INTRODUCING A SPREAD INTO THE KYLE MODEL

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Abstract. The Kyle (1985) model is extended to take into account market maker competition and the spread. It is shown that with a spread the Kyle model has a Nash equilibrium also with two market makers, not only with three or more, as shown in earlier research. The spread is endogenized, and two testable predictions of the model are generated. The first is that the spread is increasing in the standard deviation of the fundamentals. The second is that it is independent of the standard deviation of noise trades.

Keywords: Market microstructure; spread; market maker

JEL Classification: D53; D82; G12; G14

1. Introduction

A well known robustness problem with the Kyle model, due to Kyle (1985), is that it has no equilibrium when there are fewer than three market makers. This was shown first by Dennert (1993) in the one period setting, and then by Bondarenko (2001) in the multiperiod setting. In this paper it is shown that the Kyle model has an equilibrium also with two market makers if the model is extended to incorporate the spread.

The rationale for introducing the spread into the Kyle model is threefold. Firstly, spreads are indeed a reality on all markets.[1]

Secondly, although Bernhardt and Hughson (1997) have shown that an equilibrium exists with both one and two market makers if noise trader demand falls as prices increase - provided it does not fall too much, in practice this may often not be the case. For example, in the literature on predatory trading, e.g. Brunnermeier and Pedersen (2005), it has been noted that traders may be pushed to exit their positions if prices move too far in the harmful direction. An example would be that somebody who is short an asset may have to buy it back if the price is pushed sufficiently high, which of course would mean that noise trader demand is increasing in the price. Thus, especially in times of crisis, Bernhardt and Hughson's (1997) approach may suggest that markets are much more vulnerable than they really are.

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[1] Recently, Bollen et al (2004) showed empirically that the spread during three recent periods on the Nasdaq dependeded on the minimum tick size, the order processing cost, the level of competition, inventory holding costs, and adverse selection costs.

Thirdly, introducing spreads into the Kyle model improves our understanding of how market makers cover costs. In the original Kyle model they do it by using pricing rules that results in overshooting prices, whereas they do it by using the spread in Glosten and Milgrom (1985). In this merger of the two models, we see that they use a combination of both approaches.

The perfect competition between market makers assumed in Kyle (1985) has been interpreted in at least two ways. First, as argued in Kyle (1984), it can be interpreted as an ideal case resulting from competition between infinitely many market makers. Second, as argued in Bernhardt and Hughson (1997), it can be interpreted as the result from a winner-takes-all contest between two, or more, market makers.

Dennert (1993) looks at both possibilities. First he looks at market maker competition in the setting of Glosten and Milgrom (1985). In that setting the market makers announce bid ask prices and the quantities they offer at those prices. The noise traders then take the best offer available, given their exogenous need to trade. It is assumed that the market maker with the best price can always satisfy the noise traders' entire liquidity need. As a result, the other market makers will not trade at all with the noise traders. This implies that there exists no pure strategy equilibrium in this setting. However, Dennert shows that a symmetric mixed strategy equilibrium always exists. In this mixed equilibrium all market makers make zero profits. In addition, the best prices are actually offered to noise traders when there are only two market makers. The reason is that as the number of market makers increase, the risk of only trading with the informed trader - who is unconstrained when it comes to the size of the order he can trade - increases. To compensate, the market makers must use a wider spread.

Second, Dennert looks at market maker competition within the Kyle (1985) model. Since market makers announce linear pricing rules, orders will always be split among market makers - as long as they use the same intercept. As a result, the winner-takes-all structure, that existed in the Glosten and Milgrom setting, is no longer relevant. Instead there exists a symmetric pure equilibrium when there are more than two market makers - and no equilibrium otherwise. In addition, the noise traders get better and better prices as the number of market makers increase. In addition, as the number of market makers approach infinity, their own profits approaches zero.

Bernhardt and Hughson (1997) expands on the winner-takes-all case discussed by Dennert and considers the effect if applied on the Kyle (1985) model. They show that if the noise trader is not allowed to split trades between market makers, then the model has a mixed equilibrium - just as in the Glosten and Milgrom (1985) setting. However, they argue that this result is not robust in the sense that if it is allowed for the noise traders to split their orders, then the mixed equilibrium breaks down. Instead, when we have two market makers, we get the no equilibrium result that Dennert established.

The results in this paper are consistent with the notion that perfect competition is interpreted as an ideal case resulting from competition between infinitely many market makers. Thus, noise traders are allowed to split their trades and the equilibrium is a pure strategy equilibrium.

The paper is organized as follows. The baseline model is defined in Section 2. There is an arbitrary number of noise traders, one informed trader, and two market makers. The market makers use a pricing rule that is linear in the net order flow and where an exogenous bid-ask spread has been added. To make the model more tractable the intercept in the pricing rule is set equal to the expected fundamental value of the asset, which is an equilibrium result in both Kyle (1985) and Dennert (1993). Since the spread is exogenously given the market makers compete only by choosing the slope of the pricing rule.

In Section 3 the model is solved and it is shown that a Nash equilibrium exists even with only two market makers. The reason is that the introduction of a spread increases the market makers' potential profit per trade, and thus increases the competition between market makers. The perpetual overbidding found in Dennert (1993) will thus eventually stop. Some comparative statics are then performed and it is demonstrated that if the spread is sufficiently high, then the noise traders would gain from pooling their trades and, if possible, clearing them with each before approaching the market makers.

In Section 4 the spread is endogenized and it is again shown that a Nash equilibrium exists. Furthermore, it is shown that the spread is increasing in the volatility of the fundamentals. This is a prediction of the model that, to my knowledge, has neither been shown formally in previous theoretical models, nor been tested empirically. Another prediction, possibly less robust, of the model is that the optimal spread is independent of the volatility of noise trades.

In Section 5 the baseline model is extended to allow for any arbitrary number of market makers. As soon as we have more than two market makers there is sufficient competition for an equilibrium to exist even without a spread. However, when the spread is also taken into account, the competition increases even further, and the prices are pushed even lower. Some comparative statics when going from two to three market makers are considered. The market makers always lose as a group, while the noise traders and the informed traders always gain. Finally, in Section 6 the related literature is discussed and in Section 7 the paper is concluded.

2. The Model

The model is the Kyle model extended to take into account competition between market makers, and to incorporate a spread. The timing is as follows.

1. Two market makers simultaneously announce their pricing functions. The market maker $m \in \{1, 2\}$ announces a pair of pricing functions

$$\begin{aligned} p_m^+ &= \alpha + \beta_m \tilde{y}_m + \Delta \\ p_m^- &= \alpha + \beta_m \tilde{y}_m - \Delta, \end{aligned}$$

   where he sells at the higher price and buys at the lower one. The net order flow, $\tilde{y}_m$, is given by $\tilde{y}_m = \tilde{x}_m + \sum_n \tilde{z}_{nm}$, where $\tilde{x}_m$ is the order from the informed trader and $\tilde{z}_{nm}$ is the order from noise trader n. Both $\alpha \in \mathbb{R}_+$ and $\Delta \in \mathbb{R}_{++}$ are exogenous. Let market maker m's strategy be $\beta_m \in \mathbb{R}_{++}$. The market makers' strategy profile is written $\boldsymbol{\beta} = (\beta_1, \beta_2)$.

2. Nature draws $\tilde{v} \sim N(\alpha, \sigma_v^2)$ and $\tilde{u}_n \sim N(0, \sigma_u^2/N)$ independently.

3. The informed trader and the noise traders trade simultaneously.

   - The informed trader $i \in \{1\}$ gets information on the realization of the fundamental value $\tilde{v}$, and submits an order $\tilde{x}_m \in \mathbb{R}$ to each market maker m. Let his strategy $\xi : \mathbb{R}^2_{++} \times \mathbb{R} \to \mathbb{R}^2$ be defined by

$$\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2) = \xi(\beta_1, \beta_2, \tilde{v}). \tag{1}$$

   - The noise trader $n \in \{1, ..., N\}$ observes $\tilde{u}_n$ and submits an order $\tilde{z}_{nm}$ to each market maker where $\tilde{z}_{n1}$ and $\tilde{z}_{n2}$ both add up to, and have the same sign as, $\tilde{u}_n$. Let his strategy $\phi_n : \mathbb{R}^2_{++} \times \mathbb{R} \to \mathbb{R}^2$ be defined by

$$\tilde{\mathbf{z}}_n = (\tilde{z}_{n1}, \tilde{z}_{n2}) = \{\phi_n(\beta_1, \beta_2, \tilde{u}_n) : \tilde{z}_{n1} + \tilde{z}_{n2} = \tilde{u}_n \ \cap \ |\tilde{z}_{n1}|, |\tilde{z}_{n2}| \leq |\tilde{u}_n|\}. \tag{2}$$

   The noise traders' strategy profile is the $N \times 2$ matrix $\mathbf{z}$.

