Outer measure and utility

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Abstract. Starting from an intuitive and constructive approach for countable domains, and combining this with basic measure theory, we obtain an upper semi-continuous utility function based on outer measure. Whenever preferences over an arbitrary domain can at all be represented by a utility function, our outer-measure function does the job. Moreover, whenever the preference domain is endowed with a topology that makes the preferences upper semi-continuous, so is the outer-measure utility function. Although links between utility theory and measure theory have been pointed out before, to the best of our knowledge, this is the first time that the present — more elementary — route has been taken.

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1. Introduction

In most economics textbooks there is a gap between the potential non-existence of utility functions for complete and transitive preference relations on non-trivial connected Euclidean domains — usually illustrated by lexicographic preferences (Debreu, 1954) — and the existence of continuous utility functions for complete, transitive and continuous preferences on connected Euclidean domains; see, e.g. Mas-Colell, Whinston, and Green (1995). Yet, for many purposes, in particular for the existence of a best alternative in a compact set of alternatives, a weaker property — upper semi-continuity — suffices. Hence, the reader of such a textbook treatment might wonder if there exist upper semi-continuous utility functions, and whether this is true even if the domain is not connected. We here fill this gap providing *necessary and sufficient* conditions for the existence of upper semi-continuous utility functions on arbitrary domains; see Theorem 3.1 and the text following it. Our approach is intuitive, constructive, and easily accessible also to readers without any knowledge of measure theory.

Measure theory is the branch of mathematics that deals with the question of how to define the "size" (area/volume) of sets. We here formalize a direct intuitive link with utility theory: given a binary preference relation on a set of alternatives, the "better" an alternative is, the "larger" is its set of worse alternatives. So if one can measure the "size" of the set of worse elements, for each given alternative, one obtains a utility function.

To be a bit more precise, measure theory starts out by first defining the "size" — measure — of a class of "simple" sets, such as bounded intervals on the real line or rectangles in the plane, and then extends this definition to other sets by way of approximation in terms of simple sets. The outer measure is the best such approximation "from above". This is illustrated in Figure 1, where a set S in the plane is covered by rectangles. The outer measure S is the infimum, over

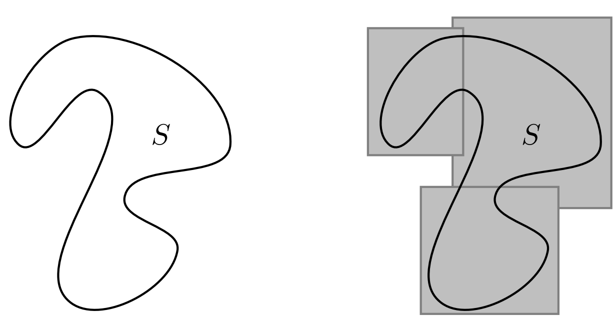

Figure 1: A set S and an approximation of its size using a covering.

all coverings by a countable number of rectangles, of the sum of the rectangles' areas. In more general settings, the outer measure is defined likewise as the infimum over coverings whose sizes have been defined; see, for instance, Rudin (1976, p. 304), Royden (1988, Sec. 3.2), Billingsley (1995, Sec. 3), Ash (2000, p. 14).

We follow this approach by way of defining the utility of an alternative as the outer measure of its set of worse alternatives. We start by doing this for a countable set of alternatives, where this is relatively simple and then proceed to arbitrary sets.

Our paper is not the first to use tools from measure theory to address the question of utility representation: pioneering papers are Neuefeind (1972) and Sondermann (1980). See Bridges and Mehta (1995, sections 2.2 and 4.3) for a textbook treatment. However, our approach differs fundamentally from these precursors. Firstly, we only use the basic notion of outer measure, while the mentioned studies impose additional topological and/or measure-theoretic constraints.[2] To the best of our knowledge, the logical connection between outer measure and utility has never been made before. This link between utility theory and measure theory is more explicit, intuitive and mathematically elementary than the above-mentioned approaches. Let us stress the generality of this result. Although the outer-measure function is simple and intuitive, it delivers the most general results possible. First, *whenever* preferences over an arbitrary set of alternatives can at all be represented by a utility function, the outer-measure function does the job. Secondly, *whenever* the set of alternatives is endowed with a topology that makes preferences upper semi-continuous, also the outer-measure utility function becomes upper semi-continuous.

The rest of the paper is organized as follows. Section 2 recalls definitions and provides notation. Our general representation theorem is given in Section 3. Its proof is in the appendix.

2. Definitions and notation

Preferences. Let preferences on an arbitrary set X be defined in terms of a binary relation $\succsim$ ("weakly preferred to") which is:

complete: for all $x, y \in X : x \succsim y, y \succsim x$, or both;

transitive: for all $x, y, z \in X$: if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

[2] Neuefeind (1972) restricts attention to finite-dimensional Euclidean spaces and assumes that indifference sets have Lebesgue measure zero. Sondermann (1980) assumes that preferences are defined on a probability space or a second countable topological space; see also Corollary 3.3 below.

