conditions under which firm H chooses to remain active on the market.

Proposition 3 *If it is the case that*

$$bq_L < \max\left\{a + x_L - c_H, \min\left\{x_H - x_L, \frac{(a + x_H - c_H)^2}{4(x_H - x_L)}\right\}\right\}, \tag{19}$$

then firm H's program admits a unique solution, and this solution satisfies the first-order condition. If $x_H - x_L > a + x_L - c_H$, then for $q_L = \frac{(a+x_H-c_H)^2}{4b(x_H-x_L)}$, firm H's program admits two solutions, one interior solution satisfying the first-order condition and one corner solution. In all other cases, the unique solution to firm H's problem is $q_H = 0$.

Proof. The question of whether $\pi_H(q_L, q_H)$ assumes some positive values on $(0, +\infty)$ for a fixed q_L can be settled by studying the "margin function"

$$\mu_H(q_L, q_H) = a + \frac{q_L x_L + q_H x_H}{q_L + q_H} - b(q_L + q_H) - c_H, \tag{20}$$

a well-defined rational function of q_H on $\mathbb{R}_+$, for any $q_L > 0$.

If μ_H takes positive values on $(0, +\infty)$, then there is an interior solution to the firm's problem as firm H can make a positive profit in that range.

Observe that on $[0, +\infty)$:

$$\frac{\partial \mu_H}{\partial q_H} = \frac{q_L(x_H - x_L)}{(q_L + q_H)^2} - b, \tag{21}$$

$$\frac{\partial^2 \mu_H}{(\partial q_H)^2} < 0, \tag{22}$$

$$\frac{\partial \mu_H(q_L, 0)}{\partial q_H} = \frac{x_H - x_L}{q_L} - b, \tag{23}$$

and

$$\mu_H(q_L, 0) = a + x_L - bq_L - c_H \tag{24}$$

(i) If $\mu_H(q_L, 0)$ is strictly greater then zero (i.e. if $bq_L < a + x_L - c_H$), then by continuity π_H must achieve a positive interior maximum. There is a unique interior solution to firm H's problem.

(ii) If $bq_L \geq a + x_L - c_H$, then one must distinguish two cases.

(a) If $\frac{\partial \mu_H(q_L,0)}{\partial q_H} \leq 0$ (i.e. if $bq_L \geq x_H - x_L$), then μ_H assumes strictly negative values on $(0, +\infty)$ by strict concavity. There is a unique corner solution.

(b) If $\frac{\partial \mu_H(q_L,0)}{\partial q_H} > 0$ (i.e. if $bq_L < x_H - x_L$), then it is possible that μ_H increases sufficiently to reach the positive range before starting decreasing inexorably. In any case, from the first-order condition, the maximum is reached at $q_H = \sqrt{\frac{q_L(x_H - x_L)}{b}} - q_L$, leading to an average quality equal to $x_H - \sqrt{bq_L(x_H - x_L)}$, and a margin equal to $a + x_H - c_H - 2\sqrt{bq_L(x_H - x_L)}$. Thus, there is an interior solution if and only if this latter expression is positive, that is, if and only if $bq_L \leq \frac{(a+x_H-c_H)^2}{4(x_H-x_L)}$.

Note that when $x_H - x_L \leq (a + x_H - c_H)/2 \leq a + x_L - c_H$, the previous inequality is mechanically satisfied once $bq_L < x_H - x_L$. So the only instance in which the condition can strictly bind is when $x_H - x_L > (a + x_H - c_H)/2 > a + x_L - c_H$.

In that case, when bq_L exactly equals $\frac{(a+x_H-c_H)^2}{4(x_H-x_L)}$, the best interior margin equals zero. The quantity q_H associated with this margin is given by

$$\frac{a + x_H - c_H}{2b} \left(1 - \frac{a + x_H - c_H}{2(x_H - x_L)}\right) \tag{25}$$

and must correspond to a local maximum of the profit function, reaching zero at that point, negative everywhere else on $(0, +\infty)$. Observe that this quantity is strictly positive since $x_H - x_L > a + x_L - c_H$. Thus, firm H's best-response correspondence at this point is pair-valued: the optimal response comprises the interior solution as well as the corner solution. So bq_L must be strictly smaller than $(a + x_H - c_H)^2 / [4(x_H - x_L)]$ in order for the first-order condition necessarily to be verified at a solution to the firm's problem. ■

Denote ρ_L the supremum of the set of quantities q_L that elicit a strictly positive response from firm H.

4.1.1 Numerical examples

We illustrate the conditions above by looking at some numerical examples.

Let $x_L = 1$, $x_H = 2$, $c_L = 1$, $c_H = 3$, $a = 5$, and $b = 1$. We can compute the value of the tresholds:

$$\begin{aligned} a + x_L - c_H &= 3 \\ x_H - x_L &= 1 \\ \frac{(a + x_H - c_H)^2}{4(x_H - x_L)} &= 4. \end{aligned}$$

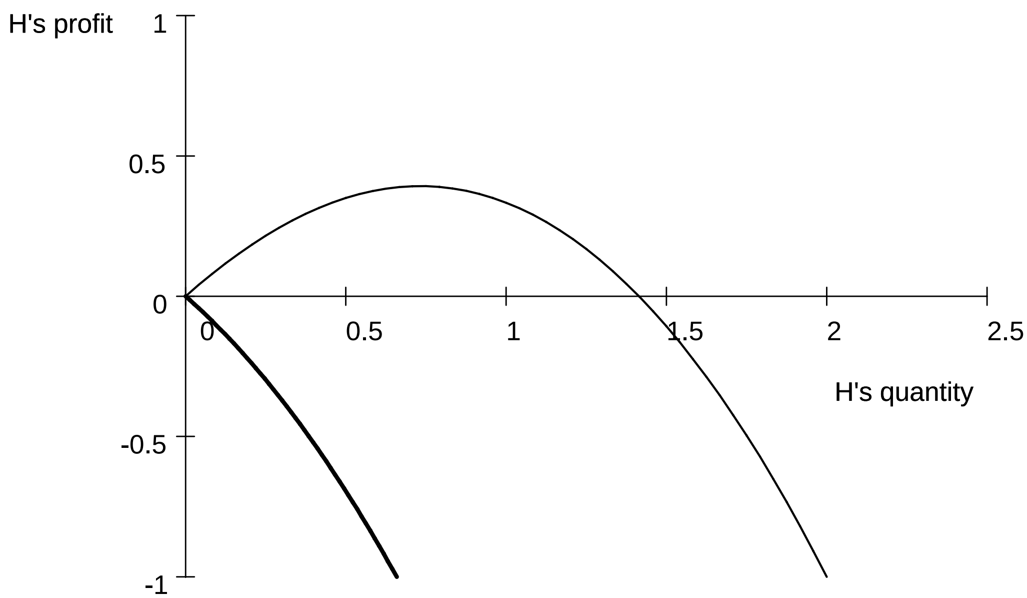

Figure 1: Strict concavity of firm H's problem

On figure 1, the thin line is for $q_L = 2$. The thick line is for $q_L = 4$. The general shapes of these two curves are the only possible here as the quality differential is too small to generate some convex sections.

Contrast with the following example, depicted on figure 2: $x_L = 1$, $x_H = 10$, $c_L = 0$, $c_H = 1$, $a = 5$, and $b = 1$, giving

$$\begin{aligned} a + x_L - c_H &= 5 \\ x_H - x_L &= 9 \\ \frac{(a + x_H - c_H)^2}{4(x_H - x_L)} &\simeq 5.44. \end{aligned}$$

The thin line is for $q_L = 4.8$, the thicker line for $q_L = 5.35$, and the thickest line for $q_L = 5.5$. One can observe the convex sections of the curve that the relatively large quality differential creates near zero. A curve for

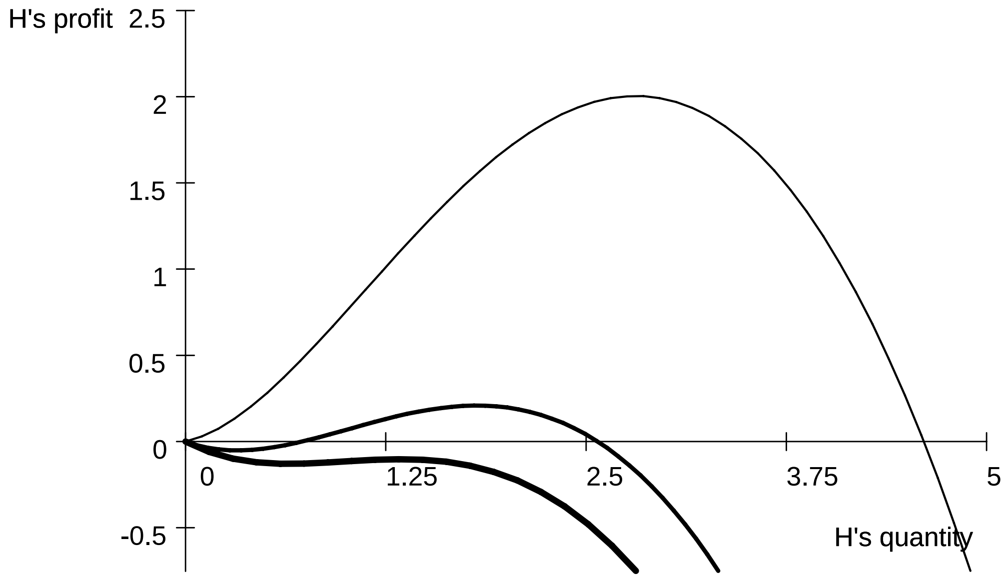

Figure 2: An example of the possible non-concavity of firm H's problem

$q_L > 9$, not shown on the figure, would be downward-sloping and concave everywhere on $[0, +\infty)$.

We now attempt at describing the incentives that firms face and the general shape of the the best-response curves to which these incentives give rise.

4.2 Firms' best responses

Consider the first-order conditions necessarily satisfied at the interior solutions to the firms' programs:

$$a + \frac{q_L x_L + q_H x_H}{q_L + q_H} - b\left(q_L + q_H\right) - c_L + \frac{q_L q_H \left(x_L - x_H\right)}{\left(q_L + q_H\right)^2} - bq_L = 0 \qquad \text{(A)}$$

$$a+\frac{q_L x_L + q_H x_H}{q_L+q_H} - b\left(q_L+q_H\right) - c_H + \frac{q_L q_H \left(x_H - x_L\right)}{\left(q_L+q_H\right)^2} - bq_H = 0. \qquad \text{(B)}$$

4.2.1 Incentives to produce

It is possible to rewrite conditions (A) and (B) in a more illustrative manner. After some manipulation (A) gives

$$a - 2bq_L - bq_H + \left[1 - \left(\frac{q_H}{q_L+q_H}\right)^2\right] x_L + \left[\left(\frac{q_H}{q_L+q_H}\right)^2\right] x_H = c_L. \qquad (26)$$

This expression shows that the first-order condition for firm L is the same as the one in the standard Cournot model, except that the demand-intercept term not only depends upon the quantities chosen by the players but, for those quantities, is also a downward-distorted linear combination of x_L and x_H. Therefore, the fact that consumers immediately observe the average quality available on the market decreases firm L's marginal revenue, as any additional output brought to the market not only depresses the market price along the current demand curve but also impacts this average, thus shifting the demand curve down.