4. The market makers observe their respective net order flows and set the prices according to the prespecified rule.

5. The payoffs are realized. Let us use the notation

$$\begin{aligned} \tilde{x}_m^+ &= \max\{0, \tilde{x}_m\} \\ \tilde{x}_m^- &= \min\{0, \tilde{x}_m\} \\ \tilde{z}_{nm}^+ &= \max\{0, \tilde{z}_{nm}\} \\ \tilde{z}_{nm}^- &= \min\{0, \tilde{z}_{nm}\}. \end{aligned}$$

   - Market maker m's payoff $\tilde{\pi}_m : \mathbb{R}_{++} \to \mathbb{R}$ is defined by

$$\begin{aligned} \tilde{\pi}_m(\beta_m) &= \left(p_m^+ - \tilde{v}\right)\left(\tilde{x}_m^+ + \sum_n \tilde{z}_{nm}^+\right) + \left(p_m^- - \tilde{v}\right)\left(\tilde{x}_m^- + \sum_n \tilde{z}_{nm}^-\right) \\ &= (p_m - \tilde{v})\left(\tilde{x}_m + \sum_n \tilde{z}_{nm}\right) + \Delta\left(|\tilde{x}_m| + \sum_n |\tilde{z}_{nm}|\right), \end{aligned}$$

   where

$$p_m = \alpha + \beta_m \tilde{y}_m.$$

   - The informed trader's payoff $\tilde{\pi}_i : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\begin{aligned} \tilde{\pi}_i(\tilde{x}_1, \tilde{x}_2) &= \sum_{m=1}^{2} \left[\left(\tilde{v} - p_m^+\right)\tilde{x}_m^+ + \left(\tilde{v} - p_m^-\right)\tilde{x}_m^-\right] \\ &= \sum_{m=1}^{2} \left[(\tilde{v} - p_m)\tilde{x}_m - \Delta|\tilde{x}_m|\right]. \end{aligned}$$

- The noise trader's payoff $\tilde{\pi}_n : \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined by

$$\begin{aligned}
\tilde{\pi}_n\left(\tilde{z}_{n1}, \tilde{z}_{n2}\right) &= \sum_{m=1}^{2} \left[\left(\tilde{v} - p_m^+\right) \tilde{z}_{nm}^+ + \left(\tilde{v} - p_m^-\right) \tilde{z}_{nm}^-\right] \\
&= \sum_{m=1}^{2} \left[\left(\tilde{v} - p_m\right) \tilde{z}_{nm} - \Delta \left|\tilde{z}_{nm}\right|\right].
\end{aligned}$$

Thus, the strategy profile is $S = (\boldsymbol{\beta}, \tilde{\mathbf{x}}, \tilde{\mathbf{z}})$. The approach will be to propose that a certain strategy profile is a Nash equilibrium, and then prove that nobody can unilaterally gain by deviating if everybody else is playing the proposed strategies. If the proposed Nash equilibrium is $S^* = (\boldsymbol{\beta}^*, \tilde{\mathbf{x}}^*, \tilde{\mathbf{z}}^*)$, then we will use the notational convention that $S = \left(\boldsymbol{\beta}_{-m}^*, \tilde{\mathbf{x}}^*, \tilde{\mathbf{z}}^*\right)$ means that everybody except market maker m is playing the proposed strategy. Similarly, $\tilde{\mathbf{z}}_{-n\cdot}^*$ will denote the situation when every noise trader except noise trader n plays the proposed strategy.

3. Analysis

Let us propose a candidate for a Nash equilibrium, $S^* = (\boldsymbol{\beta}^*, \tilde{\mathbf{x}}^*, \tilde{\mathbf{z}}^*)$, where

$$\begin{aligned}
\beta_m^* &= 4\frac{A(\Delta)}{B(\Delta, N)} && (3) \\
\tilde{x}_m^* &= \left\{ \begin{array}{lll} \frac{\tilde{v}-(\alpha-\Delta)}{2\beta_m} & \text{if} & \tilde{v} < \alpha - \Delta \\ 0 & \text{if} & \alpha - \Delta < \tilde{v} < \alpha + \Delta \\ \frac{\tilde{v}-(\alpha+\Delta)}{2\beta_m} & \text{if} & \alpha + \Delta < \tilde{v} \end{array} \right\} && (4) \\
\tilde{z}_{nm}^* &= \frac{\beta_{-m}}{\beta_m + \beta_{-m}} \tilde{u}_n, && (5)
\end{aligned}$$

and the functions $A : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ and $B : \mathbb{R}_{++} \times \mathbb{N} \rightarrow \mathbb{R}_{++}$ are given by

$$\begin{aligned}
A(\Delta) &= \frac{\left(\sigma_v^2 + \Delta^2\right)}{4} \left(\operatorname{erf c}\left(\Delta\sqrt{\frac{1}{2\sigma_v^2}}\right) + \frac{\sqrt{\Delta^2\sigma_v^2}}{\sqrt{2}\left(\sigma_v^2 + \Delta^2\right)} \frac{\partial \operatorname{erf c}\left(\Delta\sqrt{\frac{1}{2\sigma_v^2}}\right)}{\partial\left(\Delta\sqrt{\frac{1}{2\sigma_v^2}}\right)} \right) && (6) \\
B(\Delta, N) &= \Delta\sqrt{\frac{2N\sigma_u^2}{\pi}}. && (7)
\end{aligned}$$

The complementary error function is given by

$$\operatorname{erf c}(\gamma) = \frac{2}{\sqrt{\pi}} \int_{\gamma}^{\infty} e^{-t^2} dt. \quad (8)$$

The main result in this section is that this strategy profile is a Nash equilibrium Formally, we have the following proposition:

Proposition 1. *The strategy profile $S^* = (\boldsymbol{\beta}^*, \tilde{\mathbf{x}}^*, \tilde{\mathbf{z}}^*)$ is a Nash equilibrium.*

The proof can be found in the Appendix. In the remainder of this section, we outline the intuition behind the results through some comparative statics.

3.1. Comparative statics.

The market makers' best replies. Driving the results is that the market maker's best reply will be given by

$$\frac{A(\Delta)}{\beta_m^2} + \frac{\beta_{-m}^{*2}\left(\beta_{-m}^* - \beta_m\right)}{\left(\beta_m + \beta_{-m}^*\right)^3}\sigma_u^2 - \frac{\beta_{-m}^*}{\left(\beta_m + \beta_{-m}^*\right)^2}B(\Delta, N) = 0 \tag{9}$$

instead of as in Dennert (1993) where it was given by

$$\frac{\sigma_v^2}{4\beta_m^2} + \frac{\beta_{-m}^{*2}\left(\beta_{-m}^* - \beta_m\right)}{\left(\beta_m + \beta_{-m}^*\right)^3}\sigma_u^2 = 0. \tag{10}$$

Thus, in Dennert (1993) it is always best for the market maker to announce a higher slope than the competitor, whereas this is not the case with a spread. This is reflected in the figure below where we plot the best replies both with no spread and with a strictly positive spread. It can be seen that without a spread the best replies never intersect, whereas they do with a spread.

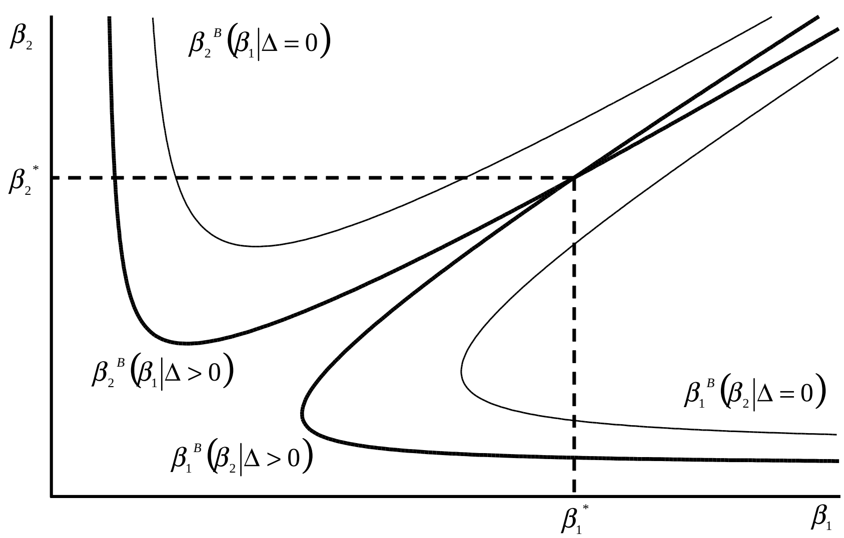

Figure 1: The best replies with and without a spread.

Thus, even an infinitesimal spread will increase the competition between the two market makers so that an equilibrium exists.

The equilibrium slope. The equilibrium slope varies depending on the spread. As the spread increases, the market makers' potential profit increases, which lead to increased competition, and thus a lower slope. However, the relationship is not linear. Instead, the spread's effect on the equilibrium slope is diminishing, as the figure below suggests.

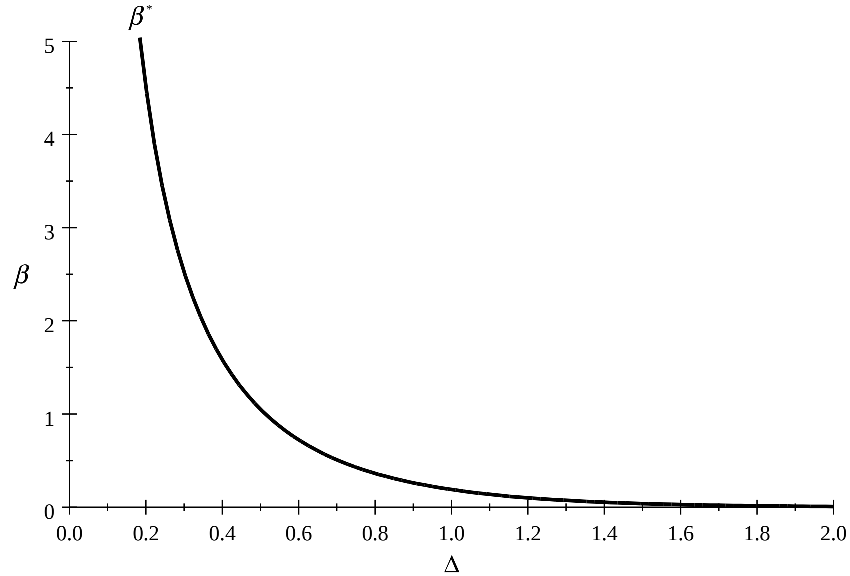

Figure 2: The equilibrium slope as the spread changes, for $\sigma_v^2 = \sigma_u^2 = N = 1$.