As usual, $x \succ y$ means $x \succsim y$, but not $y \succsim x$, whereas $x \sim y$ means that both $x \succsim y$ and $y \succsim x$. The sets of elements strictly worse and strictly better than $y \in X$ are denoted

$$W(y) = \{x \in X : x \prec y\} \text{ and } B(y) = \{x \in X : x \succ y\}.$$

For $x, y \in X$ with $x \prec y$, the "open interval" of alternatives better than x but worse than y is denoted

$$(x, y) \quad = \quad \{z \in X : x \prec z \prec y\}.$$

Topology. Given a topology on X, preferences $\succsim$ are:

continuous if for each $y \in X$, $W(y)$ and $B(y)$ are open;

upper semi-continuous (usc) if for each $y \in X$, $W(y)$ is open.

Similarly, a function $u : X \to \mathbb{R}$ is usc if for each $r \in \mathbb{R}$, $\{x \in X : u(x) < r\}$ is open.

Three important topologies are, firstly, the *order topology*, generated by (i.e., the smallest topology containing) the collections $\{W(y) : y \in X\}$ and $\{B(y) : y \in X\}$; secondly, the *lower order topology*, generated by the collection $\{W(y) : y \in X\}$, and thirdly, for any subset $D \subseteq X$, the *D-lower order topology*, generated by the collection $\{W(y) : y \in D\}$. By definition, the order topology is the coarsest topology in which $\succsim$ is continuous; the lower order topology is the coarsest topology in which $\succsim$ is usc.

As mentioned in the introduction, although one often appeals to continuity to establish existence of most preferred alternatives, the weaker requirement of upper semi-continuity suffices. A short proof: consider a complete, transitive, usc binary relation $\succsim$ over a compact set X. If X has no most preferred element, then for each $x \in X$, there is a $y \in X$ with $y \succ x$, i.e., the collection $\{W(y) : y \in X\}$ is a covering of X with (by usc) open sets. By compactness, there are finitely many $y^1, \ldots, y^k \in X$ such that $W(y^1), \ldots, W(y^k)$ cover X. Let y^j be the most preferred element of $\{y^1, \ldots, y^k\}$. Then $W(y^j)$ covers the entire set X, a contradiction.

Utility. A preference relation $\succsim$ is *represented* by a utility function $u : X \to \mathbb{R}$ if

$$\forall x, y \in X : \quad \begin{cases} x \sim y & \Rightarrow \quad u(x) = u(y), \\ x \succ y & \Rightarrow \quad u(x) > u(y). \end{cases} \tag{1}$$

3. Upper semi-continuous utility via outer measures

A complete, transitive binary relation $\succsim$ on a set X can be represented by a utility function if and only if it is *Jaffray order separable*[3] (Jaffray, 1975): there is a countable set $D \subseteq X$ such

[3]See Fishburn (1970, Sec. 3.1) or Bridges and Mehta (1995, Sec. 1.4) for alternative separability conditions.

that for all $x, y \in X$:

$$x \succ y \qquad \Rightarrow \qquad \exists d, d' \in D : x \succsim d \succ d' \succsim y. \tag{2}$$

Roughly speaking, countably many alternatives suffice to keep all pairs $x, y \in X$ with $x \succ y$ apart: x lies on one side of d and d', whereas y lies on the other. To make our search for a (usc) utility representation at all meaningful, we will henceforth focus on preference relations that are Jaffray order separable.

The set D in the definition of Jaffray order separability is countable, so let $n : D \to \mathbb{N}$ be an injection. Finding a utility function on D is easy. Give each element d of D a positive weight such that weights have a finite sum and use the total weight of the elements weakly worse than d as the utility of d. For instance, give weight $\frac{1}{2}$ to the alternative d with label $n(d) = 1$, weight $\frac{1}{4}$ to the alternative d with label $n(d) = 2$, and inductively, weight $w(d) = 2^{-k}$ to the alternative d with label $n(d) = k$. In general, let $(\varepsilon_k)_{k=1}^{\infty}$ be a summable sequence of positive weights; without loss of generality its sum $\sum_{k=1}^{\infty} \varepsilon_k$ is one. Assign to each $d \in D$ weight $w(d) = \varepsilon_{n(d)}$.[4] Define $u_0 : D \to \mathbb{R}$ for each $d \in D$ by $u_0(d) = \sum_{d' \precsim d} w(d')$. Clearly, (1) is satisfied.

