Epistemic robustness of sets closed under rational behavior*

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Abstract

This paper provides two conditions of epistemic robustness, *robustness to alternative best replies* and *robustness to non-best replies*, and uses these to characterize variants of CURB sets in finite games, including the set of rationalizable strategies.

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1 Introduction

A CURB set — mnemonic for 'closed under rational behavior' — is a Cartesian product of pure-strategy sets, one for each player, that includes all best replies to all probability distributions over the strategies in the set. Hence, if a player believes that her opponents stick to strategies from their components of a CURB set, then her component contains all her best replies, so she'd better stick to her strategies as well.

CURB sets and variants were introduced by Basu and Weibull (1991) and became of importance in the literature on strategic adaptation in finite games, since a variety of arguably plausible adjustment processes eventually settle down in a minimal (w.r.t. set inclusion) CURB set; cf. Hurkens (1995), Sanchirico (1996), Young (1998), and Fudenberg and Levine (1998). Such sets give appealing results in communication games (Hurkens, 1996; Blume, 1998) and network formation games (Galeotti, Goyal, and Kamphorst, 2006). For closure properties under generalizations of the best-reply correspondence and implications for evolutionary dynamics, see Ritzberger and Weibull (1995).

A Cartesian product of pure-strategy sets is fixed under rational behavior (FURB) if each player's component not only contains, but is identical with the set of best replies to all probability distributions over the set. Hence, FURB sets are the natural set-valued generalization of strict Nash equilibria. Basu and Weibull (1991) — who refer to FURB sets as 'tight' CURB sets — show that minimal CURB sets and the product set of rationalizable strategies (Bernheim, 1984; Pearce, 1984) are important special cases of FURB sets.

Allowing for set-valued solution concepts such as CURB sets and its variants is a way to avoid the 'epistemic criticism' of the Nash equilibrium concept. It is by now well-known that Nash equilibrium is not implied by players' knowledge or beliefs about the game and each others' rationality; it requires additional stringent assumptions about the consistency of players' conjectures about each other's actions, assumptions that seem hard to justify; see Bernheim (1984), Pearce (1984), Aumann and Brandenburger (1995). In particular, it requires (a) that a player with multiple best replies is conjectured, by all other players, to pick a particular best reply, for no better reason than to induce the others to be willing to play their parts of the equilibrium, and (b) that players believe that others never err: they play best replies with probability one. By contrast, closedness under rational behavior requires neither (a) nor (b). The purpose of this study is to define epistemic robustness in these two respects, and to establish a precise formal link with CURB sets. More exactly, the two kinds of epistemic robustness are:

> **Robustness to alternative best replies:** Player i may hold *any* belief over opponent profiles where each opponent j chooses some best reply to a belief for player j that player i deems possible.
>
> **Robustness to non-best replies:** Player i may assign a small positive probability to opponent profiles where not each opponent j chooses a best reply given some belief for player j that player i deems possible.

Links with CURB sets are established in Propositions 1 to 3. Roughly speaking, robustness to alternative best replies allows player i to have arbitrary beliefs in which all other players best-reply to whatever i deems possible, rather than pinpointing specific best replies as in epistemic conditions for Nash equilibria (Aumann and Brandenburger, 1995). Robustness to non-best replies allows player i to have beliefs that assign positive, but small probability to 'irrational' behavior of the opponents. The reason why the CURB property implies the latter type of robustness is that if a player i is absolutely sure that the others use strategies in a certain CURB set, then — by definition — each of his pure strategies outside the CURB set is strictly worse than some pure strategy inside it. In finite games, by continuity of expected payoffs with respect to beliefs, this remains true if i is *sufficiently* sure that his fellow players will use strategies in the CURB set, i.e., if his belief assigns a sufficiently large probability to this event (Ritzberger and Weibull, 1995).

In order to illustrate this line of reasoning, consider first the two-player game

$$\begin{array}{ccc} & l & c \\ u & 3,1 & 1,2 \\ m & 0,3 & 2,1 \end{array} \tag{1}$$

In its unique Nash equilibrium, player 1 uses her first pure strategy with probability 2/3 and player 2 uses his first pure strategy with probability 1/4. However, even if player 1, say, would expect player 2 to play his equilibrium strategy, $(1/4, 3/4)$, player 1 would be indifferent between her two pure strategies. Hence, any pure or mixed strategy would be optimal for her, under the equilibrium expectation about player 2. For all other beliefs about her opponent's behavior, only one of her pure strategies would be optimal, and likewise for player 2. The unique CURB set in this game is the full set $S = S_1 \times S_2$ of pure-strategy profiles.