Unconditional expected profits when $N = 1$. Let us now consider the unconditional profits of the three types of players as the spread changes. Note that we have

$$NE\left[\pi_n \mid S = (\boldsymbol{\beta}^*, \tilde{\mathbf{x}}^*, \tilde{\mathbf{z}}^*)\right] = -2\frac{A(\Delta)}{B(\Delta, N)}\sigma_u^2 - B(\Delta, N) \tag{11}$$

$$E\left[\tilde{\pi}_i \mid S = (\boldsymbol{\beta}^*, \tilde{\mathbf{x}}^*, \tilde{\mathbf{z}}^*)\right] = \frac{B(\Delta, N)}{2} \tag{12}$$

$$2E\left[\tilde{\pi}_m \mid S = (\boldsymbol{\beta}^*, \tilde{\mathbf{x}}^*, \tilde{\mathbf{z}}^*)\right] = 2\frac{A(\Delta)}{B(\Delta, N)}\sigma_u^2 + \frac{B(\Delta, N)}{2}. \tag{13}$$

The noise trader's loss can thus be decomposed into two component. The first is the loss due to the slope. This part of the loss goes to the market makers. The second part of the loss is due to the spread. This part of the loss is evenly split between the informed trader and the market makers. As we have seen, the optimal slope is inversely proportional to the spread. This results in the informed trader's expected profit actually being positively related to the spread. The reason is that as the spread increases, the optimal slope will decrease. The total effect is a gain for the informed trader. Below we plot the unconditional expected profits in equilibrium, aggregated for each of the three types, when $N = 1$.

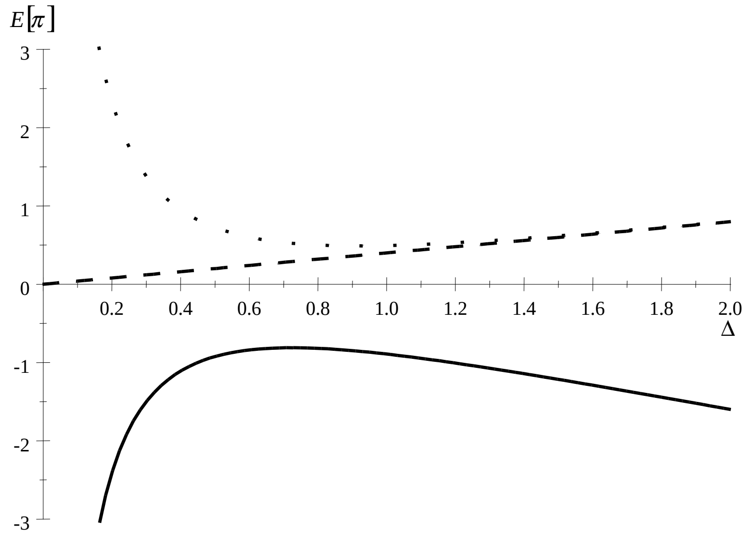

Figure 3: The unconditional expected profits, aggregated over type, when $\sigma_v^2 = \sigma_u^2 = N = 1$. The dotted line is the aggregated profits of market makers. The dashed line is the informed trader's profit. The solid line is the noise trader's profit.

Note that when the spread is very low, the noise trader's loss is very high. The reason is that the market makers' prices will be very sensitive to order flow. Since the noise trader has to trade at all prices, his loss will be very high. The informed trader's profit, on the other hand, will be very low. He can chose when to trade, but will trade in small quantities since the prices are so sensitive to order flow. As the spread increases, competition between the market makers increase and the prices become less sensitive to order flow. As a consequence, the noise trader's loss decreases. However, at a certain level the costs the spread entails for the noise traders outweigh the less sensitive prices. As a result the noise traders' loss increases again.

Unconditional expected profits when N changes from 1 to 2. Let us look at how a change in the number of noise traders affect the profits of the three types of agents. In the figure below we plot the aggregated unconditional expected profits when we go from $N = 1$ to $N = 2$.

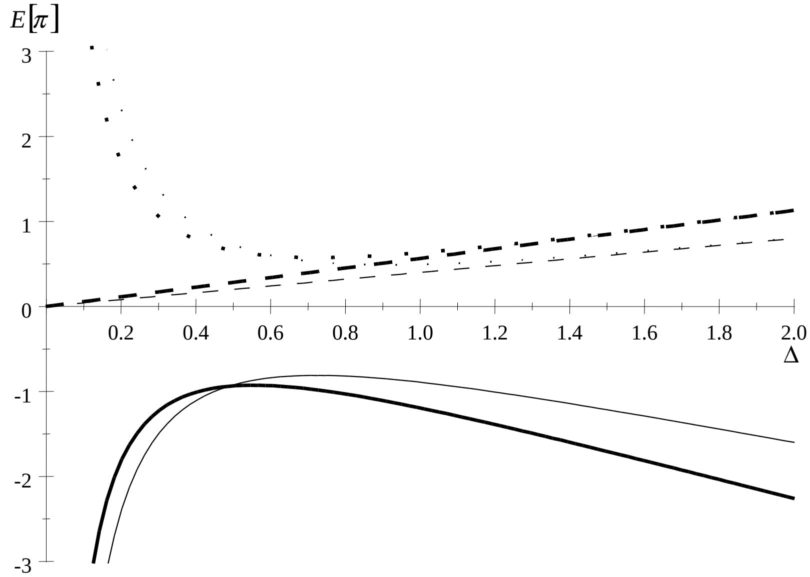

Figure 4: Aggregated unconditional expected profits. The thin curves correspond to $N = 1$. The thick curves correspond to $N = 2$.

Increasing the number of noise traders, we see that the noise traders benefit when the spread is low. Their total cost will then be lower compared to when there was only one noise trader. The reason is that with two noise traders there exists a possibility for the market makers to offset two opposing trades with each other and thus make a net gain. This results in a higher collective loss for the noise traders. However, when the spread is low, the net effect is actually positive. The reason is that the possibility of making a larger gain increases the competition between the market makers, and the prices become less sensitive to order flow. However, as the spread increases, the effect on competition, and thus on the price sensitivity, diminishes, while the noise traders still make their collective loss. Thus, the overall effect is that when the spread is high, then the noise traders as a group are actually worse off the more numerous they are.

Individually, however, a noise trader gains if more noise traders join the market. The reason is that the cost the spread entails remains constant for the individual noise trader, whereas the effect on the price sensitivity results in a gain. Below we plot the case with one and two noise traders. The solid curve is the noise trader's unconditional expected profit when there is only one noise trader. The dashed curve is the same noise trader's cost when there are two noise traders. The dotted curve is the aggregated cost for noise traders when there are two noise traders. Thus, we clearly see that an individual noise trader gains from having another noise trader joining the market, at a constant total noise trading variance. However, as a group, the noise traders are worse off if the spread is sufficiently high.

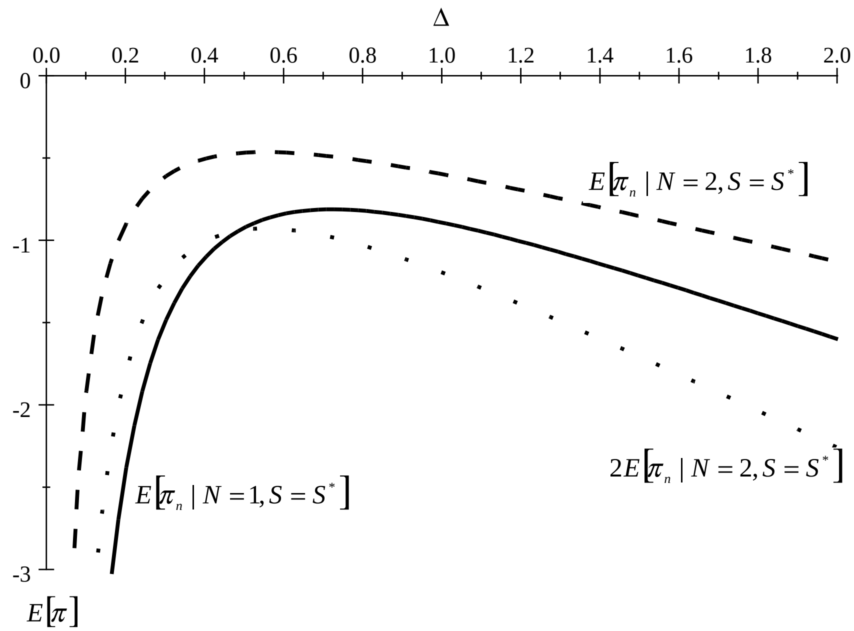

Figure 5: Comparing the cost of being a noise trader when we go from one to two noise traders, and $\sigma_u = \sigma_v = 1$.

In the next section we will endogenize the spread. It then turns out that the optimal spread in this case would be 0.8, which is large enough to imply that the noise traders as a group is worse off if another noise trader joins the market. This then suggests that the noise traders would gain from pooling their trades and first try to offset them with each other before they approach the market makers.

4. Endogenous spread

We will now endogenize the spread. Thus each market maker m can choose a spread $\Delta_m \in \mathbb{R}_+$, which implies that market maker m's strategy is now $(\beta, \Delta)_m \in \mathbb{R}^2_+$. The informed trader's strategy $\xi : \mathbb{R}^4_{++} \times \mathbb{R} \to \mathbb{R}^2$ is now defined by

$$\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2) = \xi\left(\beta_1, \beta_2, \Delta_1, \Delta_2, \tilde{v}\right), \tag{14}$$

while noise trader n's strategy $\phi_n : \mathbb{R}^4_{++} \times \mathbb{R} \to \mathbb{R}^2$ is defined by

$$\tilde{\mathbf{z}}_n = (\tilde{z}_{n1}, \tilde{z}_{n2}) = \left\{\phi_n\left(\beta_1, \beta_2, \Delta_1, \Delta_2, \tilde{u}_n\right) : \tilde{z}_{n1} + \tilde{z}_{n2} = \tilde{u}_n \;\cap\; \left|\tilde{z}_{n1}\right|, \left|\tilde{z}_{n2}\right| \leq \left|\tilde{u}_n\right|\right\}. \tag{15}$$

Again we will propose a strategy profile to be a Nash equilibrium, and then show that it is indeed the case.

