From preferences to Cobb-Douglas utility

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Abstract. We provide characterizations of preferences representable by a Cobb-Douglas utility function.

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1. Introduction

Due to its analytical tractability, the Cobb-Douglas utility function

$$u : \mathbb{R}^n_+ \to \mathbb{R} \quad \text{with} \quad u(x) = x_1^{a_1} \cdots x_n^{a_n} = \prod_{i=1}^{n} x_i^{a_i} \qquad (n \in \mathbb{N}, a_1, \ldots, a_n > 0)$$

is among the most commonly used in economics. Its name credits Cobb and Douglas (1928), but its roots can be traced back a few more decades to, among others, Mill, Pareto, and Wicksell. See Lloyd (2001) for an historical overview.

The Cobb-Douglas utility function is so commonplace that its use is hardly ever motivated or just accompanied by a statement that it concerns a "standard" utility function. If it is motivated at all, it often uses functional equations: it presumes the existence of a function, imposes some properties the function must satisfy, and derives that it must be of Cobb-Douglas form. Lloyd (2001) gives an informal discussion, Eichhorn (1978) the mathematical details.

Rather than simply assuming that a utility function with desirable properties exists, this note takes things one step back and derives Cobb-Douglas utility functions from first principles: what properties of an economic agent's *preferences* guarantee that they can be represented by a utility function of Cobb-Douglas type?

This makes the functional equation approach difficult to apply. Ordinal properties of preference relations need not translate to well-defined functional equations on a corresponding utility function: the latter are determined only up to a monotonic transformation.

Section 2 fixes our notation. Section 3 contains our characterizations of preference relations representable by Cobb-Douglas utility functions and a discussion of related literature.

2. Preliminaries

Define preferences on a set X in terms of a binary relation $\succsim$ ("weakly preferred to") which is:

complete: for all $x, y \in X : x \succsim y, y \succsim x$, or both;

transitive: for all $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

We call a complete, transitive relation $\succsim$ a ***weak order***. As usual, $x \succ y$ means $x \succsim y$, but not $y \succsim x$, whereas $x \sim y$ means that both $x \succsim y$ and $y \succsim x$. Preferences $\succsim$ are ***represented*** by utility function $u : X \to \mathbb{R}$ if

$$\forall x, y \in X : \qquad x \succsim y \quad \Leftrightarrow \quad u(x) \geq u(y).$$

Let $n \in \mathbb{N}$. For vectors $x, y \in \mathbb{R}^n$, write $x \leq y$ if $x_i \leq y_i$ for all $i \in \{1, \ldots, n\}$ and $x < y$ if $x_i < y_i$ for all $i \in \{1, \ldots, n\}$. Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}$ and $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : x > 0\}$. For $i \in \{1, \ldots, n\}$, let $e_i \in \mathbb{R}^n$ be the i-th standard basis vector with i-th coordinate one and all other coordinates zero; $e = \sum_{i=1}^n e_i$ is the vector of ones. Endow $\mathbb{R}^n$ with its standard topology and subsets with the relative topology.

Let $\succsim$ be a binary relation on X, where X equals $\mathbb{R}^n$, $\mathbb{R}^n_+$, or $\mathbb{R}^n_{++}$, and define:

continuity: for each $x \in X$, $\{y \in X : y \prec x\}$ and $\{y \in X : y \succ x\}$ are open.

upper semicontinuity: for each $x \in X$, $\{y \in X : y \prec x\}$ is open.

additivity: for all $x, y, z \in X$, if $x \succsim y$, then $x + z \succsim y + z$.

homotheticity: for all $x, y \in X$ and all scalars $t > 0$, if $x \succsim y$, then $tx \succsim ty$.

homotheticity in coordinate $i \in \{1, \ldots, n\}$***:*** for all $x, y \in X$ and all scalars $t > 0$, if $x \succsim y$, then $(x_1, \ldots, x_{i-1}, tx_i, x_{i+1}, \ldots, x_n) \succsim (y_1, \ldots, y_{i-1}, ty_i, y_{i+1}, \ldots, y_n)$.

monotonicity: for all $x, y \in X$, if $x \leq y$, then $x \precsim y$.

strict monotonicity: for all $x, y \in X$, if $x < y$, then $x \prec y$.

sensitivity: for each $i \in \{1, \ldots, n\}$, there exist $x, y \in X$ with $x_j = y_j$ whenever $j \neq i$ and $x \not\sim y$.

substitutability: for each $x \in X$, there is a scalar α such that $\alpha e \in X$ satisfies $x \sim \alpha e$.