We can extend this procedure from D to X as follows. Let $\mathcal{W}$ be the collection of subsets $\{W(d) : d \in D\} \cup \{\varnothing, X\}$ and define $\rho : \mathcal{W} \to [0, 1]$ as follows: $\rho(\varnothing) = 0, \rho(X) = 1$ and for $d \in D$:

$$\rho(W(d)) = \sum_{d' \in D : d' \precsim d} w(d'). \tag{3}$$

Notice that $\mathcal{W}$ is countable and that it is a covering of X. Extend ρ to an *outer measure* μ^* on X in the usual way (recall Figure 1): for each set $A \subseteq X$, define $\mu^*(A)$ as the smallest total size of sets in $\mathcal{W}$ covering A. Formally, a countable collection $\{W_i\}$ of sets W_i from $\mathcal{W}$ *covers* A if $A \subseteq \cup_i W_i$. Now define

$$\mu^*(A) = \inf \textstyle\sum_i \rho(W_i),$$

where the infimum is taken over all countable collections $\{W_i\}$ that cover A.

Define $u : X \to \mathbb{R}$ for each $x \in X$ as the outer measure of the set of elements worse than x:

$$u(x) = \mu^*(W(x)). \tag{4}$$

This outer measure gives the desired utility representation:

[4] If there is a worst element in X (an $x^0 \in X$ with $x^0 \precsim x$ for all $x \in X$), one may assume without loss of generality that D contains one such element, say $\underline{d}$. Its weight can be normalized to zero: $w(\underline{d}) = 0$. This will assure that $\rho(W(\underline{d})) = \rho(\varnothing) = 0$ in (3).

Theorem 3.1 *Consider a complete, transitive, Jaffray order separable binary relation* $\succsim$ *on an arbitrary set* X*. The outer-measure utility function* u *in* (4) *represents* $\succsim$ *and is usc in the* D*-lower order topology.*

Let us stress the generality of this result. The outer-measure utility function is based on basic measure-theoretic intuition. Yet it delivers the most general results possible. First, *whenever* preferences $\succsim$ over an arbitrary set X can at all be represented by a utility function (i.e., they are complete, transitive, Jaffray order separable), the outer-measure function does the job. Secondly, *whenever* X is endowed with a topology that makes the preferences $\succsim$ usc, also the outer-measure utility function becomes usc.

Corollaries 3.2 and 3.3 below provide applications of this result. Consider preferences $\succsim$ over a topological space X with countable base.[5] If $\succsim$ is usc in this topology, it is Jaffray order separable (Rader, 1963). By assumption, $W(y)$ is open for each $y \in X$, so the topology on X is finer than the D-lower order topology. Hence, Theorem 3.1 applies:

Corollary 3.2 *If* $\succsim$ *is a complete, transitive, usc binary relation over a topological space* X *with countable base, the outer-measure utility function in* (4) *represents* $\succsim$ *and is usc.*

Also Rader (1963) establishes existence of a usc utility function under the conditions of Corollary 3.2. However, we obtain the result as a special case of Theorem 3.1, which holds under weaker conditions and gives a specific usc utility function building upon basic measure-theoretic intuition.

Sondermann (1980) calls a preference relation $\succsim$ on a set X *perfectly separable* if there is a countable set $C \subseteq X$ such that for all $x, y \in X$, with $x \not\sim c$ and $y \not\sim c$ for all $c \in C$, the following holds:

$$x \succ y \qquad \Rightarrow \qquad \exists c \in C : x \succ c \succ y.$$

Perfect separability implies Jaffray order separability (Jaffray, 1975), so we obtain the following result, due to Sondermann (1980), as a special case:

Corollary 3.3 [Sondermann, 1980, Corollary 2] *Consider a complete, transitive, perfectly separable binary relation* $\succsim$ *on a set* X*. Then there is a utility function representing* $\succsim$*, usc in any topology equal to or finer than the lower order topology.*

Also here, the "value added" of Theorem 3.1 is that it provides a specific usc utility function building upon basic measure-theoretic intuition.

[5] E.g., consumer preferences over a commodity space $X = \mathbb{R}^n_+$ ($n \in \mathbb{N}$) with its standard Euclidean topology.

Appendix: Proof of Theorem 3.1

Preliminaries. By definition,

$$\forall d \in D: \quad u(d) = \mu^*(W(d)) = \rho(W(d)) = \sum_{d' \in D: d' \precsim d} w(d'), \tag{5}$$

and the outer measure μ^* is monotonic: if $A \subseteq B \subseteq X$, then $\mu^*(A) \leq \mu^*(B)$.

Representation. We prove (1). Let $x, y \in X$. If $x \sim y$, then $W(x) = W(y)$ by transitivity of $\succsim$, so $u(x) = u(y)$. If $x \succ y$, there are $d, d' \in D$ with $x \succsim d \succ d' \succsim y$ by (2). By monotonicity of μ^* and (5): $u(x) = \mu^*(W(x)) \geq \mu^*(W(d)) > \mu^*(W(d'^*(W(y)) = u(y)$.