The proposed strategy profile is $S^{**} = ((\boldsymbol{\beta}^{**}, \boldsymbol{\Delta}^{**}), \tilde{\mathbf{x}}^{**}, \tilde{\mathbf{z}}^{**})$, where

$$\beta_m^{**} = 4\frac{A(\Delta_m)}{B(\Delta, N)} \tag{16}$$

$$\Delta_m^{**} = \left\{ \Delta_m^{**} = 4\left(\frac{A(\Delta_m^{**})}{\Delta_m^{**}} - A'(\Delta_m^{**})\right) : \Delta_m^{**} \in \mathbb{R}_+ \right\} \tag{17}$$

$$x_m^{**} = \left\{ \begin{array}{lll} \frac{v-\alpha+\Delta_m}{2\beta_m} & \text{if} & v < \alpha - \Delta_m \\ 0 & \text{if} & \alpha - \Delta_m < v < \alpha + \Delta_m \\ \frac{v-\alpha-\Delta_m}{2\beta_m} & \text{if} & \alpha + \Delta_m > v \end{array} \right\} \tag{18}$$

$$z_{nm}^{**} = \frac{\beta_{-m}}{\beta_m + \beta_{-m}}u - \frac{1}{2}\frac{\Delta_m - \Delta_{-m}}{\beta_m + \beta_{-m}}\frac{u}{|u|}. \tag{19}$$

The main result in this section is the following proposition

Proposition 2. *The strategy profile $S^{**} = ((\boldsymbol{\beta}^{**}, \boldsymbol{\Delta}^{**}), \tilde{\mathbf{x}}^{**}, \tilde{\mathbf{z}}^{**})$ is a Nash equilibrium.*

The proof can be found in the Appendix. Again we will outline the intuition behind the results through some comparative statics in the remainder of this section.

4.1. Comparative statics.

The optimal slope and spread as functions of the standard deviations. In the figure below we plot the optimal slope and spread as a function of the standard deviation of fundamentals. As we can see, they are both increasing in σ_v.

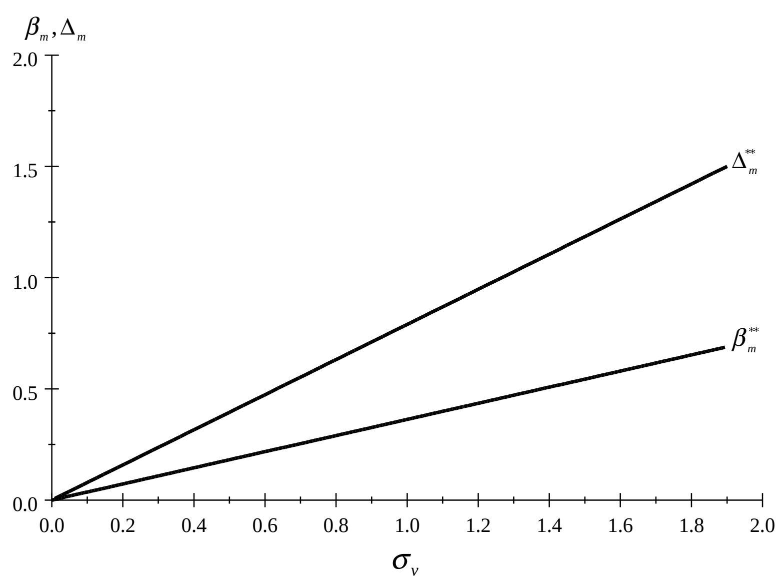

Figure 6: The endogenous spread and slope as functions of the standard variation of fundamentals when $\sigma_u = 1$. Both the spread and the slope increase as the standard variation of fundamentals increases.

As far as I know, it has not been shown in earlier theoretical models that the spread is increasing in the standard variation of fundamentals, nor has it been tested empirically.[2]

In the figure below we plot the optimal slope and spread as a function of the standard deviation of noise trades. The optimal slope is downward sloping. However, the optimal spread is independent of the standard variation of noise trades. The reason for this result has its roots in the noise trader's strategy (19) . In a symmetric equilibrium the spreads will not influence the noise traders' choice of market maker. As a result, the market maker will not care about noise trader demand when setting the spread, which makes the optimal spread independent of the standard deviation of noise trades.

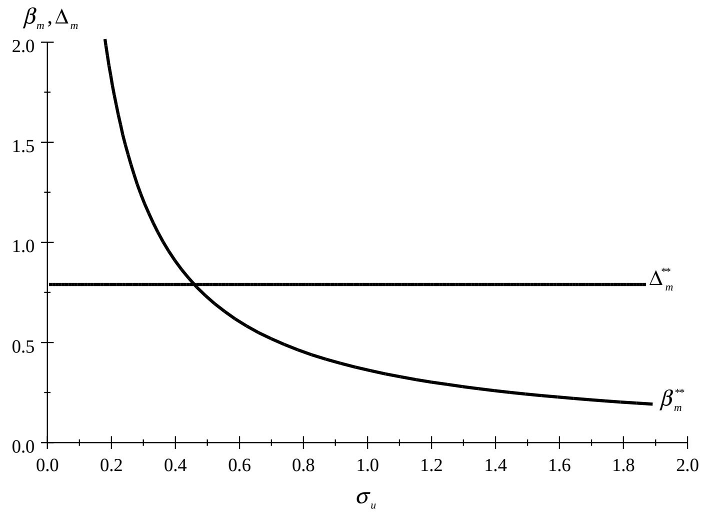

Figure 7: The endogenous spread and slope as functions of the standard variation of noise trades when $\sigma_v = 1$. Note that the spread does not depend on the standard variation of noise trades, whereas the slope is inversely proportional to the standard variation of noise trades.

It should be noted that this result may not hold if noise trader demand is price elastic. Then the absolute level of the spread, instead of only the relative level, may influence noise trader demand. However, this issue is outside the scope of this paper.

The effects of changes in the number of noise traders. In the figure below we have plotted β_m^{**} and Δ_m^* as N changes. Note that the optimal slope is decreasing in N. The reason is that with more noise traders, the market makers can clear more trades against each other, netting the profit. As a result, they will compete more fiercely, and the slope will be pushed down. The optimal spread, by contrast, is invariant to the number of noise traders. The reason for this is again that the spreads does not influence the noise traders' choice of market maker, which results in the market makers not caring about noise trader demand when setting the spread.

[2]A somewhat related empirical paper is Jayaraman (2008). He tests the relationship between the difference between the volatility of earnings and the volatility of cash flows, and the spread, and finds a U-shaped relationship. However, he does not test directly whether the spread is positively correlated with either the volatility of earnings of cash flows.

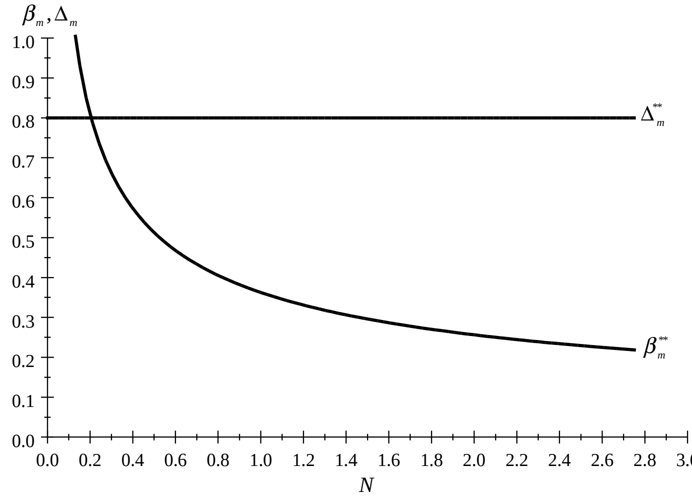

Figure 8: β_m^{**} and Δ_m^{**} as functions of N for $\sigma_v^2 = \sigma_v^2 = 1$.

5. Extension and robustness

In this section we will first extend the baseline model to take into account an arbitrary number of market makers. The assumption that the intercept is equal to the expected value of the fundamental value is then briefly discussed.

5.1. M **market makers.** Let us now extend the baseline model, i.e. with an exogenous spread, to allow for M market makers. The market makers are indexed by $m \in \{1, ..., M\}$. We will again propose a strategy profile that will be played in a Nash equilibrium. The proposed strategy profile is $S^{***} = (\boldsymbol{\beta}^{***}, \tilde{\mathbf{x}}^{***}, \tilde{\mathbf{z}}^{***})$, where

$$\beta_m^{***} = \left\{ \begin{array}{ll} 4\frac{A(\Delta)}{B(\Delta,N)} & \text{if} \quad M=2 \\ -\frac{M(M-1)}{2(M-2)\sigma_u^2} B(\Delta,N) + \sqrt{\frac{M^3}{(M-2)\sigma_u^2} A(\Delta) + \frac{M^2(M-1)^2}{4(M-2)^2\sigma_u^4} B(\Delta,N)^2} & \text{if} \quad M>2 \end{array} \right\}$$

$$x_m^{***} = \left\{ \begin{array}{lll} \frac{v-\alpha+\Delta}{2\beta_m} & \text{if} & v < \alpha - \Delta \\ 0 & \text{if} & \alpha - \Delta < v < \alpha + \Delta \\ \frac{v-\alpha-\Delta}{2\beta_m} & \text{if} & \alpha + \Delta > v \end{array} \right\}$$

$$z_{nm}^{***} = \frac{\prod_{j \neq m}^{M} \beta_j}{\sum_k^M \prod_{j \neq k}^{M} \beta_j} u_n.$$

The main result in this section is the following proposition.

Proposition 3. *The strategy profile* $S^{***} = (\boldsymbol{\beta}^{***}, \tilde{\mathbf{x}}^{***}, \tilde{\mathbf{z}}^{***})$ *is a Nash equilibrium.*

Comparative statics. In the figure below we plot the equilibrium slope depending on the number of market makers. As we can see, prices are very sensitive to order flows when there are only two market makers, whereas the sensitivity decrease substantially as soon as there are at least three market makers. However, as the number of market makers increase the sensitivity again increases. The reason is that a larger number of market makers have to divide a given number of noise trading among themselves.