Similarly, one can rewrite (B) as

$$a - bq_L - 2bq_H + \left[\left(\frac{q_L}{q_L+q_H}\right)^2\right] x_L + \left[1 - \left(\frac{q_L}{q_L+q_H}\right)^2\right] x_H = c_H. \qquad (27)$$

This expression shows that for the high-quality firm, the weights on the qualities distort the average upwards. Therefore, for given quantity levels, firm H's marginal revenue from increasing output is higher than in the standard Cournot model, as this increase has the additional effect of increasing the average quality.

4.2.2 Strategic relationship between the firms' actions

One can also rewrite (A) as:

$$a + x_L - c_L - bq_H - 2bq_L = \left(\frac{q_H}{q_L+q_H}\right)^2 \left(x_L - x_H\right). \qquad (28)$$

For a given value of q_H, the left-hand side (LHS) is a linear function of q_L with negative slope. The right-hand side (RHS) assumes only negative values and monotonically increases towards 0. These geometrical considerations confirm what was infered from the strict concavity of the high-cost firm's problem, i.e. that its best response is unique, and interior as long as $bq_H \leq a + x_H - c_L$. It is clearly continuous in q_H. A change in q_H shifts the straight line down but also pushes the hyperbola down (in the bottom-right quadrant) so that the total effect is a priori undeterminate.

An application of the implicit-function theorem in the neighborhood of the best interior response to q_H gives

$$\frac{dq_L}{dq_H} = \frac{\frac{q_L q_H}{(q_L+q_H)^3}(x_H - x_L) - \frac{1}{2}b}{\frac{(q_H)^2}{(q_L+q_H)^3}(x_H - x_L) + b} \tag{29}$$

So, if b is small, and the quality differential is large, an increase in q_H can drive the average quality sufficiently high for firm L to find it profitable to increase its own quantity. Figure 3 illustrates this possibility. It depicts firm L's best response (measured along the horizontal axis) as a function of q_H for the case when $a = 9$, $b = 1$, $x_L = 2$, $x_H = 100$, and $c_L = 1$.

This graph hides the fact that firm L's best response is initially decreasing in q_H (at the rate of $1/2$, like in the standard Cournot model). Figure 4 is a zoom around the point $\left(q_L^M, 0\right)$.

Thus, it is not always the case that the two producers' quantities are strategic substitutes, as in the standard Cournot model with linear demand. Observe that

$$\frac{dq_L}{dq_H} \geq 0 \Longleftrightarrow \frac{q_L q_H}{(q_L + q_H)^3}(x_H - x_L) - \frac{1}{2}b \geq 0. \tag{30}$$

The first term in the right-most inequality is the product of firm H's market share, denoted z, with the effect of a change in q_H on the average quality $\partial\bar{x}/\partial q_H$, itself equal to $(1-z)\frac{1}{Q}(x_H - x_L)$. From this observation stream several features of firm L's best response. First, BR_L cannot be upward-sloping at points that are too close to the axes or too distant from the origin. Second, being continuous, BR_L has to cross the 45-degree line. When it does, we have

$$\frac{dq_L}{dq_H} = \frac{\frac{(x_H - x_L)}{4Q} - \frac{1}{2}b}{\frac{(x_H - x_L)}{4Q} + b}, \tag{31}$$

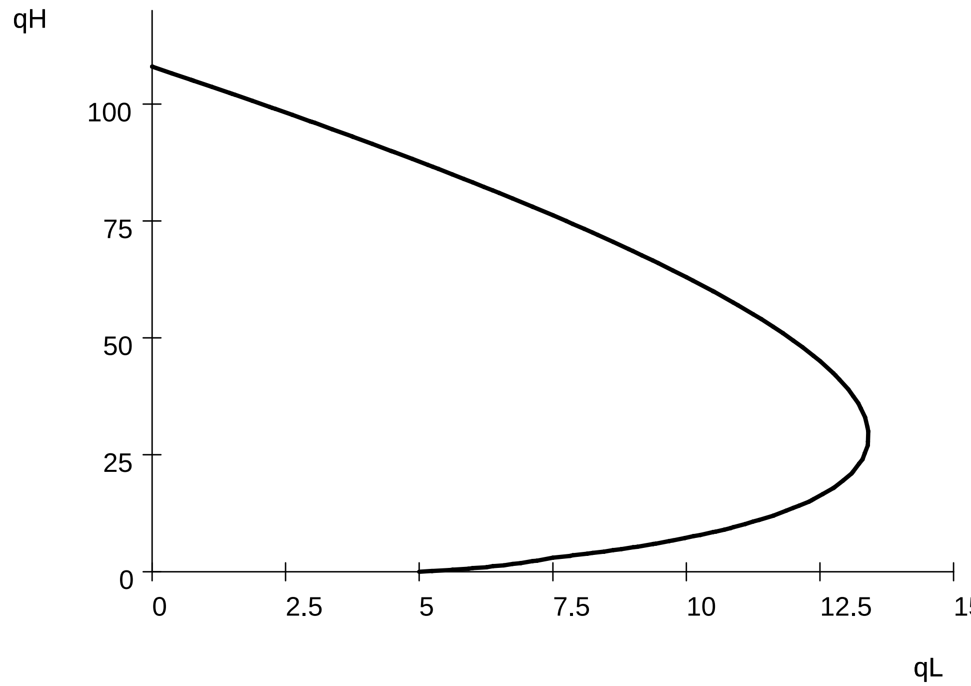

Figure 3: Firm L's best response when the quality differential is big

which is greater than $-1/2$ in all events. So because BR_L is moving away from the main diagonal *and* from the origin at this point, dq_L/dq_H must be decreasing in this neighborhood, implying that BR_L is concave. Once the main diagonal is crossed, because the curve moves away and q_H increases, the slope in (29) must converge to $-1/2$ from above, ruling out any inflexion and upward-sloping portion. So, if an upward-sloping portion exists, the inflexion must take place before the curve crosses the 45-degree line.

Similarly, one can rewrite (B) as

$$a + x_H - c_H - bq_L - 2bq_H = \left(\frac{q_L}{q_L + q_H} \right)^2 (x_H - x_L) \tag{32}$$

This time, for a given value of q_L, the LHS is a linear function of q_H with negative slope while the RHS is a downward-sloping hyperbola assuming only

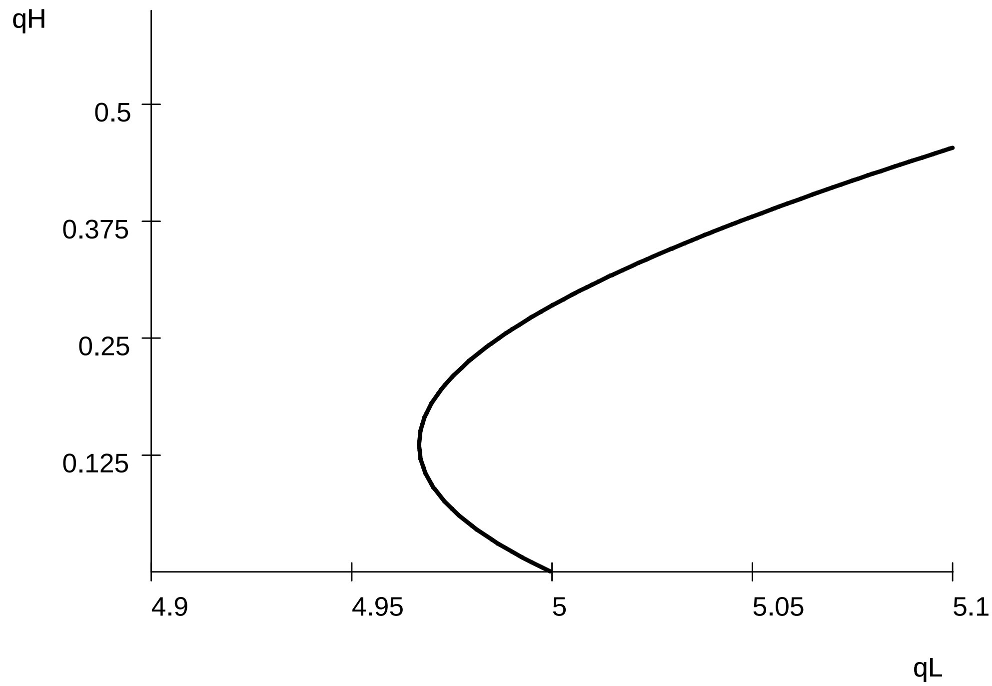

Figure 4: Firm L's best response when the quality differential is big (zoom)

positive values. Right of $-q_L$, it is thus possible for the curves to intersect once, twice, or not at all. The circumstances in which they intersect only once *in the positive orthant* are of course the same as the ones ensuring the global concavity of π_H on $\mathbb{R}_+$. The circumstances in which they don't intersect at all in the positive orthant are the reverse of the ones ensuring that there is an interior best-response. Observe that in the case where the curves intersect twice, there is no need for formally checking the second-order condition as the higher quantity always corresponds to the local maximum of the profit function on $(0, +\infty)$. Observe also that when q_L goes up, the line shifts down while the hyperbola shifts up. As a result, the q_H-coordinate of the left-most intersection goes up while the q_H-coordinate of the right-most intersection goes down.

Thus, firm H's unique interior best-response is locally strictly decreasing in q_L and concave. Unfortunately, firm H's best-response correspondence is not a continuous function on $\mathbb{R}_+$. Indeed, recall that, when the quality differential is large ($x_H - x_L > a + x_L - c_H$), firm H is indifferent between $q_H = 0$ and $q_H = \frac{a+x_H-c_H}{2b}\left(1 - \frac{a+x_H-c_H}{2(x_H-x_L)}\right) > 0$ for $q_L = \frac{(a+x_H-c_H)^2}{4b(x_H-x_L)}$. Thus, at this point, the best-response correspondence is not singleton-valued. It is upper-hemicontinuous, though, and each "branch" of the correspondence graph is nonincreasing in q_L.

We summarize these results in the following claim.[4]

Claim 4 *Firm L's best response $BR_L(q_H)$ is a continuous, possibly non-monotonic function of q_H, although it is strictly decreasing in the neighborhood of* 0 *and ρ_H. Firm H's best-response correspondence, $BR_H(q_L)$, is singleton-valued at all points, with the possible exception of the point $q_L = \frac{(a+x_H-c_H)^2}{4b(x_H-x_L)}$ in the event where $x_H - x_L > a + x_L - c_H$ (in which case the firm's problem admits both an interior and a corner solution). BR_H is nevertheless upper-hemicontinuous and non-increasing in q_L. On $[0, \rho_L)$, it is a strictly decreasing and concave function.*

Again, because of these features of firms' best-responses, one cannot use the tools commonly applied in oligopoly theory to show the general existence, unicity, or stability of Nash equilibria. In particular, owing to the "discontinuity" in BR_H, one can not resort to traditional fixed-point theorems to prove existence. Similarly, owing to the non-monotonicity of BR_L one cannot redefine strategic variables in a way that transforms the model into a supermodular game, as in the standard Cournot duopoly.

4.3 Monopolistic equilibria

The conditions for the existence of interior best-responses derived in the previous section allow us to study the existence of monopolistic equilibria in which one firm produces a strictly positive quantity (in fact, the monopoly output) and the other one withdraws from the market. There are two possible classes of monopolistic equilibria: one in which firm L is inactive and one in which firm H is inactive.

[4] In all rigor, by "non-increasing" we mean that if $g \in BR_H(q_L)$ and $h \in BR_H(q'_L)$ for any pair (q_L, q'_L) such that $0 \leq q_L < q'_L$, then $g \geq h$. Because this last inequality is true for any two *selections* from the correspondence at the two points, this property is sometimes referred to as "strong monotonicity".