Most properties are standard. Sensitivity avoids trivialities: each coordinate matters in preference relation $\succsim$. Substitutability is a weak compensation principle: for each alternative, improvements due to changes in one set of variables can compensate for deteriorations in others — say, a little more of coordinate i might compensate for a little less of coordinate j — to "smoothen out" any differences in the coordinates. In decision theory under uncertainty, where coordinates correspond with payoffs in different states of nature, this property is known as the "fair price" principle (Diecidue and Wakker, 2002): each alternative has an equivalent, constant price α. The proof of Theorem 3.1 uses:

Lemma 2.1 [Diecidue and Wakker, 2002, Thm. 2] *Consider a binary relation $\succsim$ on $\mathbb{R}^n$. The following statements are equivalent:*

(a) *There are nonnegative numbers $a_1, \ldots, a_n$ adding up to one such that $\succsim$ is represented by the utility function $u : \mathbb{R}^n \to \mathbb{R}$ with $u(x) = \sum_{i=1}^n a_i x_i$.*

(b) $\succsim$ *is a weak order satisfying strict monotonicity, additivity, and substitutability.*

Lemma 2.2 [Dow and Werlang, 1992, Thm. 2.1] *If a weak order $\succsim$ on $\mathbb{R}^n_+$ is upper semicontinuous, monotonic, and homothetic, then it is continuous.*

3. Representation theorem

Theorem 3.1 provides two characterizations of preferences that can be represented by Cobb-Douglas utility functions. Normalizing its coefficients $a_1, \ldots, a_n$ to add up to one entails no loss of generality: utilities are determined only up to a monotonic transformation.

Theorem 3.1 *Consider a binary relation $\succsim$ on $\mathbb{R}^n_+$. The following statements are equivalent:*

(a) *There are nonnegative numbers $a_1, \ldots, a_n$ adding up to one such that $\succsim$ is represented by the Cobb-Douglas utility function $u : \mathbb{R}^n \to \mathbb{R}$ with $u(x) = \prod_{i=1}^n x_i^{a_i}$.*

(b) $\succsim$ *is a weak order satisfying strict monotonicity, homotheticity in each coordinate, and substitutability.*

(c) $\succsim$ *is a weak order satisfying strict monotonicity, homotheticity in each coordinate, and upper semicontinuity.*

The numbers in (a) *are positive if and only if $\succsim$ satisfies sensitivity.*

Proof. (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c): The function u is strictly monotonic ($x < y \Rightarrow u(x) < u(y)$), homogeneous in each coordinate, continuous, and $\{u(\alpha e) : \alpha \in \mathbb{R}_+\} = \{u(x) : x \in \mathbb{R}^n_+\} = \mathbb{R}_+$.
(b) $\Rightarrow$ (a): Assume (b) holds. We use Lemma 2.1 to show that $\succsim$ can be represented by a Cobb-Douglas utility function on $\mathbb{R}^n_{++}$. The domain is then extended to $\mathbb{R}^n_+$.
Step 1, domain $\mathbb{R}^n_{++}$: Define $f : \mathbb{R}^n \to \mathbb{R}^n_{++}$ for each $x \in \mathbb{R}^n$ by $f(x) = (\exp x_1, \ldots, \exp x_n)$. As f and its inverse $f^{-1} : \mathbb{R}^n_{++} \to \mathbb{R}^n$ with $f^{-1}(y) = (\ln y_1, \ldots, \ln y_n)$ are continuous, f is a homeomorphism. Given the weak order $\succsim$ on $\mathbb{R}^n_{++}$, define a weak order $\succsim_f$ on $\mathbb{R}^n$ as follows:

$$\forall x, y \in \mathbb{R}^n : \qquad x \succsim_f y \quad \Leftrightarrow \quad f(x) \succsim f(y). \tag{1}$$

The exponential function is strictly increasing, so by substitution in (1), properties imposed on $\succsim$ carry over in a straightforward way to properties of $\succsim_f$: it is a weak order satisfying strict monotonicity, and substitutability. Applying coordinatewise homotheticity n times, it follows that

$$\forall x, y, t \in \mathbb{R}^n_{++} : \qquad x \succsim y \quad \Rightarrow \quad (t_1 x_1, \ldots, t_n x_n) \succsim (t_1 y_1, \ldots, t_n y_n).$$

Hence, by definition (1), $(\ln x_1, \ldots, \ln x_n) \succsim_f (\ln y_1, \ldots, \ln y_n)$ implies that

$$(\ln x_1, \ldots, \ln x_n) + (\ln t_1, \ldots, \ln t_n) \succsim_f (\ln y_1, \ldots, \ln y_n) + (\ln t_1, \ldots, \ln t_n).$$

As f is bijective, it follows that $\succsim_f$ is additive.