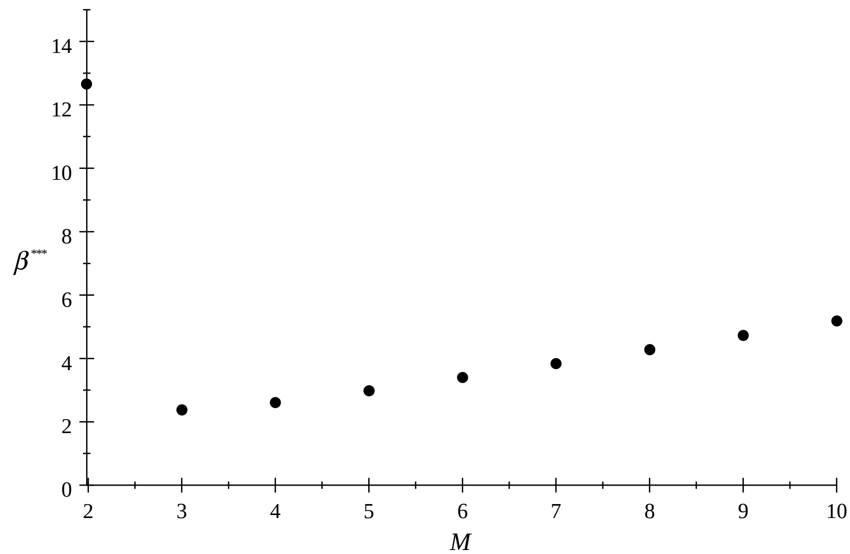

Figure 9: The equilibrium slope β^{***} as M changes, for $\sigma_u^2 = \sigma_v^2 = N = 1$, $\Delta = 0.1$.

Let us now look at the total profits for each type as we go from two to three market makers, and as the spread changes. The total payoff of each of the three types are

$$\begin{aligned} NE\left[\pi_n \mid S = (\boldsymbol{\beta}^{***}, \tilde{\mathbf{x}}^{***}, \tilde{\mathbf{z}}^{***})\right] &= -\frac{1}{M}\beta^{***}\sigma_u^2 - B\left(\Delta, N\right) \\ E\left[\tilde{\pi}_i \mid S = (\boldsymbol{\beta}^{***}, \tilde{\mathbf{x}}^{***}, \tilde{\mathbf{z}}^{***})\right] &= M\frac{A\left(\Delta\right)}{\beta^{***}} \\ ME\left[\tilde{\pi}_m \mid S = (\boldsymbol{\beta}^{***}, \tilde{\mathbf{x}}^{***}, \tilde{\mathbf{z}}^{***})\right] &= \frac{1}{M}\beta^{***}\sigma_u^2 + B\left(\Delta, N\right) - M\frac{A\left(\Delta\right)}{\beta^{***}}. \end{aligned}$$

Note again the noise traders' loss can be decomposed into two components, one depending on the slope, the other depending on the spread. Both components corresponds to gains for the market makers. However, the market makers also make a loss to the informed trader. This loss is increasing in the number of market makers.

Below we have plotted the aggregated unconditional expected profits when $M = 2$ and $M = 3$. With three market makers there is an equilibrium also without a spread. As a result the market makers' payoff is always lower when there are three market makers rather than two. However, when the spread is low, the competition is fairly weak, which makes it possible for the market makers as a group to get a higher profit than the informed trader. Nevertheless, as the spread increases, the market makers' profit initially decreases, while the informed trader's profit increases linearly. As a result the informed trader eventually receives a higher profit than the market makers.

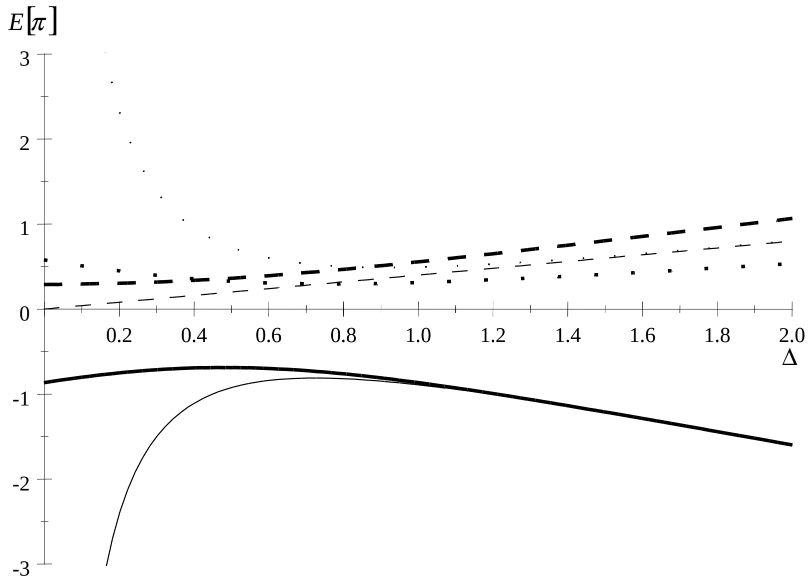

Figure 10: Comparing the aggregated unconditional expected profits when we go from $M = 2$ (thin lines) to $M = 3$ (thick lines), keeping the number of noise traders constant at 1.

5.2. The intercept in the pricing function. In the baseline model we made the simplifying assumption that the intercept in the pricing function is equal to the expected fundamental value. As already noted, this an equilibrium result in both Kyle (1985) and Dennert (1993). In the present model the noise traders are only allowed to perform limited arbitrage. For example a noise trader is not allowed to buy from one market maker and sell to the other. However, even when the noise traders' best replies are independent of the intercept, as they are in this model, the market makers will not have an incentive to change the intercept. The reason is that the informed trader performs some arbitrage. From (4) we can see that if a market maker increases his intercept, then the informed trader will sell more often to that market maker and buy less often from him. As a result, the profit expression of the market maker is independent of the intercept. It is thus not possible for a market maker to improve his payoff by unilaterally changing the intercept, which means that the intercept being equal to the expected value of the fundamentals is indeed a Nash equilibrium.

This is not a full analysis of the problem though. Ideally, we would let the noise traders buy from one market maker and sell to another, with best replies depending also on the intercept. However, such an extension of the model is outside the scope of this paper.

6. Discussion

The two workhorse models in market microstructure, Kyle (1985) and Glosten and Milgrom (1985), both assumed perfect competition between market makers. As discussed in Section 1, this assumption was relaxed in Dennert (1993), Bondarenko (2001), and Bernhardt and Hughson (1997).[3] Dennert showed that if noise traders did not split

[3] Glosten (1989) compared perfect competition with a supervised monopoly. He noted that a supervised monopolist may sometimes trade at a loss due to the fact that he can average gains and losses

trades, then a perfectly competitive mixed equilibrium existed with only two market makers. However, in the Kyle model, where the noise traders may split trades, no equilibrium exists with only two market makers. An equilibrium exists though with three or more market makers, and the perfect equilibrium corresponds to the ideal case when the number of market makers approach infinity. Bernhardt and Hughson (1997) showed that if the noise trader demand was decreasing in price then an equilibrium exists in the Kyle model both with one and two market makers. However, noise trader demand can not be too price elastic, then trade breaks down. While it is in most cases realistic to assume that noise trader demand is decreasing in price, it is unrealistic to assume that it is always is the case - for example under predatory trading as described by Brunnermeier and Pedersen (2005).

A quite substantial literature has been devoted to studying the spread and its components. Typically, three components are stressed. The first is an order processing cost. The second is an inventory holding cost, as described by Demsetz (1968), Stoll (1978), Ho and Stoll (1981), and Ho and Stoll (1983). The third component is adverse selection as demonstrated by Bagehot (1971), Copeland and Galai (1983), and Glosten and Milgrom (1985).

Empirical studies of the composition of the spread has been made by Glosten and Harris (1988), Stoll (1989), George et al (1991), Lin et al (1995), Huang and Stoll (1997), Ahn et al (2002), and Bollen et al (2004). Bollen et al (2004) show that the bid-ask spread is a function of the minimum tick size, the inverse of the trading volume, competition between market makers, and the expected inventory holding premium.

The effect of decimalization has been investigated by Bacidore et al (2001), Bessembinder (2003), Chung et al (2004), Gibson et al (2003), and Serednyakov (2005), who all found that the spread decreased substantially when decimalization was introduced on the NYSE. According to Serednyakov (2005) it appears to be primarily due to order processing and inventory holding costs going down, while Gibson et al (2003) find that the reduction in spreads is due to lower order processing costs. Giouvris and Philippatos (2008) studied the components of the bid-ask spread when the London Stock Exchange changed from a quote driven to an order driven market and found that the adverse selection component was reduced.

Another issue is that collusion between market makers may be a factor, as demonstrated by Christie and Schultz (1994) and Christie, Harris and Schultz (1994). Godek (1996) argues that preference trading may result in collusion, whereas Kandel and Marx (1997) show that market makers can use odd-tick avoidance as a coordination device to increase spreads. Their argument is especially valid if the tick size is large relative to the spreads being charged. Dutta and Madhavan (1997), on the other hand, show that market makers may engage in implicit collusion if they are sufficiently patient and if there are barriers to entry. Price discreteness is thus not necessary in their model.

7. Conclusion

In this paper it was shown that a Nash equilibrium exists in the Kyle model with two market makers if it is extended to take into account the spread. A side effect is that the

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over time. This may help restore trading when it has broken down.

Kyle model has also been extended to take into account gross order flow instead of only considering the net order flow. Thus, whereas market makers in the original Kyle model only cares about adverse selection, they here also care about whether they can offset opposing trades with each other. Conceptually this means that the price sensitivity can no longer be interpreted as a measure of the spread, which it often is in the original Kyle model. This also opens up for further developments of the Kyle model. We briefly looked at how the number of noise traders will influence how sensitive prices will be to order flow, but other extensions may also be possible. For example, although we did extend the baseline model, i.e. the one with an exogenous spread, to take into account an arbitrary number of market makers, one could also envision such an extension with an endogenous spread. In addition, a dynamic extension of this static model is also called for.

Appendix A

A.1. Proof of Proposition 1. We now proceed to show that the proposed Nash equilibrium is indeed a Nash equilibrium. This is achieved by showing that no player can gain by unilaterally deviating, given that all other players are playing the proposed strategy.