4.3.1 Inactive low-quality firm

Take for our candidate equilibrium the situation in which the high-quality firm produces $q_H > 0$ and the low-quality firm shuts down: $q_L = 0$. For this latter behavior to be optimal one needs:

$$q_H \geq \frac{a + x_H - c_L}{b}. \tag{33}$$

The question is: Does firm H find it optimal to provide so large a quantity under the assumption that firm L leaves the market? In equilibrium, firm H should take firm L's withdrawal for granted and best-respond by producing the monopoly-profit-maximizing quantity, that is

$$q_H^M = \frac{a + x_H - c_H}{2b}. \tag{34}$$

This volume is greater than the treshold iff

$$\frac{a + x_H - c_H}{2b} \geq \frac{a + x_H - c_L}{b}, \tag{35}$$

or, equivalently

$$c_L - c_H \geq a + x_H - c_L. \tag{36}$$

Under our profitable supply assumption, this inequality is impossible to satisfy for $c_H \geq c_L$ but if marginal cost decreases sufficiently with quality, then there exits a monopolistic equilibrium in which firm L is inactive. The condition resembles the one in the standard Cournot model, in that the cost differential must be sufficiently unfavorable to firm L as to turn negative its margin on the first unit sold.

4.3.2 Inactive high-quality firm

Our next candidate equilibrium corresponds to the situation in which firm L produces $q_L^M = \frac{a+x_L-c_L}{2b}$ and the firm H shuts down: $q_H = 0$. For this latter behavior to be optimal one needs:

$$q_L^M \geq \frac{\max\left\{a + x_L - c_H, \min\left\{x_H - x_L, \frac{(a+x_H-c_H)^2}{4(x_H-x_L)}\right\}\right\}}{b}. \tag{37}$$

It is immediately observed that $\frac{a+x_H-c_H}{2}$ is the arithmetic average of $a + x_L - c_H$ and $x_H - x_L$. So we need to distinguish only two cases, according to the ranking of these two magnitudes.

Large quality differential Suppose that the quality differential is relatively big, so that

$$x_H - x_L \geq \frac{a + x_H - c_H}{2} \geq a + x_L - c_H. \tag{38}$$

Then

$$\min\left\{x_H - x_L, \frac{(a + x_H - c_H)^2}{4(x_H - x_L)}\right\} = \frac{(a + x_H - c_H)^2}{4(x_H - x_L)} \tag{39}$$

The relevant quantity treshold for q_L is then either $(a + x_L - c_H)/b$ or $\frac{(a+x_H-c_H)^2}{4b(x_H-x_L)}$. Note that for $x_H > x_L$:

$$a+x_L-c_H \leq \frac{(a + x_H - c_H)^2}{4(x_H - x_L)} \Longleftrightarrow (a + x_L - c_H)(x_H - x_L) \leq \left(\frac{a + x_H - c_H}{2}\right)^2. \tag{40}$$

If $a + x_L - c_H < 0$, then this proposition is always true, since $x_H > x_L$ by assumption. Else,

$$a+x_L-c_H \leq \frac{(a + x_H - c_H)^2}{4(x_H - x_L)} \Longleftrightarrow \sqrt{(a + x_L - c_H)(x_H - x_L)} \leq \frac{a + x_H - c_H}{2}. \tag{41}$$

As the geometric mean of two positive numbers is no greater than their arithmetic mean, we have

$$\sqrt{(a + x_L - c_H)(x_H - x_L)} \leq \frac{a + x_H - c_H}{2}. \tag{42}$$

Hence it is always true that

$$a + x_L - c_H \leq \frac{(a + x_H - c_H)^2}{4(x_H - x_L)} \tag{43}$$

As a result, the relevant treshold when the quality differential is high is always $\frac{(a+x_H-c_H)^2}{4b(x_H-x_L)}$. In a sense, all that matters for firm H is whether, in the event that the margin on the first unit sold is negative (owing to firm L's large production), it can restaure its profitability by driving the average quality up. We thus have

$$bq_L^M \geq \frac{(a + x_H - c_H)^2}{4(x_H - x_L)} \Longleftrightarrow \left(\frac{a + x_L - c_L}{2}\right)(x_H - x_L) \geq \left(\frac{a + x_H - c_H}{2}\right)^2, \tag{44}$$

which is true whenever $a+x_L-c_L$ is not much smaller than $a+x_H-c_H$. The exact treshold depends on the difference between $(x_H - x_L)$ and $\frac{a+x_H-c_H}{2}$. A sufficient condition for the inequality to hold is $c_H - c_L \geq x_H - x_L$ (but if a big difference between $a + x_L - c_H$ and $x_H - x_L$ exists, then firm L needs less of a cost advantage).

Small quality differential Suppose that

$$x_H - x_L < \frac{a + x_H - c_H}{2} < a + x_L - c_H. \tag{45}$$

Then the relevant treshold is $(a + x_L - c_H)/b$. This occurs whenever the demand for firm L's product is relatively high in comparison to the quality differential. Then, all that matters for firm H is whether the margin on the first unit sold is positive or not because whenever it is possible to enjoy a positive margin by driving quality up through mass production, it is also the case that there is profit to be made on the very first unit. In a monopolistic equilibrium, one has

$$q_L^M \geq \frac{a + x_L - c_H}{b} \iff \frac{a + x_L - c_L}{2} \geq a+x_L-c_H \iff c_H-c_L \geq a+x_L-c_H. \tag{46}$$

That is, once again, the cost differential must be large enough for a monopolistic equilibrium to exist. A necessary (but not sufficient) condition for it to hold is $c_H - c_L \geq x_H - x_L$.

We summarize the observations made in that section by providing a sufficient condition for the existence of a monopolistic equilibrium. In the standard Cournot model, monopolistic equilibria were sustained by a cost differential larger than the margin on the first unit sold by the evicted firm. This condition carries over but one also has to account for the fact that even if this margin is negative, firm H can (sometimes) drive prices up by increasing its quantity and thus average quality. In that instance, eviction is always possible if the cost differential is larger than the quality differential.

Claim 5 *If the cost differential is sufficiently unfavorable to firm i, that is if*

$$c_i - c_{-i} \geq \max\{a + x_{-i} - c_i \ , \ x_i - x_{-i}\}, \tag{47}$$

then there always exists an equilibrium in which firm $-i$ produces the monopoly output $q_{-i}^M = (a + x_{-i} - c_{-i}) / (2b)$ and firm i withdraws from the market.

4.4 Duopolistic equilibria

Claim 13 above established that firm L's best-response correspondence is in fact a continuous function, although possibly non-monotone, while firm H's best-response correspondence is upper semi-continuous, and singleton-valued at all points except, sometimes, at $q_L = \frac{(a+x_H-c_H)^2}{4b(x_H-x_L)}$. It remains to be shown that there are circumstances in which the best-response curves intersect away from the axes, thus proving the existence of a pure-strategy duopolistic equilibrium. As it happens, there are instances in which such an equilibrium fails to exist. We do not immediately tackle this question but instead look at the multiplicity issue. In a fashion that parallels our treatment of cost heterogeneity, we introduce x, taken to be the arithmetic average of qualities x_L and x_H, and let $\varepsilon = (x_H - x_L)/2$. This way, $x_L = x - \varepsilon$ and $x_H = x + \varepsilon$. We are now in the position to state our result on the number of duopolistic equilibria.

Proposition 6 *There exist at most two pure-strategy duopolistic equilibria.*

Proof. Suppose that there exists a duopolistic equilibrium in pure-strategies. The system of necessary conditions (A) and (B) can then be equivalently rewritten as

$$\left\{ \begin{array}{c} q_L = \frac{a+x_L+z(x_H-x_L)+c_H-2c_L+3z(z-1)(x_H-x_L)}{3b} \\ q_H = \frac{a+x_L+z(x_H-x_L)+c_L-2c_H+3z(1-z)(x_H-x_L)}{3b} \\ z = \frac{q_H}{q_L+q_H} \end{array} \right. . \tag{48}$$

Observe that q_L and q_H are uniquely determined by z and the parameters of the model.

By substitution of q_L and q_H, as given by the first two equations, into the third one, and by definition of x, ε, c, and δ, one obtains the following equation in z:

$$z = \frac{a + x + (2z-1)\varepsilon - c + 3\left[2\varepsilon z(1-z) - \delta\right]}{2\left[a + x + (2z-1)\varepsilon - c\right]}. \tag{49}$$

In a duopolistic equilibrium, $Q = q_L + q_H > 0$. So one can multiply both sides of equation (49) by its denominator to obtain a quadratic equation in z. Its discriminant equals $15\varepsilon^2 + M^2 - 30\delta\varepsilon$. If it is negative, then there cannot exist a solution to the system of equations (A) and (B), which contradicts our initial assumption. Suppose it is positive, then. The roots are given by

$$z^* = \frac{1}{2} + \frac{\sqrt{15\varepsilon(\varepsilon - 2\delta) + M^2} - M}{10\varepsilon}, \tag{50}$$

$$z^{**} = \frac{1}{2} - \frac{\sqrt{15\varepsilon(\varepsilon - 2\delta) + M^2} + M}{10\varepsilon}. \tag{51}$$

Each root need not correspond to an equilibrium but the associated quantities are the only candidate equilibria. Therefore, there can be at most two duopolistic equilibria. ■

As a by-product, the proposition singles out the candidate equilibria.

Substituting z^* back into the two first equations, we have:

$$\begin{array}{l} q_H^* = \frac{1}{3b}\left\{\frac{4M+\sqrt{15\varepsilon(\varepsilon-2\delta)+M^2}}{5} + \frac{3}{2}(\varepsilon - 2\delta) - \frac{3\left(\sqrt{15\varepsilon(\varepsilon-2\delta)+M^2}-M\right)^2}{50\varepsilon}\right\} \\ q_L^* = \frac{1}{3b}\left\{\frac{4M+\sqrt{15\varepsilon(\varepsilon-2\delta)+M^2}}{5} - \frac{3}{2}(\varepsilon - 2\delta) + \frac{3\left(\sqrt{15\varepsilon(\varepsilon-2\delta)+M^2}-M\right)^2}{50\varepsilon}\right\} \end{array}. \tag{52}$$

Thus

$$Q^* = \frac{2}{3b}\frac{4M + \sqrt{15\varepsilon(\varepsilon - 2\delta) + M^2}}{5}. \tag{53}$$

Plugging this quantity into the inverse demand curve gives

$$P^* = c + \frac{1}{3}\frac{4M + \sqrt{15\varepsilon(\varepsilon - 2\delta) + M^2}}{5}. \tag{54}$$

Substituting z^{**} back into the two first equations, we have:

$$\begin{array}{l} q_H^{**} = \frac{1}{3b}\left\{\frac{4M-\sqrt{15\varepsilon(\varepsilon-2\delta)+M^2}}{5} + \frac{3}{2}(\varepsilon - 2\delta) - \frac{3\left(\sqrt{15\varepsilon(\varepsilon-2\delta)+M^2}+M\right)^2}{50\varepsilon}\right\} \\ q_L^{**} = \frac{1}{3b}\left\{\frac{4M-\sqrt{15\varepsilon(\varepsilon-2\delta)+M^2}}{5} - \frac{3}{2}(\varepsilon - 2\delta) + \frac{3\left(\sqrt{15\varepsilon(\varepsilon-2\delta)+M^2}+M\right)^2}{50\varepsilon}\right\} \end{array}. \tag{55}$$

Thus

$$Q^{**} = \frac{2}{3b}\frac{4M - \sqrt{15\varepsilon(\varepsilon - 2\delta) + M^2}}{5}. \tag{56}$$

Plugging this quantity into the inverse demand curve gives

$$P^{**} = c + \frac{1}{3}\frac{4M - \sqrt{15\varepsilon(\varepsilon - 2\delta) + M^2}}{5}. \tag{57}$$

Several remarks are in order. First, in constrast with the standard Cournot duopoly, the market outcomes are not independent on the level of cost heterogeneity: the (candidate) equilibrium prices and quantities are functions of δ, and a change in firms' cost parameters (keeping the average constant) does more than shifting production and profit from one firm to the other.