By Lemma 2.1, there are $a_1, \ldots, a_n \geq 0$ with $\sum_{i=1}^n a_i = 1$ such that $\succsim_f$ is represented by the utility function $x \mapsto \sum_{i=1}^n a_i x_i$. By (1), for all $x, y \in \mathbb{R}^n_{++}$:

$$x \succsim y \quad \Leftrightarrow \quad (\ln x_1, \ldots, \ln x_n) \succsim_f (\ln y_1, \ldots, \ln y_n) \quad \Leftrightarrow \quad \sum_{i=1}^n a_i \ln x_i \geq \sum_{i=1}^n a_i \ln y_i.$$

Taking exponentials, $\succsim$ is represented by utility function u with $u(x) = \prod_{i=1}^n x_i^{a_i}$ on $\mathbb{R}^n_{++}$.

Step 2, domain $\mathbb{R}^n_+$: To see that u represents $\succsim$ on the entire domain $\mathbb{R}^n_+$, we must establish that $x \sim 0$ for each $x \in \mathbb{R}^n_+$ with some, but not all, coordinates equal to zero. Pick such an x. As $x + (1/n)e \in \mathbb{R}^n_{++}$ for each $n \in \mathbb{N}$, strict monotonicity implies $0 \prec x + (1/n)e$. By substitutability, there is an $\varepsilon_n > 0$ with $x + (1/n)e \sim \varepsilon_n e$. As at least one coordinate of $x + (1/n)e$ goes to zero:

$$0 = \lim_{n \to \infty} u(x + (1/n)e) = \lim_{n \to \infty} u(\varepsilon_n e) = \lim_{n \to \infty} \varepsilon_n^{a_1 + \cdots + a_n} = \lim_{n \to \infty} \varepsilon_n.$$

By substitutability, $x \sim \alpha e$ for some $\alpha \geq 0$. Positive α are ruled out: $x \prec x + (1/n)e \sim \varepsilon_n e$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} \varepsilon_n = 0$. So α must be zero.

(c) $\Rightarrow$ (b): Assume (c) holds. We show that $\succsim$ satisfies substitutability. Applying coordinate-wise homotheticity n times, it follows that $\succsim$ is homothetic. Moreover, $\succsim$ is monotonic: let $x, y \in \mathbb{R}^n_+$ have $x \leq y$. Then $x < y + (1/n)e$ for all $n \in \mathbb{N}$, so $x \precsim y + (1/n)e$ (even strictly) by strict monotonicity. Letting $n \to \infty$ and using that the set of alternatives weakly better than x is closed by upper semicontinuity, it follows that $x \precsim y$. So $\succsim$ is continuous by Lemma 2.2.

Substitutability now follows from a standard separation argument: for each $x \in \mathbb{R}^n_+$, $0e \precsim x \precsim \max\{x_1, \ldots, x_n\}e$ by monotonicity. As the diagonal $D = \{\alpha e : \alpha \geq 0\}$ is a connected set and $\succsim$ is continuous, there is an $\alpha \geq 0$ with $x \sim \alpha e$: otherwise, the sets $\{y \in \mathbb{R}^n_+ : y \prec x\}$ and $\{y \in \mathbb{R}^n_+ : y \succ x\}$, open in the relative topology on $\mathbb{R}^n_+$ by continuity, separate D.

Sensitivity: Let $i \in \{1, \ldots, n\}$. Sensitivity in the i-th coordinate excludes $a_i = 0$. Conversely, if $a_i \neq 0$, then $u(\varepsilon e_i + \sum_{j \neq i} e_j) = \varepsilon^{a_i}$ establishes sensitivity in the i-th coordinate. □

To my knowledge, the results above are new. Bossert and Weymark (2004, Theorem 11.1), in a social choice setting, give a special case of the characterization in (c): they assume continuity, rather than upper semicontinuity. They also refer to related results under additional assumptions and on the easier domain $\mathbb{R}^n_{++}$; cf. Moulin (1988, Theorem 2.3) and Trockel (1989). The domain $\mathbb{R}^n_{++}$ avoids the complication that the indifference curve through the origin has a decidedly different shape than indifference curves through points in $\mathbb{R}^n_{++}$ and essentially allows one to skip part (in particular, Step 2) of our proof above.

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