The noise traders. If everybody else is playing the proposed Nash equilibrium, then noise trader n's expected profit is

$$E\left[\pi_n \mid S=\left(\boldsymbol{\beta}^*, \tilde{\mathbf{x}}^*, \tilde{\mathbf{z}}^*_{-n\cdot}\right), \tilde{u}_n=u_n,\right] = -\beta_1^* z_{n1}^2 - \Delta\left|z_{n1}\right| - \beta_2^*\left(u-z_{n1}\right)^2 - \Delta\left|u-z_{n1}\right|. \tag{A.20}$$

The first order condition is

$$\frac{\partial E\left[\pi_n \mid S=\left(\boldsymbol{\beta}^*, \tilde{\mathbf{x}}^*, \tilde{\mathbf{z}}^*_{-n\cdot}\right), \tilde{u}_n=u_n\right]}{\partial z_{n1}} = -2\beta_1^* z_{n1}^B + 2\beta_2^*\left(u_n - z_{n1}^B\right) = 0. \tag{A.21}$$

The second order condition is

$$\frac{\partial^2 E\left[\pi_n \mid S=\left(\boldsymbol{\beta}^*, \tilde{\mathbf{x}}^*, \tilde{\mathbf{z}}^*_{-n\cdot}\right), \tilde{u}_n=u_n\right]}{\partial z_{n1}^2} = -2\left(\beta_1^*+\beta_2^*\right) < 0,$$

which is always satisfied. Thus, the best replies are

$$\left(z_{n1}^B, z_{n2}^B\right) = \left(\frac{\beta_2^*}{\beta_1^*+\beta_2^*}u_n, \frac{\beta_1^*}{\beta_1^*+\beta_2^*}u_n\right), \tag{A.22}$$

which are identical to the strategy in the proposed Nash equilibrium.

The informed trader. If everybody else is playing the proposed Nash equilibrium, then the informed trader's expected profit is

$$E\left[\tilde{\pi}_i\left(\tilde{x}_1,\tilde{x}_2\right) \mid S=\left(\boldsymbol{\beta}^*, \mathbf{x}, \tilde{\mathbf{z}}^*\right), \tilde{v}=v\right] = \sum_{m=1}^{2}\left[\left(v-\alpha\right)x_m - \beta_m^* x_m^2 - \Delta\left|x_m\right|\right]. \tag{A.23}$$

The first order condition in

$$\frac{\partial E\left[\tilde{\pi}_i\left(\tilde{x}_1,\tilde{x}_2\right) \mid S=\left(\boldsymbol{\beta}^*, \mathbf{x}, \tilde{\mathbf{z}}^*\right), \tilde{v}=v\right]}{\partial x_m} = v - \alpha - 2\beta_m^* x_m^B - \Delta\frac{x_m^B}{\left|x_m^B\right|} = 0.$$

The second order condition is

$$\frac{\partial^2 E\left[\tilde{\pi}_i\left(\tilde{x}_1,\tilde{x}_2\right)\mid S=\left(\boldsymbol{\beta}^*,\mathbf{x},\tilde{\mathbf{z}}^*\right),\tilde{v}=v\right]}{\partial x_m^2}=-2\beta_m^*<0,$$

which is always satisfied. Thus, the best reply is

$$x_m^B=\left\{\begin{array}{lll}\frac{v-\alpha+\Delta}{2\beta_m^*} & \text{if} & v<\alpha-\Delta\\ 0 & \text{if} & \alpha-\Delta<v<\alpha+\Delta\\ \frac{v-\alpha-\Delta}{2\beta_m^*} & \text{if} & \alpha+\Delta<v\end{array}\right\},$$

which again is identical to the strategy in the proposed Nash equilibrium.

The market makers. Let us derive market maker m's expected profit. First it is straight-forward to show that

$$E\left[\tilde{\pi}_m|S=\left(\boldsymbol{\beta}_{-m}^*,\tilde{\mathbf{x}}^*,\tilde{\mathbf{z}}^*\right)\right]=\left\{\begin{array}{c}E\left[\left(\alpha+\beta_m\tilde{x}_m^*-\tilde{v}\right)\tilde{x}_m^*\right]+\Delta E\left[\left|\tilde{x}_m^*\right|\right]\\ +\beta_m\sum_n E\left[\tilde{z}_{nm}^{*2}\right]+\Delta\sum_n E\left[\left|\tilde{z}_{nm}^*\right|\right]\end{array}\right\}. \qquad \text{(A.24)}$$

Let us consider the terms in this expression. Using the proposed strategy for noise trader n, (5), we get

$$\sum_n E\left[\tilde{z}_{nm}^{*2}\right]=\frac{\beta_{-m}^2}{\left(\beta_m+\beta_{-m}\right)^2}\sigma_u^2, \qquad \text{(A.25)}$$

and

$$\sum_n E\left[\left|\tilde{z}_{nm}^*\right|\right]=\frac{\beta_{-m}}{\beta_m+\beta_{-m}}\sqrt{\frac{2N\sigma_u^2}{\pi}}. \qquad \text{(A.26)}$$

Using the proposed strategy for the informed trader, (4), we get

$$E\left[\left|\tilde{x}_m^*\right|\right]=\frac{1}{2\beta_m^*\sqrt{2\pi\sigma_v^2}}\left(\begin{array}{c}\int_{-\infty}^{\alpha-\Delta}-\left(\tilde{v}-\left(\alpha-\Delta\right)\right)\exp\left(-\frac{\left(\tilde{v}-\alpha\right)^2}{2\sigma_v^2}\right)d\tilde{v}\\ +\int_{\alpha+\Delta}^{\infty}\left(\tilde{v}-\left(\alpha+\Delta\right)\right)\exp\left(-\frac{\left(\tilde{v}-\alpha\right)^2}{2\sigma_v^2}\right)d\tilde{v}\end{array}\right) \qquad \text{(A.27)}$$

and

$$E\left[\left(\alpha+\beta_m^*\tilde{x}_m^*-\tilde{v}\right)\tilde{x}_m^*\right]=\frac{1}{2\beta_m^*\sqrt{2\pi\sigma_v^2}}\left(\begin{array}{c}\int_{-\infty}^{\alpha-\Delta}\left(\alpha+\frac{\tilde{v}-\alpha+\Delta}{2}-\tilde{v}\right)\left(\tilde{v}-\alpha+\Delta\right)\exp\left(-\frac{\left(\tilde{v}-\alpha\right)^2}{2\sigma_v^2}\right)d\tilde{v}\\ +\int_{\alpha+\Delta}^{\infty}\left(\alpha+\frac{\tilde{v}-\alpha-\Delta}{2}-\tilde{v}\right)\left(\tilde{v}-\alpha-\Delta\right)\exp\left(-\frac{\left(\tilde{v}-\alpha\right)^2}{2\sigma_v^2}\right)d\tilde{v}\end{array}\right) \qquad \text{(A.28)}$$

Using the expressions $(A.27)$ and $(A.28)$ we get

$$E\left[\left(\alpha+\beta_m^*\tilde{x}_m^*-\tilde{v}\right)\tilde{x}_m^*\right]+\Delta E\left[\left|\tilde{x}_m^*\right|\right]=-\frac{A\left(\Delta\right)}{\beta_m^*}. \qquad \text{(A.29)}$$

Inserting the terms $(A.25)$, $(A.26)$, and $(A.29)$ into $(A.24)$ we get

$$E\left[\tilde{\pi}_m|S=\left(\boldsymbol{\beta}^*_{-m},\tilde{\mathbf{x}}^*,\tilde{\mathbf{z}}^*\right)\right]=-\frac{A(\Delta)}{\beta_m}+\beta_m\frac{\beta^{*2}_{-m}}{\left(\beta_m+\beta^*_{-m}\right)^2}\sigma_u^2+\frac{\beta^*_{-m}}{\beta_m+\beta^*_{-m}}B(\Delta,N). \tag{A.30}$$

The first order condition is

$$\frac{\partial E\left[\tilde{\pi}_m|S=\left(\boldsymbol{\beta}^*_{-m},\tilde{\mathbf{x}}^*,\tilde{\mathbf{z}}^*\right)\right]}{\partial\beta_m}=\frac{A(\Delta)}{\beta_m^2}+\frac{\beta^{*2}_{-m}\left(\beta^*_{-m}-\beta_m\right)}{\left(\beta_m+\beta^*_{-m}\right)^3}\sigma_u^2-\frac{\beta^*_{-m}}{\left(\beta_m+\beta^*_{-m}\right)^2}B(\Delta,N)=0. \tag{A.31}$$

Inserting the proposed strategy for market maker $-m$ and rewriting the first order condition as a cubic equation, we get

$$(4A(\Delta)-B(\Delta,N)\beta_m)\Psi(\beta_m)=0,$$

where $\Psi(\beta_m)$ is given by

$$\Psi(\beta_m)=a_2\beta_m^2+a_1\beta_m+a_0,$$

and

$$\begin{aligned}a_2&=16A(\Delta)\sigma_u^2+3B(\Delta,1)^2\\a_1&=16A(\Delta)B(\Delta,1)\\a_0&=16A(\Delta)^2.\end{aligned}$$

Note that $\Psi(0)=a_0$ and

$$\frac{\partial\Psi(0)}{\partial\beta_m}=a_1,$$

and that all three coefficients are positive. This implies that the two roots must be negative and thus not admissible.

Thus, the only admissible solution to the first order condition is

$$\beta_m=4\frac{A(\Delta)}{B(\Delta,N)},$$

i.e. the proposed solution.

Furthermore, when $\beta_m=\beta_{-m}=\beta^*$, the second order condition is

$$\frac{\partial^2E\left[\tilde{\pi}_m|S=(\boldsymbol{\beta}^*,\tilde{\mathbf{x}}^*,\tilde{\mathbf{z}}^*)\right]}{\partial\beta_m^2}=-\frac{1}{64}\frac{2A(\Delta)\sigma_u^2+B(\Delta,N)^2}{A(\Delta)^2}B(\Delta,N)<0$$

which is always true. Thus, the proposed strategy is indeed a global maximum - on the admissible interval - and thus a best reply.

A.2. Proof of Proposition 2.