Second, from the proof of Proposition 15, it is readily observed that $z^{**} < 1/2$ for any choice of parameters (as long as $M > 0$), and that $z^* > 1/2$ if and only if $\varepsilon > 2\delta$. So we have the following corollary.

Corollary 7 *There can exist a pure-strategy duopolistic equilibrium in which firm H produces a higher quantity than firm L ($q_H^* > q_L^*$) only if* $2(c_H - c_L) < x_H - x_L$*, or, equivalently,* $2\delta < \varepsilon$.

A sufficient condition for the last inequality to hold is that

$$\mu'(x) < \frac{1}{2} \tag{58}$$

The condition requires the marginal cost of production not to increase too quickly with quality.

Third, in those cases where firm H's problem is not concave, the pair (q_L^{**}, q_H^{**}) (assuming it involve positive quantities) may not be an equilibrium, as q_H^{**} can then correspond to a local minimum or a non-global maximum of firm H's profit function. In the first case, (q_L^{**}, q_H^{**}) lies on the lower branch of the locus of points satisfying firm H's first-order condition. In the second case, it lies on its upper branch but not on firm H's best-global-response curve, BR_H, because the corner solution is preferred.

Now, we have already argued that BR_H, in the range of quantities that elicit a strictly positive response from firm H (i.e. for $0 \leq q_L < \rho_L$) is a continuous and strictly decreasing function. From the expressions for the candidate equilibrium quantities, it is easy to see that $q_H^{**} < q_H^*$. Thus, if (q_L^{**}, q_H^{**}) lies on BR_H, then (q_L^*, q_H^*) lies on it as well.

Corollary 8 *If* $(q_L^{**}, q_H^{**}) \in \mathbb{R}_{++}^2$ *is an equilibrium, then* (q_L^*, q_H^*) *is an equilibrium as well, provided* $(q_L^*, q_H^*) \in \mathbb{R}_+^2$.

In particular, whenever $\varepsilon > 2\delta$, if there exists a duopolistic equilibrium in which $q_L^{**} > q_H^{**}$, then there exists another duopolistic equilibrium (q_L^*, q_H^*) in which $q_L^* < q_H^*$, provided $q_L^* > 0$.

The previous corollary bears on the issue of equilibrium selection. Indeed, recall that firm H's best-interior-response is a strictly increasing and concave function of q_L on $[0, \rho_L)$. So, if (q_L^{**}, q_H^{**}) lies on the upper "branch" of BR_H and therefore is an equilibrium, then it must be the case that BR_L is flatter than BR_H there (in the $q_L \times q_H$ space). That implies that (q_L^{**}, q_H^{**}) is always an unstable equilibrium under the usual (i.e. alternate) best-reply dynamics. For the reverse reason, (q_L^*, q_H^*) is always stable.

Corollary 9 *If an equilibrium, (q_L^*, q_H^*) is always stable under the usual best-reply dynamics. If an equilibrium, (q_L^{**}, q_H^{**}) is always unstable.*

In our detailed treatment below, we will thus focus on the monopolistic equilibria on one hand, and on (q_L^*, q_H^*) on the other hand.

5 A taxonomy of equilibria

We now want to describe the possible equilibrium outcomes in relation to the levels of cost heterogeneity and quality heterogeneity. The case where $\varepsilon = 0$ corresponds to the standard Cournot model and was covered in Section 3. We take in turn the cases where the cost function μ does not depend on quality, decreases with quality, and increases with it.

5.1 Identical marginal costs

Suppose that $c_H = c_L = c$, or $\delta = 0$.

There does not exist any monopolistic equilibrium in this situation as a firm cannot deter its rival from entering the market, which always requires a cost advantage. This is just another way of saying that $\rho_L > q_L^M$ and $\rho_H > q_H^M$ for all values of ε. So we focus on duopolistic equilibria. We claim that, independently on the level of quality heterogeneity, a unique, pure-strategy duopolistic equilibrium always exists when costs are identical.

Proposition 10 *For $\delta = 0$ and for any ε, there exists a unique stable, pure-strategy equilibrium, which is duopolistic and characterized by the following* *market share for firm H:*

$$z^* = \frac{1}{2} + \frac{\sqrt{15\varepsilon^2 + (a+x-c)^2} - (a+x-c)}{10\varepsilon}, \tag{59}$$

the following equilibrium quantity:

$$Q^* = \frac{2}{3b}\frac{\sqrt{15\varepsilon^2 + (a+x-c)^2} + 4(a+x-c)}{5}, \tag{60}$$

and equilibrium price:

$$P^* = c + \frac{1}{3}\frac{\sqrt{15\varepsilon^2 + (a+x-c)^2} + 4(a+x-c)}{5}. \tag{61}$$

Proof. From Section 3.4.3, we know there cannot exist any monopolistic equilibrium with $\delta = 0$.

From Section 3.4.4, there is only one stable, duopolistic candidate equilibria, characterised by:

$$z^* = \frac{1}{2} + \frac{\sqrt{15\varepsilon^2 + M^2} - M}{10\varepsilon}, \tag{62}$$

The equilibrium quantities must be:

$$\begin{aligned} q_H^* &= \frac{1}{3b}\left\{\frac{\sqrt{15\varepsilon^2 + M^2} + 4M}{5} + \left[\frac{3}{2}\varepsilon - \frac{3\left(\sqrt{15\varepsilon^2 + M^2} - M\right)^2}{50\varepsilon}\right]\right\}, \\ q_L^* &= \frac{1}{3b}\left\{\frac{\sqrt{15\varepsilon^2 + M^2} + 4M}{5} - \left[\frac{3}{2}\varepsilon - \frac{3\left(\sqrt{15\varepsilon^2 + M^2} - M\right)^2}{50\varepsilon}\right]\right\}, \end{aligned}$$

giving

$$Q^* = \frac{2}{3b}\frac{\sqrt{15\varepsilon^2 + M^2} + 4M}{5}, \tag{63}$$

and

$$P^* = c + \frac{1}{3}\frac{\sqrt{15\varepsilon^2 + M^2} + 4M}{5}. \tag{64}$$

The point (q_L^*, q_H^*) will not correspond to an equilibrium in the circumstances where either (i) it lies outside the positive orthant, or (ii) the first-order conditions do not characterize the firm's best *global* responses.

As for (i), observe that it is not possible that both q_L^* and q_H^* be simultaneously negative since $\frac{\sqrt{15\varepsilon^2+M^2}+4M}{5} > 0$. Therefore, if (q_L^*, q_H^*) lies outside the positive orthant, then $\frac{q_H^*}{q_L^*} = \frac{z^*}{1-z^*} \leq 0$, which occurs only if $z^* \leq \frac{1}{2}$. This cannot happen since $z^* > \frac{1}{2}$ for any $\varepsilon > 0$.

As for (ii), there is no issue if the quality differential is small, for then BR_H is a continuous function which coincides with the locus of points satisfying the first-order condition for all $q_L \in [0, \rho_L]$. If the quality differential is large (i.e. if $a+x_L-c_H \leq x_H - x_L$, or, equivalently, $\varepsilon > M/3$), then BR_H exhibits a non-convexity at $\tilde{q}_L = \frac{(a+x_H-c_H)^2}{4b(x_H-x_L)} = \frac{M+\varepsilon}{2b}\frac{M+\varepsilon}{4\varepsilon}$, at which point both $q_H = 0$ and $\tilde{q}_H = \frac{M+\varepsilon}{2b}\left(1 - \frac{M+\varepsilon}{4\varepsilon}\right)$ are optimal responses. A simple computation shows that at the best interior response,

$$\tilde{z} = 1 - \frac{M+\varepsilon}{4\varepsilon} = \frac{3\varepsilon - M}{4\varepsilon}. \tag{65}$$

If we can show that $z^* \geq \tilde{z}$ for any choice of parameters, then we will be done as we will be reassured that the intersection of firm L's best *interior* response curve with firm H's best *interior* response curve lies to the left of the possible "jump" in firm H's best *global* response. Now,

$$z^* - \tilde{z} = \frac{3M - 5\varepsilon + 2\sqrt{15\varepsilon^2 + M^2}}{20\varepsilon}, \tag{66}$$

which a simple computation shows is positive for any M and ε. ■

Several remarks are in order before we proceed with illustrations. First, by L'Hospital rule, q_L^* and q_H^* tend to $(a+x-c)/3b$ as ε tends to 0. In that sense, the standard Cournot result with identical marginal costs is robust to the homogeneity assumption, as the introduction of a small quality differential leads to an equilibrium that is "close" to the usual Cournot outcome.

Second, z^* monotonically increases toward $1/2 + \sqrt{15}/10 \simeq 0.89$ as ε increases. So, firm H's market share always goes up as this firm's quality advantage increases. Nonetheless, even in the case of an extreme advantage, firm L's always secures at least 10% of the sales. It is never possible for the quality leader to reduce its competitor to insignificance on this market.

Third, this pattern can be explained by the rates of change in the firms' equilibrium quantities. q_H^* is an increasing function of ε for any M and b. Asymptotically, it increases linearly, which implies that it grows with ε at a smaller and smaller rate. q_L^* first decreases then increases with ε. In the

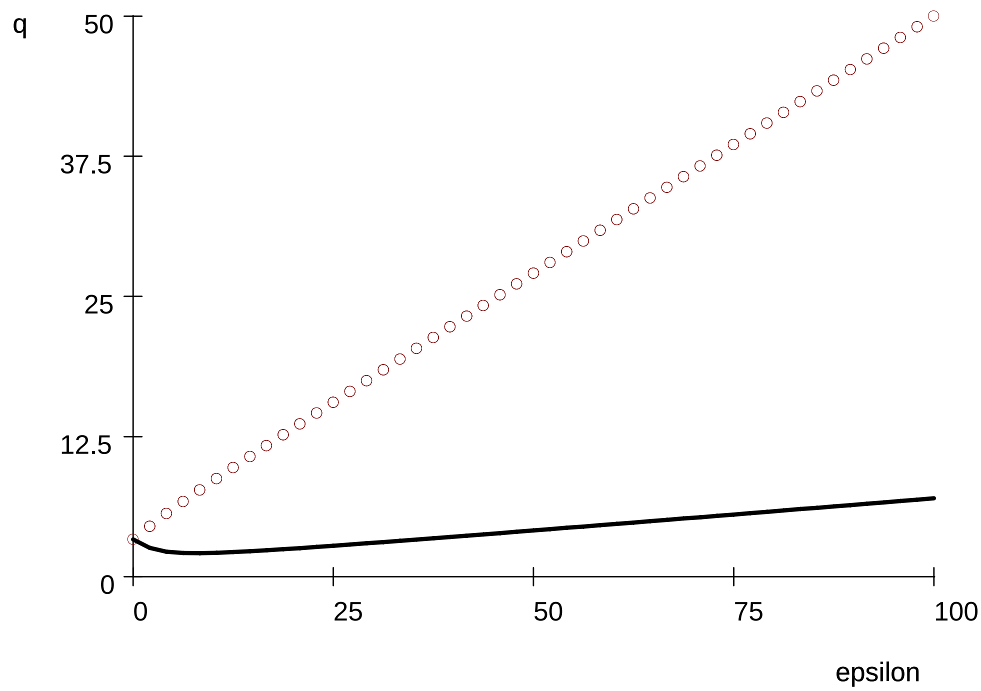

Figure 5: Equilibrium quantities for $M = 10$ and $b = 1$

limit, it also tends to increase linearly. The convergence of z^* implies that the growth rates of both equilibrium quantities must themselves converge.