Semi-continuity. Let $r \in \mathbb{R}$. We show that $\{x \in X : u(x) < r\}$ is open. To avoid trivialities, assume that $\{x \in X : u(x) < r\}$ equals neither $\varnothing$ nor X. Hence, there is a $y^* \in X$ with $r \leq u(y^*) \leq 1$. Let $x \in X$ have $u(x) < r$. In particular, $y^* \succ x$. It suffices to show that there is an open neighborhood V of x with $u(v) < r$ for each $v \in V$.

Case 1: There is no $d \in D$ with $d \sim x$. As D may be assumed to contain a worst element of X, if such exists (see footnote 4), $W(x) \neq \varnothing$. By definition of μ^*, there are $\{W_i\}_{i \in \mathbb{N}} \subseteq \mathcal{W}$ with $W(x) \subseteq \cup_{i \in \mathbb{N}} W_i$ and $\mu^*(W(x)) \leq \sum_{i \in \mathbb{N}} \rho(W_i) < r \leq 1$. As $W(x) \neq \varnothing$, the set $J = \{i \in \mathbb{N} : W_i \neq \varnothing\}$ is nonempty. As $\rho(X) = 1$ and $\sum_{i \in \mathbb{N}} \rho(W_i) < 1$, $W_i \neq X$ for each $i \in J$. So for each $i \in J$ there is a $d_i \in D$ with $W_i = W(d_i)$. We show that $d_i \succ x$ for some $i \in J$. Suppose, to the contrary, that $d_i \prec x$ for each $i \in J$. For each $j \in J$, the set $\{d_i \in D : i \in J, d_i \succsim d_j\}$ is infinite: otherwise, it has a best element d^*, but then $\cup_{i \in \mathbb{N}} W_i = \cup_{i \in J} W(d_i) = W(d^*)$ is a proper subset of $W(x)$ by Jaffray order separability, contradicting $W(x) \subseteq \cup_{i \in \mathbb{N}} W_i$. Let $j \in J$ with $\rho(W(d_j)) := \varepsilon > 0$. By the above, there are infinitely many $i \in J$ with $\rho(W_i) = \rho(W(d_i)) \geq \rho(W(d_j)) = \varepsilon$, contradicting that $\sum_{i \in \mathbb{N}} \rho(W_i) < 1$. We conclude that $d_i \succ x$ for some $i \in J$. So $x \in W(d_i)$, an open set in the D-lower order topology, and for each $v \in W(d_i)$: $u(v) < u(d_i) = \rho(W(d_i)) < r$.

Case 2: There is a $d \in D$ with $d \sim x$. Using (2) and $y^* \succ x$: $B(d) \cap D = \{d' \in D : d' \succ d\} \neq \varnothing$.

Case 2A: There is a $d' \in B(d) \cap D$ with $(d, d') = \varnothing$. Then $\{z \in X : z \precsim d\} = \{z \in X : z \prec d'\} = W(d')$ is open in the D-lower order topology, contains x, and for each $z \in W(d') : u(z) \leq u(d) = u(x) < r$.

Case 2B: For each $d' \in B(d) \cap D$, $(d, d') \neq \varnothing$. Then by (2), there is, for each $d' \in B(d) \cap D$, a $d'' \in B(d) \cap D$ that is strictly worse: $d'' \prec d'$. So $B(d) \cap D$ is infinite. Since the sequence of weights $(\varepsilon_k)_{k=1}^{\infty}$ is summable, there is a $k \in \mathbb{N}$ such that $\sum_{\ell=k}^{\infty} \varepsilon_\ell < r - u(x)$. Since there are only finitely many $d' \in D$ with $n(d') < k$, there is a $d^* \in B(d) \cap D$ such that $n(d') \geq k$ for each $d' \in B(d) \cap D$ with d'^*.

Since $d^* \in B(d) \cap D$, $x \in W(d^*)$, which is open in the D-lower order topology. Using $x \sim d$ and the construction of d^*:

$$u(x) = \sum_{d' \in D : d' \precsim d} w(d')$$

and

$$\sum_{d' \in B(d) \cap D : d'^*} w(d') = \sum_{d' \in B(d) \cap D : d'^*} \varepsilon_{n(d')} \leq \sum_{\ell = k}^{\infty} \varepsilon_{\ell} < r - u(x).$$

Hence, for each $v \in W(d*)$,

$$u(v) < u(d^*) = \rho(W(d^*)) = \sum_{d' \in D : d' \precsim d} w(d') + \sum_{d' \in B(d) \cap D : d'^*} w(d') < u(x) + r - u(x) = r.$$

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