The noise trader. If everybody else is playing the proposed Nash equilibrium, then the noise trader's expected profit is

$$E\left[\pi_n \mid S = \left(\left(\boldsymbol{\beta}^{**}, \boldsymbol{\Delta}^{**}\right), \tilde{\mathbf{x}}^{**}, \tilde{\mathbf{z}}^{**}_{-n\boldsymbol{.}}\right), \tilde{u} = u\right] = -\beta_1^{**} z_{n1}^2 - \Delta_1^{**} \left|z_{n1}\right| - \beta_2^{**} \left(u - z_{n1}\right)^2 - \Delta_2^{**} \left|u - z_{n1}\right|. \tag{A.32}$$

The first order condition w.r.t. z_{n1} is

$$-2\beta_1^{**} z_{n1}^B + 2\beta_2^{**}\left(u - z_{n1}^B\right) - \Delta_1^{**}\frac{z_{n1}^B}{\left|z_{n1}^B\right|} + \Delta_2^{**}\frac{u - z_{n1}^B}{\left|u - z_{n1}^B\right|} = 0.$$

Using the assumption that

$$\frac{z_1}{|z_1|} = \frac{u - z_1}{|u - z_1|} = \frac{u}{|u|},$$

the best reply is

$$z_{n1}^B = \frac{\beta_2^{**}}{\beta_1^{**} + \beta_2^{**}} u - \frac{1}{2}\frac{\Delta_1^{**} - \Delta_2^{**}}{\beta_1^{**} + \beta_2^{**}}\frac{u}{|u|},$$

which is the proposed strategy.

The second order condition is

$$\frac{\partial^2 E\left[\pi_n \mid S = \left(\left(\boldsymbol{\beta}^{**}, \boldsymbol{\Delta}^{**}\right), \tilde{\mathbf{x}}^{**}, \tilde{\mathbf{z}}^{**}_{-n\boldsymbol{.}}\right), \tilde{u} = u\right]}{\partial z_{n1}^2} = -2\left(\beta_1^{**} + \beta_2^{**}\right) - \frac{\Delta_1^{**} - \Delta_2^{**}}{\beta_1^{**} + \beta_2^{**}}\delta\left(u\right) < 0,$$

where $\delta(u)$ is the Dirac delta function. Note that the second order condition must be satisfied when everybody else is playing the proposed Nash equilibrium. The proposed strategy is thus a best reply.

The informed trader. If everybody else is playing the proposed Nash equilibrium, then the informed trader's expected profit is

$$E\left[\tilde{\pi}_i\left(\tilde{x}_1, \tilde{x}_2\right) \mid S = \left(\left(\boldsymbol{\beta}^{**}, \boldsymbol{\Delta}^{**}\right), \mathbf{x}, \tilde{\mathbf{z}}^{**}\right), \tilde{v} = v\right] = \sum_{m=1}^{2}\left[v - \alpha - \beta_m^{**} x_m^2 - \Delta_m^{**}\left|x_m\right|\right]. \tag{A.33}$$

The first order conditions are

$$\frac{\partial E\left[\tilde{\pi}_i\left(\tilde{x}_1, \tilde{x}_2\right) \mid S = \left(\left(\boldsymbol{\beta}^{**}, \boldsymbol{\Delta}^{**}\right), \mathbf{x}, \tilde{\mathbf{z}}^{**}\right), \tilde{v} = v\right]}{\partial x_m} = v - \alpha - 2\beta_m^{**} x_m^B - \Delta_m^{**}\frac{x_m^B}{\left|x_m^B\right|} = 0.$$

The second order conditions are

$$\frac{\partial^2 E\left[\tilde{\pi}_i\left(\tilde{x}_1, \tilde{x}_2\right) \mid S = \left(\left(\boldsymbol{\beta}^{**}, \boldsymbol{\Delta}^{**}\right), \mathbf{x}, \tilde{\mathbf{z}}^{**}\right), \tilde{v} = v\right]}{\partial x_m^2} = -2\beta_m^{**} < 0,$$

which are always satisfied. Thus, the best replies are

$$x_m^B = \left\{ \begin{array}{ll} \frac{v - \alpha + \Delta_m}{2\beta_m^{**}} & \text{if} \quad v < \alpha - \Delta_m \\ 0 & \text{if} \quad \alpha - \Delta_m < v < \alpha + \Delta_m \\ \frac{v - \alpha - \Delta_m}{2\beta_m^{**}} & \text{if} \quad \alpha + \Delta_m > v \end{array} \right\}.$$

The market maker. If everybody else is playing the proposed Nash equilibrium, then market maker m's expected profit is

$$E\left[\tilde{\pi}_m | S = \left(\boldsymbol{\beta}^{**}_{-m}, \mathbf{x}^{**}, \mathbf{z}^{**}\right)\right] = -\frac{A(\Delta_m)}{\beta_m} + \beta_m \sum_n E\left[\tilde{z}^{**2}_{nm}\right] + \Delta_m \sum_n E\left[\left|\tilde{z}^{**}_{nm}\right|\right], \quad (A.34)$$

where

$$\sum_n E\left[\tilde{z}^{**2}_{nm}\right] = \frac{\beta^{**2}_{-m}\sigma_u^2 - \beta^{**}_{-m}\left(\Delta_m - \Delta^{**}_{-m}\right)\sqrt{\frac{2N\sigma_u^2}{\pi}} + \frac{1}{4}\left(\Delta_m - \Delta^{**}_{-m}\right)^2 N}{\left(\beta_m + \beta^{**}_{-m}\right)^2}, \quad (A.35)$$

and

$$\sum_n E\left[\left|\tilde{z}^{**}_{nm}\right|\right] = \frac{\beta^{**}_{-m}\sqrt{\frac{2N\sigma_u^2}{\pi}} - \frac{1}{2}\left(\Delta_m - \Delta^{**}_{-m}\right) N}{\beta_m + \beta^{**}_{-m}}. \quad (A.36)$$

Market maker m's first order condition w.r.t. β_m is

$$\frac{\partial E\left[\tilde{\pi}_m | S = \left(\left(\boldsymbol{\beta}^{**}, \boldsymbol{\Delta}^{**}\right)_{-m}, \tilde{\mathbf{x}}^{**}, \tilde{\mathbf{z}}^{**}\right)\right]}{\partial \beta_m} = \left\{ \begin{array}{c} \frac{A(\Delta_m)}{\beta_m^2} + \frac{\beta^{**}_{-m} - \beta_m}{\beta_m + \beta^{**}_{-m}} \sum_n E\left[\tilde{z}^{**2}_{nm}\right] \\ -\frac{\Delta_m}{\beta_m + \beta^{**}_{-m}} \sum_n E\left[\left|\tilde{z}^{**}_{nm}\right|\right] \end{array} \right\} = 0$$

Inserting the proposed solution $\left(\beta^{**}_m, \Delta^{**}_m\right)$, it is straight forward to see that it satisfies the first order condition.

In the proposed equilibrium, the second order condition is

$$\frac{\partial^2 E\left[\tilde{\pi}_m | S = \left(\left(\boldsymbol{\beta}^{***}, \boldsymbol{\Delta}^{***}\right)_{-m}, \tilde{\mathbf{x}}^{***}, \tilde{\mathbf{z}}^{***}\right)\right]}{\partial \beta_m^2} = -\frac{B\left(\Delta_m, N\right)^2 + 2A\left(\Delta_m\right)\sigma_u^2}{64A\left(\Delta_m\right)^2} B\left(\Delta_m, N\right) < 0$$

which is always satisfied.

The first order condition can be rewritten to obtain

$$\Omega\left(\beta_m\right) = b_3\beta_m^3 + b_2\beta_m^2 + b_1\beta_m + b_0 = 0,$$

where

$$\begin{aligned} b_3 &= -\left(\sigma_u^2\beta_{-m}^2 + B\left(\Delta_m, N\right)\beta_{-m} - A\left(\Delta_m\right) + C + \Delta_m D\right) \gtrless 0 \\ b_2 &= \left(\sigma_u^2\beta_{-m}^3 - B\left(\Delta_m, N\right)\beta_{-m}^2 + 3A\left(\Delta_m\right)\beta_{-m} + C\beta_{-m} - \Delta_m D\beta_{-m}\right) \gtrless 0 \\ b_1 &= 3A\left(\Delta_m\right)\beta_{-m}^2 > 0 \\ b_0 &= A\left(\Delta_m\right)\beta_{-m}^3 > 0, \end{aligned}$$

and

$$\begin{aligned} C &= -\beta^{**}_{-m}\left(\Delta_m - \Delta^{**}_{-m}\right)\sqrt{\frac{2N\sigma_u^2}{\pi}} + \frac{1}{4}\left(\Delta_m - \Delta^{**}_{-m}\right)^2 N \\ D &= -\frac{1}{2}\left(\Delta_m - \Delta^{**}_{-m}\right) N. \end{aligned}$$

Note that $\Omega(0) = b_0$ and

$$\frac{\partial \Omega(0)}{\partial \beta_m} = b_1,$$

and that $b_0, b_1 \in \mathbb{R}_{++}$ whereas $b_2, b_3 \in \mathbb{R}$. As a result, at maximum two roots can be positive. By continuity only one of these positive roots can correspond to a maximum. Since there is never more than one solution to the first order condition that also entails a negative second derivative, the proposed solution must also be the best reply.

The first order condition w.r.t. Δ_m is

$$\frac{\partial E\left[\tilde{\pi}_m | S = \left((\boldsymbol{\beta}^{**}, \boldsymbol{\Delta}^{**})_{-m}, \tilde{\mathbf{x}}^{**}, \tilde{\mathbf{z}}^{**}\right)\right]}{\partial \Delta_m} = \left\{ \begin{array}{c} \left(\frac{\beta^{**}_{-m}}{\beta_m + \beta^{**}_{-m}}\right)^2 B'(\Delta_m, N) - \frac{A'(\Delta_m)}{\beta_m} \\ -\frac{1}{2}\frac{\beta_m \Delta_m + \beta^{**}_{-m}\left(2\Delta_m - \Delta^{**}_{-m}\right)}{\left(\beta_m + \beta^{**}_{-m}\right)^2} \end{array} \right\} = 0. \tag{A.37}$$

where

$$A'(\Delta_m) = \frac{1}{2}\left(\Delta_m \operatorname{erf c}\left(\Delta_m \sqrt{\frac{1}{2\sigma_v^2}}\right) + \sqrt{\frac{\sigma_v^2}{2}} \frac{\partial \operatorname{erf c}\left(\Delta_m \sqrt{\frac{1}{2\sigma_v^2}}\right)}{\partial\left(\Delta_m \sqrt{\frac{1}{2\sigma_v^2}}\right)}\right)$$

$$B'(\Delta_m, N) = \sqrt{\frac{2\sigma_u^2}{\pi}}.$$

Again, inserting the proposed solution $(\beta^{**}_m, \Delta^{**}_m)$, it is straight forward to see that it satisfies the first order condition.