Figure 5 displays the equilibrium quantities as functions of ε in the case when $M = 10$ and $b = 1$. The solid line is firm L's quantity. The dotted line is firm H's quantity.

The same data, plotted against a logarithmic scale on figure 6, makes clear that after an initial divergence phase, the two quantities grow at the same rate.

We now illustrate the equilibrium itself with two examples, typical of the two possible configurations that arise when $\delta = 0$.

Figure 7 is for the case when $a = 10$, $b = 2$, $x_L = 1$, $x_H = 2$, $c_L = c_H = 1$. That corresponds to $x = 1.5$, $c = 1$, $\delta = 0$, $\varepsilon = .5$. The solid line is firm L's best response. The dotted line is firm H's best response. The dashed

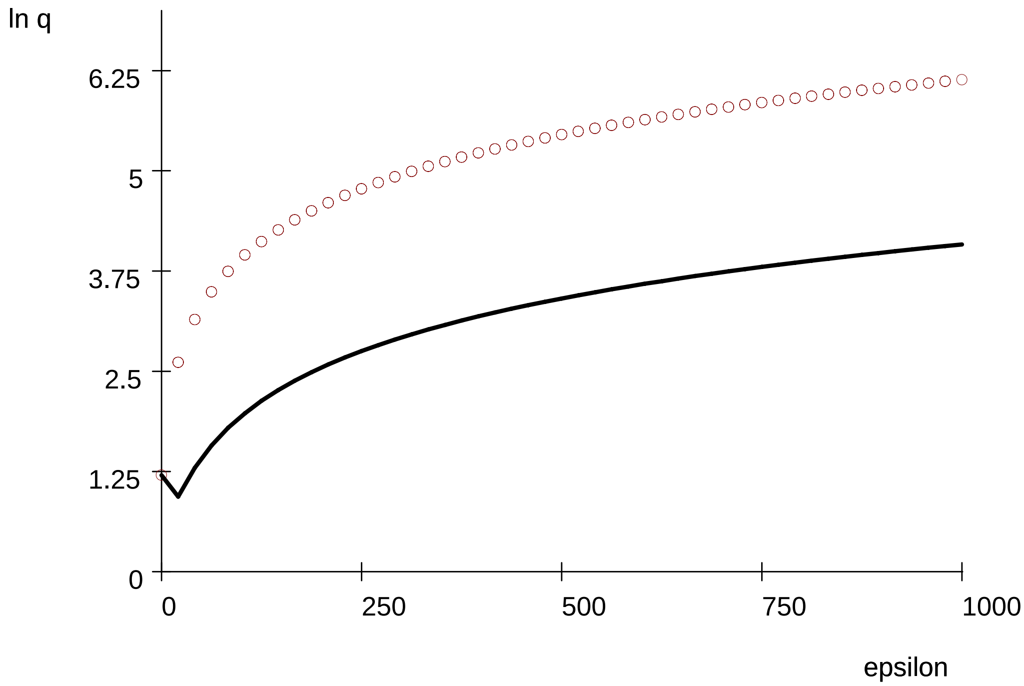

Figure 6: Equilibrium quantities for $M = 10$ and $b = 1$ - logarithmic scale

line is the main diagonal. The quality differential being small, both best responses are continuous, decreasing functions of the other firm's quantity, as in the standard Cournot model. The intersection lies to the left of the main diagonal, implying that firm H dominates the market.

Figure 8 is for the case when $a = 10$, $b = 2$, $x_L = 1$, $x_H = 101$, $c_L = c_H = 1$. That corresponds to $x = 51$, $c = 1$, $\delta = 0$, $\varepsilon = 50$. Because of the large quality differential, BR_L becomes non-monotone and BR_H exhibits a nonconvexity at 15.125. Nonetheless, the curves intersect only once, to the left of the main diagonal and the "jump" in BR_H.

5.2 Decreasing marginal cost

Consider the case now where c decreases with quality. This is not as implausible a situation as it might seem. Quality could be associated with the use of

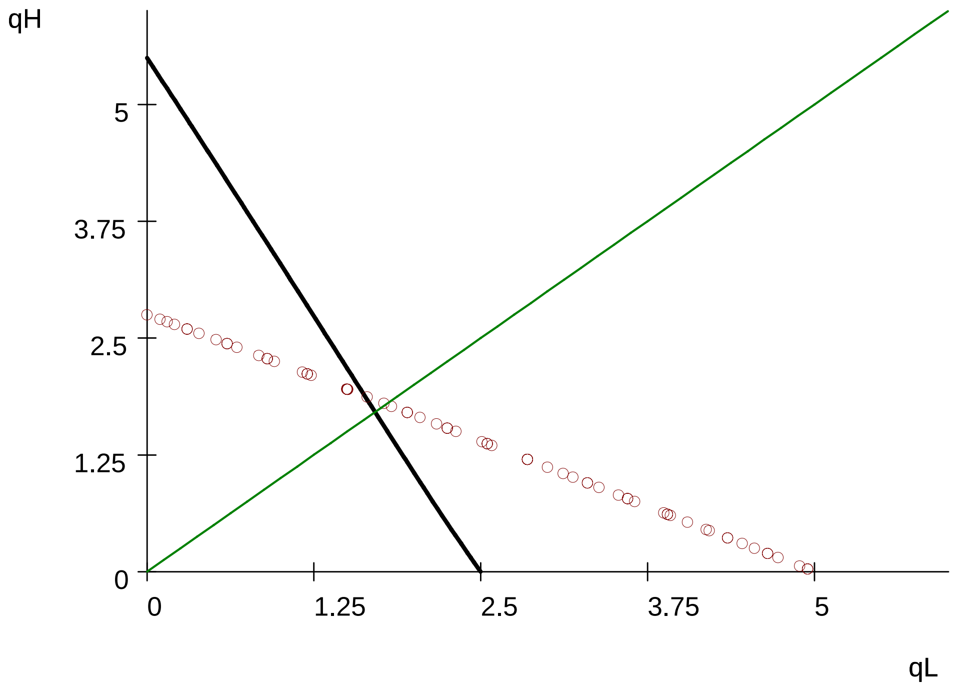

Figure 7: The unique equilibrium in the case of a small quality differential ($x_L = 1$; $x_H = 2$) and no cost differential ($c_L = c_H = 1$)

technologies requiring big set-up or fixed costs but commanding low marginal costs. In fact, any quality-improving mechanization of the production process would constitute an example of this phenomenon. The market for collected blood is also often mentioned in that context, as donors tend to self-select in such a way that the quality of the blood collected from volunteers is on average higher than the quality of the blood collected from profit-motivated donors, which generates an inverse relationship between quality and variable cost.

In any case, assume in that section that $\delta < 0$.

In this configuration, there cannot exist a monopolistic equilibrium in which only firm L is active as it is at a cost disadvantage. By contrast, from Section 4.3 we know that firm H can remain the only active firm if and only

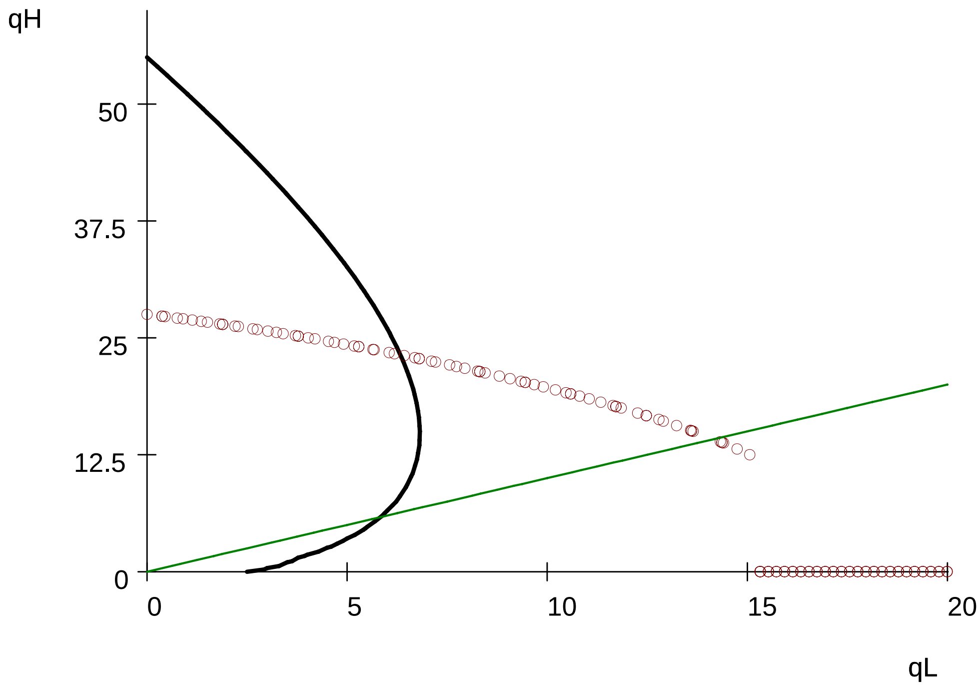

Figure 8: The unique equilibrium in the case of a large quality differential ($x_L = 1$; $x_H = 101$) and no cost differential ($c_L = c_H = 1$)

if $c_L - c_H \geq a + x_H - c_L$, or equivalently:

$$-\delta \geq \frac{M + \varepsilon}{3}. \tag{67}$$

For a given M, the right-hand side is a linear function of ε. The higher the quality differential, the bigger firm H's cost advantage must be in order for it to monopolize the market. That is, a big quality advantage makes it *harder* for the leading firm to evict its competitor. This is of course because the latter can free-ride on consumers' high valuation of the product and is therefore led to increase its quantity. Therefore, firm H must be in the position to dump a very big quantity on the market in order to preclude firm L's entry.

Two questions remain. When a monopolistic equilibrium exists, can it co-exist with a duopolistic equilibrium? When a monopolistic equilibrium does not exist, does a duopolistic equilibrium always exist? The answers are "no" and "yes", respectively.

Proposition 11 *Suppose that $\delta < 0$. (i) If $-\delta \geq \frac{M+\varepsilon}{3}$, then there is a unique stable, pure-strategy equilibrium, which is H-monopolistic. (ii) If $-\delta < \frac{M+\varepsilon}{3}$, then there is a unique stable pure-strategy equilibrium, which is duopolistic, and characterized by the following market share for firm H:*

$$z^* = \frac{1}{2} + \frac{\sqrt{15\varepsilon(\varepsilon - 2\delta) + (a + x - c)^2} - (a + x - c)}{10\varepsilon}, \tag{68}$$

the following equilibrium quantity:

$$Q^* = \frac{2}{3b} \frac{\sqrt{15\varepsilon(\varepsilon - 2\delta) + (a + x - c)^2} + 4(a + x - c)}{5}, \tag{69}$$

and equilibrium price:

$$P^* = c + \frac{1}{3} \frac{\sqrt{15\varepsilon(\varepsilon - 2\delta) + (a + x - c)^2} + 4(a + x - c)}{5}. \tag{70}$$

Proof. From Section 3.4.4, there is only one stable, pure-strategy duopolistic equilibrium candidate, characterized by

$$z^* = \frac{1}{2} + \frac{\sqrt{15\varepsilon(\varepsilon - 2\delta) + M^2} - M}{10\varepsilon} > \frac{1}{2}. \tag{71}$$

Plugging z^* back in the system of necessary conditions:

$$\left.\begin{array}{l} q_H^* = \frac{1}{3b}\left\{\frac{\sqrt{15\varepsilon(\varepsilon-2\delta)+M^2}+4M}{5} + \left[\frac{3}{2}(\varepsilon - 2\delta) - \frac{3\left(\sqrt{15\varepsilon(\varepsilon-2\delta)+M^2}-M\right)^2}{50\varepsilon}\right]\right\} \\ q_L^* = \frac{1}{3b}\left\{\frac{\sqrt{15\varepsilon(\varepsilon-2\delta)+M^2}+4M}{5} - \left[\frac{3}{2}(\varepsilon - 2\delta) - \frac{3\left(\sqrt{15\varepsilon(\varepsilon-2\delta)+M^2}-M\right)^2}{50\varepsilon}\right]\right\} \end{array}\right\}. \tag{72}$$

Because q_L^* and q_H^* cannot be simultaneously negative, one only needs to guarantee that $z^* \in \left(\frac{1}{2}, 1\right)$, which requires

$$0 \leq \frac{\sqrt{15\varepsilon^2 + M^2 - 30\delta\varepsilon} - M}{10\varepsilon} < \frac{1}{2}. \tag{73}$$

The left-most inequality is trivially satisfied for $\varepsilon > 0$ and $\delta < 0$. The right-most inequality is satisfied if and only if $-\delta < \frac{M+\varepsilon}{3}$, which is the same condition as the one leading to the non-existence of a monopolistic equilibrium. Thus, a duopolistic equilibrium and a monopolistic equilibrium cannot coexist.

It remains to show that a duopolistic equilibrium never fails to exist when $-\delta < \frac{M+\varepsilon}{3}$. So suppose this inequality holds and distinguish the cases of a small and large quality differential.

Take the case of a small quality differential first. Suppose that

$$x_H - x_L < a + x_L - c_H, \tag{74}$$

or, equivalently,

$$-\delta > 3\varepsilon - M. \tag{75}$$

Then, BR_H is a non-increasing, continuous *function* of q_L, and we have $q_L^M < \rho_L$ and $q_H^M < \rho_H$. As the two continuous best- response curves must then intersect away from the axes, the pure-strategy duopolistic equilibrium always exists.

Assume now that

$$-\delta \leq 3\varepsilon - M. \tag{76}$$

Then BR_H is no longer a continuous curve. Nevertheless, in the parameter region where there is no monopolistic equilibrium, the interior best-response curves intersect in the positive orthant and it only remains to be checked that this intersection lies left of the "jump" in BR_H.

Observe that the non-convexity in BR_H occurs at $\tilde{q}_L = \frac{(a+x_H-c_H)^2}{4b(x_H-x_L)}$, at which point the best interior reply is $\tilde{q}_H = \tilde{q}_L(1-\tilde{q}_L)$. Firm H's market share at this point is given by

$$\tilde{z} = 1 - \frac{M+\varepsilon-\delta}{4\varepsilon}. \tag{77}$$

(q_L^*, q_H^*) lies to the left of the non-convexity only if $z^* \geq \tilde{z}$. We have

$$z^* - \tilde{z} = \frac{3M - 5\varepsilon - 5\delta + 2\sqrt{15\varepsilon(\varepsilon - 2\delta) + M^2}}{20\varepsilon}. \tag{78}$$

A simple computation shows that this quantity is always positive, so that this constraint never binds. Therefore, (q_L^*, q_H^*) never lies right of the "jump" in BR_H. ■

5.3 Increasing marginal cost

We now turn to the case when $\delta > 0$, i.e. when marginal cost increases with quality.

5.3.1 Monopolistic equilibrium region

In this configuration, firm H has a quality advantage over firm L but suffers from a cost disadvantage. Hence, there cannot exist any H-monopolistic equilibrium. An L-monopolistic equilibrium may exist, under conditions that vary with the size of the quality differential. We accordingly distinguish cases.

Small quality differential Suppose that

$$x_H - x_L \leq a + x_L - c_H, \tag{79}$$

or, equivalently,

$$\delta \leq M - 3\varepsilon. \tag{80}$$

In that case, there can exist a L-monopolistic equilibrium only if

$$c_H - c_L \geq a + x_L - c_H, \tag{81}$$

or, equivalently,

$$\delta \geq \frac{M - \varepsilon}{3}. \tag{82}$$

Observe that in that parameter range, a higher quality differential corresponds to a *lower* cost differential treshold. This is caused by our symmetric measure of quality heterogeneity. When ε increases, firm L's quality diminishes, which decreases the willingness to pay on a market dominated by frm L. That makes it "easier" for firm L to evict firm H (in equilibrium), in the sense that it requires a smaller cost advantage.

Large quality differential Suppose now that

$$x_H - x_L > a + x_L - c_H, \tag{83}$$

or, equivalently,

$$\delta > M - 3\varepsilon. \tag{84}$$

In that case, there can exist a L-monopolistic equilibrium if and only if

$$\left(\frac{a+x_L-c_L}{2}\right)(x_H-x_L) \geq \left(\frac{a+x_H-c_H}{2}\right)^2, \tag{85}$$

or, equivalently,

$$\varepsilon\left(M+\delta-\varepsilon\right) \geq \frac{\left(M-\delta+\varepsilon\right)^2}{4}. \tag{86}$$

This inequation is quadratic in δ, and is always verified within a closed interval lying in $\mathbb{R}_+$, whose bounds depend on M and ε. More precisely, it is true whenever

$$M+3\varepsilon-2\sqrt{\varepsilon^2+2M\varepsilon} \leq \delta \leq M+3\varepsilon+2\sqrt{\varepsilon^2+2M\varepsilon}. \tag{87}$$

The right-most inequality is an algebraic artefact. It corresponds to a situation where the original inequality would be true only because firm H would make enormous losses, which is ruled out. So, in effect, the left-most inequality sets the lower bound on δ for firm L to enjoy monopoly in equilibrium. Asymptotically, this bound approaches $M+\varepsilon$ from below. One implication is that, even when the quality differential is extremely large, there always exist circumstances (i.e. combinations of δ and ε) in which firm H would have profitably sold its product on a separate market (although barely so) but ends up being inactive in equilibrium when buyers do not distinguish its product from firm L's product.

Figure 9 illustrates the determination of the monopolistic equilibrium region's border.

The steeper, downward-sloping, thin line delineates the region corresponding to what we have called a "small quality differential." The less steep, downward-sloping, thin line is the lower contour of the region where the margin "on the first unit sold" by firm H on a market monopolized by firm L is negative. The dashed curve stands for the lower contour of the region in which the best interior margin for firm H is negative. In case of a small quality differential area, the relevant treshold is the margin on the "first unit". By contrast, outside the small quality differential area, the best interior margin constitutes the relevant treshold. Therefore, the thick curve delineates the bottom of the L-monopolistic equilibrium region.

5.3.2 Duopolistic equilibrium region

As the argument for that case is a bit long, we present it verbally rather than in a formal proof.

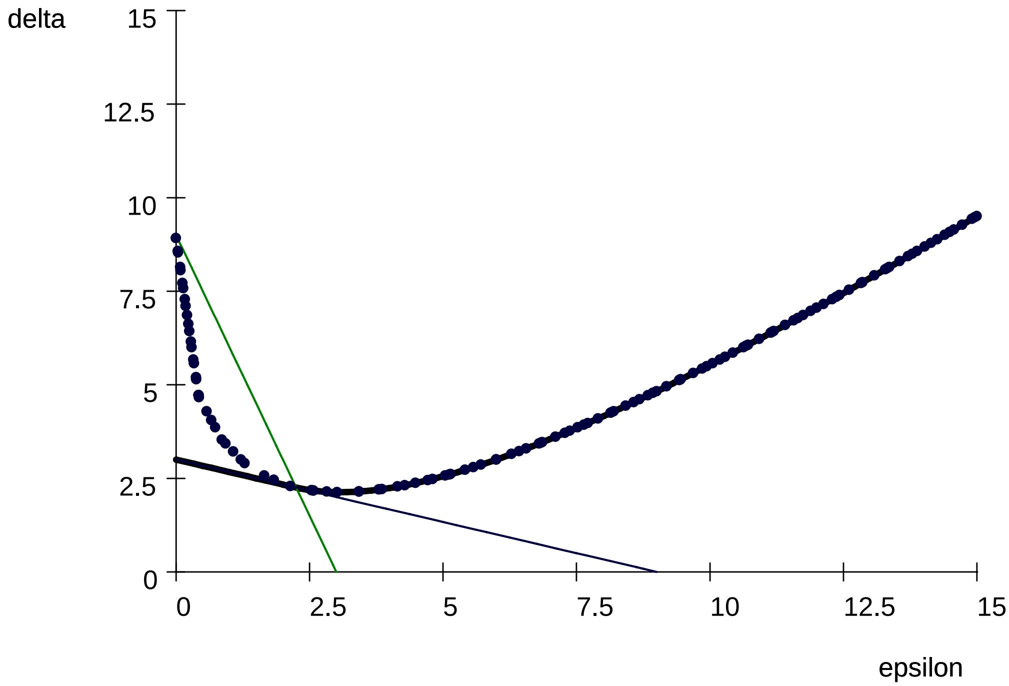

Figure 9: The border of the L-monopolistic equilibrium region $(M = 9)$

Recall that there are only two candidate duopolistic equilibria. In order to prove that they indeed correspond to equilibria, one has to check for each of them that several conditions are satisfied:

(i) that the quantities are real numbers; that is, that the best-interior-response curves do intersect;

(ii) that the quantities are strictly positive; that is, that the intersection lies in the positive orthant;

(iii) that q_L is less than $\tilde{q}_L$, the value for which BR_H might exhibit a "jump"; that is, that the intersection of the best-interior-response curves corresponds to an intersection of the best-(global)-response curves.