Note that the second derivative is always negative

$$\frac{\partial^2 E\left[\tilde{\pi}_m | S = \left((\boldsymbol{\beta}^{**}, \boldsymbol{\Delta}^{**})_{-m}, \tilde{\mathbf{x}}^{**}, \tilde{\mathbf{z}}^{**}\right)\right]}{\partial \Delta_m^2} = -\left(\frac{A''(\Delta_m)}{\beta_m} + \frac{\beta_m + 2\beta^{**}_{-m}}{2\left(\beta_m + \beta^{**}_{-m}\right)^2}\right) < 0,$$

where

$$A''(\Delta_m) = \frac{1}{2}\operatorname{erf c}\left(\Delta\sqrt{\frac{1}{2\sigma_v^2}}\right).$$

The cross derivative is

$$\frac{\partial^2 E\left[\tilde{\pi}_m | S = \left((\boldsymbol{\beta}^{***}, \boldsymbol{\Delta}^{***}), \tilde{\mathbf{x}}^{***}, \tilde{\mathbf{z}}^{***}\right)\right]}{\partial \Delta_m \partial \beta_m} = \frac{B(\Delta^{***}, 1)^2}{16 A(\Delta^{***})^2}\left(A'(\Delta^{***}) + \frac{\Delta^{***}}{8} - \frac{A(\Delta^{***})}{\Delta^{***}}\right) \tag{A.38}$$

We have already seen that the partial second order derivatives are negative, as required for the proposed strategy to be a best reply. In addition, the second order total derivative is negative if

$$A(\Delta^{**})(8A''(\Delta^{**}) + 3) > \frac{\left(8A(\Delta^{**}) - \Delta^{**2} - 8\Delta^{**}A'(\Delta^{**})\right)^2}{8\left(A(\Delta^{**})\pi + \Delta^{**2}\right)}. \tag{A.39}$$

We do not have an analytical proof that this condition is always satisfied for the proposed spread. However, a numerical proof is plotted in the figure below. We have plotted Δ^{***} in $(\sigma_v, \Delta)-$space. Δ^{***} corresponds to the thick curve, whereas the condition $(A.39)$ is satisfied below the thin curve.

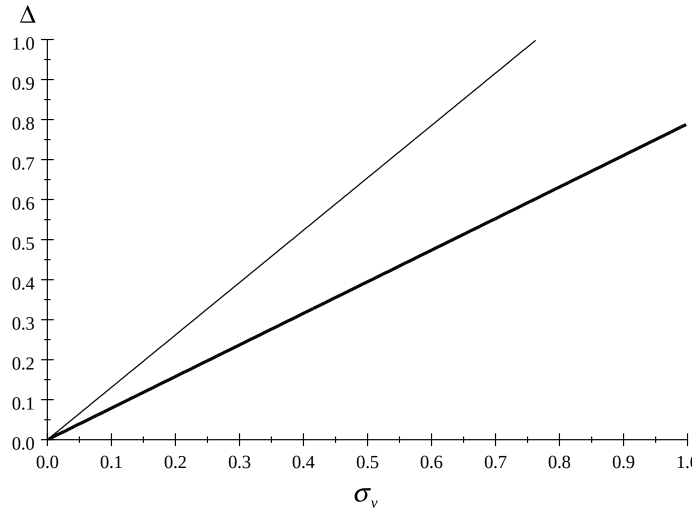

Figure 11: Representation of the numerical proof that the proposed spread always satisfies the condition $(A.39)$. The thick curve is Δ^* whereas all points below the thin curve satisfies $(A.39)$.

Thus, the proposed solution $(\beta_m^{**}, \Delta_m^{**})$ is indeed a best reply and the strategy profile S^{**} is a Nash equilibrium.

A.3. Proof of Proposition 3.

The noise traders. Noise trader n's first order condition is

$$\frac{\partial E\left[\pi_n \mid S = \left(\boldsymbol{\beta}^{***}, \tilde{\mathbf{x}}^{***}, \tilde{\mathbf{z}}_{-n,.}^{***}\right), \tilde{u} = u\right]}{\partial z_{nm}} = -2\beta_m^{***} z_{nm}^B + 2\beta_M^{***}\left(u_n - z_{nm}^B - \sum_{k\neq m}^{M-1} z_{nk}\right) = 0, \tag{A.40}$$

for $\forall m \in \{1, ..., M-1\}$. The second order condition is always satisfied since

$$\frac{\partial^2 E\left[\pi_n \mid S = \left(\boldsymbol{\beta}^{***}, \tilde{\mathbf{x}}^{***}, \tilde{\mathbf{z}}_{-n,.}^{***}\right), \tilde{u} = u\right]}{\partial z_{nm}^2} = -2\left(\beta_m^{***} + \beta_M^{***}\right) < 0,$$

$\forall m \in \{1, ..., M-1\}$.

Reshuffling the first order condition, we get

$$z_{nm}^B = \left\{ \begin{array}{lll} \frac{\beta_M^{***}\left(u_n - \sum\limits_{k\neq m}^{M-1} z_{nk}\right)}{\beta_m^{**} + \beta_M^{**}} & \text{for} & m = \{1, ..., M-1\} \\ u_n - \sum\limits_{m}^{M-1} z_{nm} & \text{for} & m = M \end{array} \right\}$$

Solving this system of equations, we get $z_{nm}^B = z_{nm}^{***}$ for $\forall m \in \{1, ..., M\}$, i.e. the best reply is indeed the proposed strategy.

The informed trader. The informed trader's expected profit is

$$E\left[\tilde{\pi}_i \mid S = \left(\boldsymbol{\beta}^{***}, \mathbf{x}, \tilde{\mathbf{z}}^{***}\right)\right] = \sum_{m=1}^{M}\left[v - \alpha - \beta_m^{***} x_m^2 - \Delta\left|x_m\right|\right]. \tag{A.41}$$

which only differs from $(A.23)$ in the number of market makers. Thus, the best replies are $x_m^B = x_m^{***}, \forall m \in \{1, ..., M\}$, i.e. the best reply is indeed the proposed strategy.

The market makers. Market maker m's expected profit is

$$E\left[\tilde{\pi}_m \mid S=\left(\boldsymbol{\beta}_{-m}^{***},\tilde{\mathbf{x}},\tilde{\mathbf{z}}^{***}\right)\right]=\left\{\begin{array}{c}-\frac{A(\Delta)}{\beta_m}+\beta_m\left(\frac{\beta^{***}}{\beta^{***}+\beta_m(M-1)}\right)^2\sigma_u^2\\+\frac{\beta^{***}}{\beta^{***}+\beta_m(M-1)}B\left(\Delta,N\right)\end{array}\right\}.$$

Taking, the first order condition, we get

$$\frac{\partial E\left[\tilde{\pi}_m \mid S=\left(\boldsymbol{\beta}_{-m}^{***},\tilde{\mathbf{x}},\tilde{\mathbf{z}}^{***}\right)\right]}{\partial\beta_m}=\left\{\begin{array}{c}\frac{A(\Delta)}{\beta_m^2}+\frac{\beta^{***3}\sigma_u^2-\beta_m\beta^{***2}(M-1)\sigma_u^2}{\left(\beta^{***}+\beta_m(M-1)\right)^3}\\-\frac{\beta^{***}}{\left(\beta^{***}+\beta_m(M-1)\right)^2}\left(M-1\right)B\left(\Delta,N\right)\end{array}\right\}=0.$$

Let us first show that the proposed strategy is a local maximum. Setting $\beta_m=\beta^{***}$, we get

$$\frac{M^3A\left(\Delta\right)-\left(M-2\right)\beta^{***2}\sigma_u^2-M\left(M-1\right)\beta^{***}B\left(\Delta,N\right)}{M^3\beta^{***2}}=0.$$

Simplifying and multiplying by M, the first order condition can be expressed as

$$M^4A\left(\Delta\right)=M\left(M-2\right)\beta^{***2}\sigma_u^2+M^2\left(M-1\right)\beta^{***}B\left(\Delta,N\right).$$

Taking the second order condition and setting $\beta_m=\beta^{***}$, we get

$$M^4A\left(\Delta\right)>\beta^{***2}\left(M-1\right)\left(M-3\right)\sigma_u^2+\beta^{***}M\left(M-1\right)^2B\left(\Delta,N\right).$$

Thus, if the first order condition is satisfied, then also the second order condition must be satisfied. The proposed strategy is thus clearly a local maximum.

Rewriting the first order condition, we get

$$\Phi\left(\beta_m\right)=c_3\beta_m^3+c_2\beta_m^2+c_1\beta_m+c_0=0,$$

where

$$c_0 = -A\beta^3 \tag{A.42}$$
$$c_1 = -3A\beta^2\left(M-1\right) \tag{A.43}$$
$$c_2 = -\beta\left(\left(3A\left(M-1\right)-\beta\right)\left(M-1\right)+\beta^2\sigma_u^2\right) \tag{A.44}$$
$$c_3 = -\left(M-1\right)\left(A-\beta\left(M-1\right)+AM\left(M-2\right)-\beta^2\sigma_u^2\right). \tag{A.45}$$

Note that $\Phi\left(0\right)=c_0$ and

$$\frac{\partial\Phi\left(0\right)}{\partial\beta_m}=c_1,$$

and that $c_0,c_1\in\mathbb{R}_{--}$ whereas $b_2,b_3\in\mathbb{R}$. As a result at maximum two roots can be positive. By continuity only one of those roots can correspond to a maximum. Thus, there can only be one solution that satisfies both the first and second order conditions. The proposed strategy is thus a best reply.

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