By means of an example, we first show that it is possible for both candidates to be equilibria, along with a third, monopolistic equilibrium, even in the case of a small quality differential. Figure 10 is for the case when

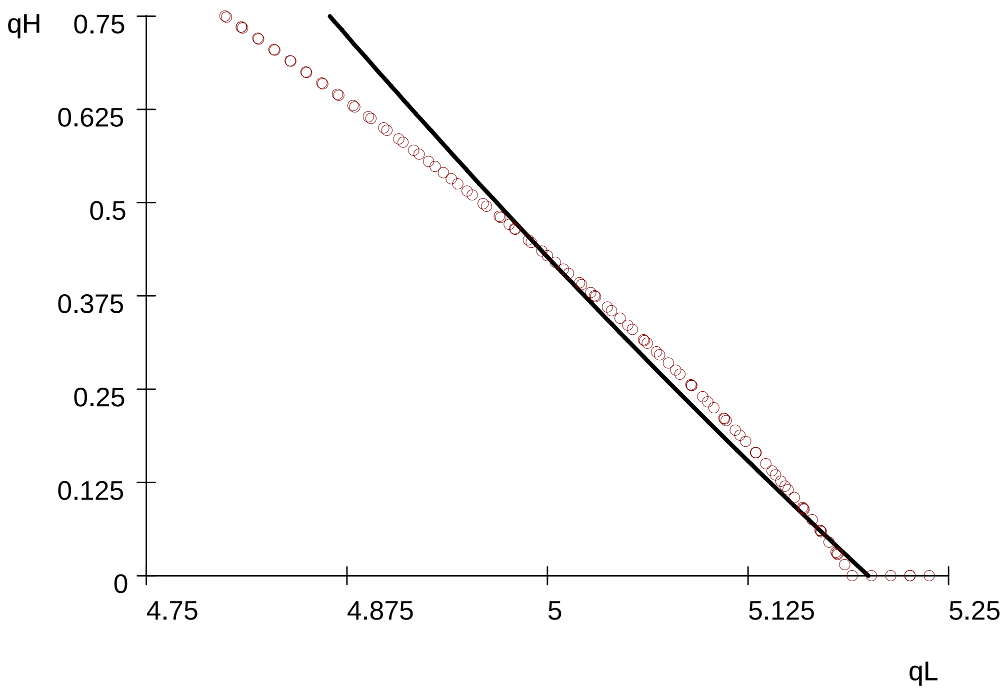

Figure 10: Three pure-strategy equilibria ($M = 9.995$, $\delta = 2.605$, $\varepsilon = 2.2$)

$a = 9.4$, $b = 1$, $x_L = 2$, $x_H = 6.4$, $c_L = 1$; and $c_H = 6.21$. That corresponds to $x = 4.2$, $c = 3.605$, $\delta = 2.605$, $\varepsilon = 2.2$.

Next, as far as condition (i) is concerned, observe from the proof of Proposition 15 that the system of equations (A) and (B) does not admit a solution if

$$15\varepsilon(\varepsilon - 2\delta) + M^2 < 0. \tag{88}$$

So, a pure-strategy duopolistic equilibrium fails to exist if

$$\delta > \frac{\varepsilon}{2} + \frac{M^2}{30\varepsilon}. \tag{89}$$

As the condition above is at times less stringent than the one corresponding to the existence of a monopolistic equilibrium, there is a whole range of parameters for which a pure-strategy equilibrium simply fails to exist. Such

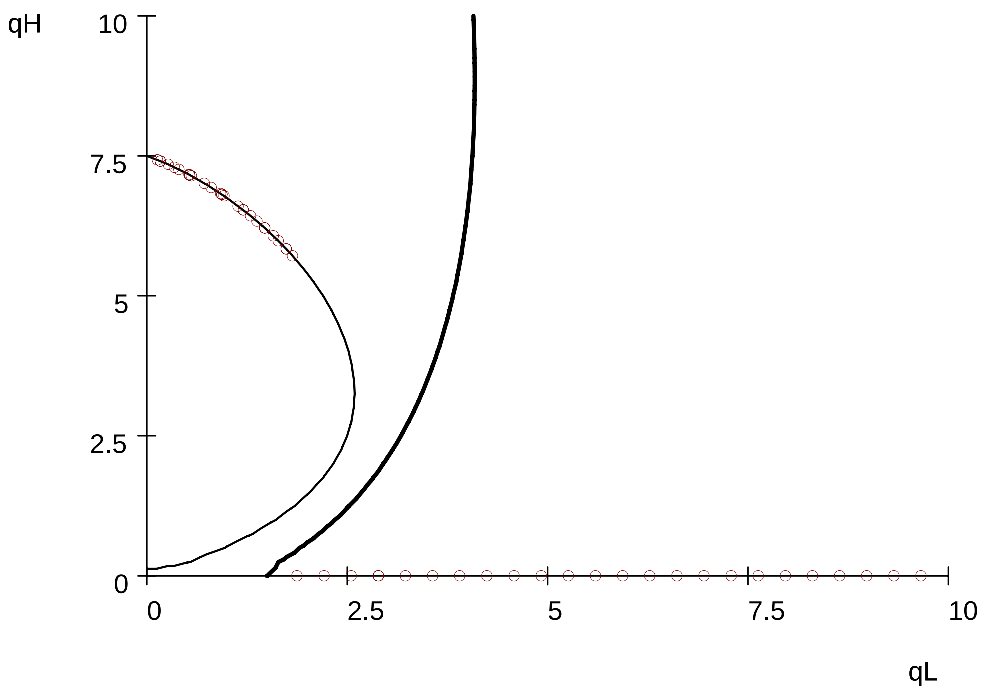

Figure 11: No equilibrium in pure strategies ($M = 9$, $\delta = 9$, $\varepsilon = 15$)

combinations of δ and ε correspond to situations where firm H's problem is not concave. The best-interior-response then decreases with q_L until it corresponds to a simple inflexion point of the profit function, after which point it is no longer well-defined.

We illustrate this possibility with the following example, displayed on figure 11: $a = 0$, $b = 1$, $x_L = 4$, $x_H = 34$, $c_L = 1$, $c_H = 19$. That corresponds to $x = 19$, $c = 10$, $M = 9$, $\delta = 9$, and $\varepsilon = 15$. The thin, rounded curve is the locus of all points satisfying the first-order condition. For low q_L, there are two of them, corresponding to a local minimum and a local maximum of π_H. For large q_L, such points do no exist. The circles stand for BR_H, which exhibits a "jump" at 1.875. Clearly, the two best-response curves do not intersect; there is no pure-strategy equilibrium.

We now check conditions (ii) and (iii) for the only stable, duopolistic candidate equilibrium, characterized by

$$z^* = \frac{1}{2} + \frac{\sqrt{15\varepsilon(\varepsilon - 2\delta) + M^2} - M}{10\varepsilon}. \tag{90}$$

Intersection in the positive orthant Even if the best-interior-response curves intersect, the intersection may lie outside the positive orthant. As it is not possible for q_L^* and q_H^* to be both negative, one only needs to check that $z^* \in (0, 1)$. We have to distinguish several cases.

If $\varepsilon = 2\delta$, then $z^* = \frac{1}{2}$, and the intersection always lies in the positive orthant.

If $\varepsilon > 2\delta$, then $z^* > \frac{1}{2}$ and we must ensure that

$$0 < \frac{\sqrt{15\varepsilon(\varepsilon - 2\delta) + M^2} - M}{10\varepsilon} < \frac{1}{2}, \tag{91}$$

which is true if and only if

$$\delta > \frac{-M - \varepsilon}{3}. \tag{92}$$

This condition is always satisfied when $\delta > 0$.

If $\varepsilon < 2\delta$, then $z^* < \frac{1}{2}$ and we must ensure that

$$0 > \frac{\sqrt{15\varepsilon(\varepsilon - 2\delta) + M^2} - M}{10\varepsilon} > -\frac{1}{2}. \tag{93}$$

If $M - 5\varepsilon \geq 0$, then the right-most inequality holds only if

$$\delta < \frac{M - \varepsilon}{3}. \tag{94}$$

This condition is the reverse of the one delineating the border of the monopolistic equilibrium region. Therefore, for $\varepsilon \leq \frac{M}{5}$, a monopolistic and a duopolistic equilibrium characterized by z^* cannot co-exist. If $M - 5\varepsilon < 0$, then the condition is always satisfied.

Intersection left of the non-convexity in BR_H Even if the best-interior-response curves do intersect in the positive orthant, their intersection might not lie on the best-global-response curves. We thus have to check that it takes place left of the possible "jump" in BR_H. In the case of a small quality differential, BR_H is a non-increasing, continuous *function* of q_L. As the two continuous best-response curves must then intersect, a pure-strategy equilibrium always exists. If $\delta < \frac{M-\varepsilon}{3}$, we also have $q_L^M < \rho_L$ and $q_H^M < \rho_H$, and this equilibrium must be duopolistic. In that case, (q_L^*, q_H^*) lies in the positive orthant. It must be an equilibrium then, since by Corollary 8 (q_L^{**}, q_H^{**}) cannot be an equilibrium without (q_L^*, q_H^*) being an equilibrium as well.

The non-convexity issue arises only in the case of a large differential. Observe that the non-convexity in BR_H then occurs at $\tilde{q}_L = \frac{(a+x_H-c_H)^2}{4b(x_H-x_L)}$, at which point the best interior reply is $\tilde{q}_H = \frac{a+x_H-c_H}{2b}\left(1 - \frac{a+x_H-c_H}{2(x_H-x_L)}\right)$. Firm H's market share at this point is given by

$$\tilde{z} = 1 - \frac{M+\varepsilon-\delta}{4\varepsilon}. \tag{95}$$

(q_L^*, q_H^*) lies to the left of the non-convexity only if $z^* \geq \tilde{z}$.

We have

$$z^* - \tilde{z} = \frac{3M - 5\varepsilon - 5\delta + 2\sqrt{15\varepsilon(\varepsilon - 2\delta) + M^2}}{20\varepsilon}. \tag{96}$$

It is possible to show that this quantity is always positive, so that this constraint never binds: (q_L^*, q_H^*) never lies right of the "jump" in BR_H.

We thus conclude with the following statement.

Proposition 12 *When $\delta > 0$, (q_L^*, q_H^*) is a pure-strategy duopolistic equilibrium if and only if the following conditions are satisfied:*
(i) $\delta \leq \frac{\varepsilon}{2} + \frac{M^2}{30\varepsilon}$;
(ii) if $\varepsilon \leq \frac{M}{5}$ and $\varepsilon < 2\delta$, then $\quad \delta < \frac{M-\varepsilon}{3}$.

5.4 Summary

We can now summarize our findings regarding the existence and nature of stable pure-strategy Nash equilibria with figure 12 (drawn to scale for the case where $M = 9$).

The quality differential, ε, is on the horizontal axis. The cost differential, δ, is on the vertical axis.

The downward-sloping line marked with circles delineates the small quality differential area. The thick, downward-sloping line that is the closest to the bottom of the figure is the border between the H-monopolistic equilibrium region and the duopolistic equilibrium region. Left of the vertical,

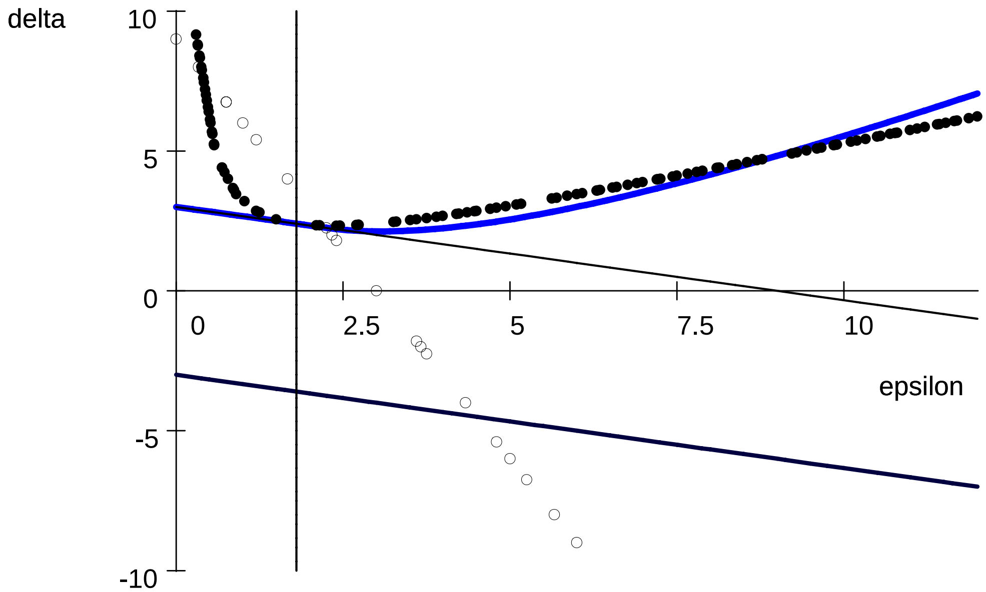

Figure 12: A typical equilibrium map ($M = 9$)

dashed line, this region extends upward until the thin, parallel, downward-sloping line is reached. Right of the vertical line, it extends until the dotted curve is reached. The thick, non-monotone curve is the border between the L-monopolistic equilibrium region and the duopolistic equilibrium region. The two latter curves are tangent to each other exactly on the vertical line. Letting the quality differential increase from that point, they delineate a region where the duopolistic and the L-monopolistic region co-exist.

The three most noticeable features of the model are the following.

First, the border of the L-monopolistic area is not monotone. When the quality differential is large, the high-quality firm can restablish its margin by flooding the market and driving quality and price up. Thus, firm L needs a big cost advantage in order to force firm H's shut-down.

Second, when the quality differential and the cost differential are of the same large order of magnitude, two stable equilibria in pure strategies coexist: one duopolistic equilibrium and one L-monopolistic equilibrium. In effect, the nature of the interection between the two firms is akin to a battle-of-the-sexes game: either the ratio of q_H over q_L is low and so are price and quality; or the ratio of q_H over q_L is high and so are price and quality.

Third, when the quality differential and the cost differential are very large, a pure-strategy equilibrium may fail to exit. This outcome is the combination of the discontinuity in firm H's optimal-response behavior and the non-monotonicity of firm L's optimal-response behavior. In terms of the usual best-reply dymanics, this is a classical instance of cycling. If H exits, then L wants to exert its monopoly power and restrict output. Given this low output, H wants to re-enter and being a large quantity to the market, driving quality and price up. L then wants to take advantage of the high margins and increase its output, forcing H's exit.

6 Welfare considerations

Rather than carrying out a detailed welfare analysis, our aim in this section is to underline the important aspects in which our strategic-quantity-setting model may depart from the conventional wisdom about the provision of quality in an environment with asymmetric information. We firstly show by means of an example that strategic behavior can mitigate the adverse selection problem. Secondly, we compare consumer surplus in two different situations to show that buyers can potentially benefit from the variation in quality at the producer level.

6.1 Market unraveling

Recall that when the cost disparity does not assume extreme values, there always exists a duopolistic equilibrium. So, in our setting where producers recognize that they have an impact on the market price, in contrast with the market for lemons, the market for melons does not completely unravel and high-quality products continue being supplied.

Consider the following lemons example, which involves only two types of products. To facilitate the comparison with our model, we assume that the good is perfectly divisible. Producers choose the quantity of cars they offer for sale. Producer H has cars whose use she values at c_H. Producer L has cars whose use she values at $c_L < c_H$. Under this constant-returns-to-scale production technology, the individual supply curves are easily derived:

$$S_L(p) = \begin{cases} 0 & \text{if } p < c_L \\ \text{any positive number} & \text{if } p = c_L \\ \infty & \text{if } p > c_L \end{cases}; \tag{97}$$

$$S_H(p) = \begin{cases} 0 & \text{if } p < c_H \\ \text{any positive number} & \text{if } p = c_H \\ \infty & \text{if } p > c_H \end{cases}. \tag{98}$$

A mass $\Theta + a$ of consumers, indexed by θ, is uniformly distributed over a closed interval $[-\Theta, a]$. We assume that Θ is sufficiently high for the demand curve beloew not to exhibit a kink. θ stands for the "baseline" (dis)utility derived from owning a car of quality zero (a loss). Consumers all have the same willingness to pay for quality improvements. That is, consumer θ's utility from buying car i at price p is given by

$$U(x_i, p; \theta) = \theta + x_i - p. \tag{99}$$

There is no utility derived from consuming more than one car and the utility from not owning any car is equal to zero. Consumers maximize expected utility so that when quality is not observable the inverse demand curve is given by

$$P = a + \bar{x} - Q. \tag{100}$$

Let us make the assumption that consumers derive more utility from owning the cars than the sellers, and the more so for high-quality cars. Let us have

$$x_L = \alpha c_L, \tag{101}$$

$$x_H = \beta c_H, \tag{102}$$

with $1 < \alpha < \beta$. Observe that

$$x_H - c_H = (\beta - 1)c_H > (\alpha - 1)\, c_L = x_L - c_L \tag{103}$$

It is thus clear that maximizing total surplus requires H cars to be sold to those consumers for which $\theta \geq -(\beta - 1)c_H$.

If the producers are price-takers, then the unique competitive equilibrium is given by:

$$p^c = c_L. \tag{104}$$

$$Q^c = a + (\alpha - 1)\, c_L. \tag{105}$$

That is, all consumers for which $\theta \geq -(\alpha - 1)\, c_L$ buy a "lemon". This of course an instance of adverse selection. Less consumers buy and they get the wrong car, so to speak. The example makes clear that the phenomenon does not arise from the fact that producers make a binary decision to offer a given capacity or not. On the contrary, it is driven by the price-taking behavior of the sellers.

By contrast, if the gap between c_L and c_H is not too pronounced, the previous analysis showed that firm H will not withdraw from the market if it and firm L strategically set their quantity. Moreover, for α or $\frac{\beta}{\alpha}$ high enough, we have $\varepsilon > 2\delta$, and firm H will actually dominate the market.

This said, on one hand, there is no value of β and α for which firm H can enjoy a monopoly position in equilibrium (as this would require $c_H - c_L < 0$, contrary to our assumption). On the other hand, if $c_H - c_L$ is high enough, there might be a L-monopolistic equilibrium that is worse (in terms of welfare) than the competitive equilibrium because firm L will exert its monopoly power. So, there is a sense in which strategic quantity-setting mitigates the adverse selection issue without solving it completely for relatively small cost differentials (but exacerbates it for large ones).

This result is an example of the well-known result in second-best theory according to which two market distorsions may be preferable to only one. Here, endowing the two competitors with some market power is a way to alleviate the problems generated by the information asymmetry between buyers and sellers.

6.2 Consumer surplus

An interesting way of looking at the stable duopolistic equilibrium outcome consists in adding (A) to (B), the necessary first-order conditions, which gives:

$$Q^* = \frac{2(a + \bar{x}^*) - (2c)}{3b} \tag{106}$$

Plugging this into the demand curve:

$$P^* = \frac{a + \bar{x}^* + (2c)}{3} \tag{107}$$

These expressions are identical to the ones in standard Cournot model, with the double caveat that the equilibrium average quality, $\bar{x}$, is endogenously determined in our model, and that $c = (c_H + c_L)/2 = [\mu(x_H) + \mu(x_L)]/2$ depends upon the specification of qualities, for a given marginal cost function μ.

Call the situation where consumers face two producers of different qualities x_L and x_H, Situation 1. Suppose that (q_L^*, q_H^*) is the unique equilibrium. (That is, suppose that the cost differential is not extreme.) Consumer surplus is straightforwardly computed:

$$CS_1 = \frac{4\left(a + \bar{x}^* - c_1\right)^2}{9b}, \tag{108}$$

where $c_1 = [\mu(x_H) + \mu(x_L)]/2$.

Imagine that, instead of facing the two firms producing x_L and x_H, consumers were in Situation 2, facing two identical firms producing the average quality x. Then, consumer surplus would be

$$CS_2 = \frac{4\left(a + x - \mu(x)\right)^2}{9b} \tag{109}$$

We have

$$CS_1 > CS_2 \Leftrightarrow \bar{x} - \frac{\mu(x_H) + \mu(x_L)}{2} > x - \mu(x). \tag{110}$$

Clearly, there are numerous instances in which CS_1 will be greater than CS_2. For a given quality differential, all that is required is that the function μ be not "too convex" (or concave enough). In fact, if μ is a linear function of quality with $\mu' < 1/2$, then the inequality holds for any quality differential. Indeed, in that case, firm H produces more than firm L (for $2\delta < \varepsilon$), so that $\bar{x} > x$, and $[\mu(x_H) + \mu(x_L)]/2 = \mu(x)$ by linearity. This is true, in particular, if marginal cost is constant, as in the case when quality impacts only fixed costs.

7 Conclusion

We have analyzed a generalization of the Cournot duopoly game where firms produce different qualities but products cannot be distinguished by consumers, whose willingness to pay for the good depends upon the average quality.

We have shown that when the quality differential is small, the game is a continuous deformation of the standard Cournot game, in the sense that there always exists a unique pure-strategy equilibrium. If the low-quality firm is at a large cost disadvantage, then the high-quality firm is a monopoly in equilibrium. If the low-quality firm is at a large cost advantage, then it enjoys monopoly in equilibrium. If it is on the par or so with its competitor, then both firms remain active in equilibrium.

When the quality differential is large, however, the two quantities are not always strategic substitutes and the high-quality firm's best-response curve may exhibit a "jump". As a result, a pure-strategy equilibrium may fail to exist, or two or three equilibria may co-exist, which raises an equilibrium selection issue. To this end, we argued that when the equilibrium is not unique, there are only two stable equilibria: one in which the low-quality firm produces its monopoly output and the high-quality firm withdraws from the market; and another one in which the high-quality firm dumps a high quantity on the market and thus sustain high levels of quality and price. In these circumstances, we would expect the high-quality firm, operating in a more complicated, multi-stage game, to take steps in order to shape the industry expectations, or commit in advance to produce high quantities, so as to end up in the duopolistic equilibrium.

Because the high-quality firm has the possibility of impacting the average quality (and therefore consumers' willingness to pay for the good) through a rise in its quantity, the range of parameters for which an L-monopolistic equilibrium exists first enlarges, then shrinks with the quality differential. However large the quality differential, there always exists a (vanishing) range of cost parameters for which the high-quality firm would have entered the market if guaranteed a monopoly position but is evicted by the low-quality firm in equilibrium. In the special case where marginal costs are equal, the high-quality firm's market share is capped by a constant that is independent of the quality differential. In other words, when quality upgrades necessitate fixed investments, there is no way for a top-quality firm to prevent a lower-quality competitor from benefiting from these investments.

In all cases where the cost differential is not extreme, there exists a stable equilibrium in which both firms remain active. This is in sharp contrast with the well-known unraveling of markets under asymmetric information and price-taking behavior. It is the sense in which strategic quantity-setting can be said to help mitigate adverse selection problems. This result has direct implications for policy-making as it suggests that a planner (or an anti-trust authority) might come to regard the horizontal concentration of same-quality producers as a way to prevent the disappearance of high-quality products.

As a matter of fact, in our model the high-cost firm ends up producing more than the low-cost firm if it enjoys a sufficiently large quality advantage. On the basis of consumer surplus, consumers may thus prefer facing two producers of unequal qualities to facing two identical producers, since in equilibrium the average quality can increase more than the price. We speculate that this result coud be reproduced under mild risk aversion.

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