Adding a third pure strategy to each player in this example, we obtain the two-player game

$$\begin{array}{cccc} & l & c & r \\ u & 3,1 & 1,2 & 0,0 \\ m & 0,3 & 2,1 & 0,0 \\ d & 5,0 & 0,0 & 6,4 \end{array} \tag{2}$$

Clearly the strategy profile $x^* = (x_1^*, x_2^*) = \left(\left(\frac{2}{3}, \frac{1}{3}, 0\right), \left(\frac{1}{4}, \frac{3}{4}, 0\right)\right)$ is a Nash equilibrium (indeed a perfect and proper equilibrium). However, if player 2 expects 1 to play x_1^*, then 2 is indifferent between his pure strategies l and c, and if 1 assigns equal probability to these two pure strategies of player 2, then 1 will play the unique best reply d, a pure strategy outside the support of the equilibrium. Moreover, if player 2 expects 1 to reason this way, then 2 will play r. By contrast, the pure-strategy profile (d, r) is a strict equilibrium. In this equilibrium, no player has any alternative best reply and each equilibrium strategy remains optimal also under some uncertainty as to the other player's action. In this game, all pure strategies are rationalizable, $S = S_1 \times S_2$ is a FURB set, and the game's unique minimal CURB set and unique minimal FURB set is $T = \{d\} \times \{r\}$. Unlike the support $\{u, m\} \times \{l, c\}$ of the Nash equilibrium x^***,** the set T is robust to all alternative best-replies and to a 'small dose' of non-best replies.

Given such robustness to alternative best replies, it is natural to follow, for instance, Asheim (2006) and Brandenburger, Friedenberg, and Keisler (2008), and model players as having beliefs about the opponents without assuming that they choose specific subsets of their best reply sets. Letting each player be characterized by his or her type, defined by a probability distribution over profiles of opponent strategy-type pairs, allows this. In particular, a player's type does not specify his or her choice as in Aumann and Brandenburger (1995).

Our results can be heuristically described as follows. Proposition 1 establishes that any CURB set can be characterized by a set of choice profiles associated with a Cartesian product Y of type sets that allow for *any* mutual belief (robustness to alternative best replies) in which it is *sufficiently likely* (robustness to non-best replies) that the others have types from Y and behave rationally. While CURB sets allow for other beliefs as well, Proposition 2 shows that FURB sets have the same epistemic robustness property, but, in addition, 'irrational' beliefs are absent. A

FURB set is characterized by a subset of types for each player, a subset that is *identical with* the set of types for which the player p-believes, for all p sufficiently close to 1, that each opponent is rational and holds a belief determined by a type in her type subset. The result reported in Proposition 3 is different in nature. It establishes that minimal CURB sets provide lower bounds, in terms of strategy subsets, for what can be epistemically robust: no proper subset of a minimal CURB set is epistemically robust. More precisely, Proposition 3 establishes that such a Cartesian product of type sets, one for each player, violates robustness to alternative best replies if the Cartesian product of the associated union of choice sets that the individual type sets give rise to — all rational choices under the corresponding beliefs — does not coincide with the smallest CURB set that includes it.

Our epistemic approach follows, e.g., Asheim (2006) and Brandenburger, Friedenberg, and Keisler (2008) by not letting player types determine strategy choices. Moreover, we consider complete epistemic models. In these respects, our modeling differs from that of Aumann and Brandenburger's (1995) characterization of Nash equilibrium. In its motivation in terms of epistemic robustness of solution concepts and in its use of p-belief, the present approach is related to Tercieux's (2006) analysis. His epistemic approach, however, is completely different from ours. Starting from a two-player game, he introduces a Bayesian game where payoff functions are perturbations of the original ones and he investigates which equilibria are robust to this kind of perturbation. By studying the robustness of non-equilibrium concepts in terms of mutual belief, our analysis is related to Zambrano (2008). The latter, however, restricts attention to rationalizability and probability-1 beliefs. His main result follows from our Proposition 2. Also Hu (2007) restricts attention to rationalizability, but allows for p-beliefs, where $p < 1$. In his games, the compact strategy sets are permitted to be infinite. By contrast, our analysis is restricted to finite games, but under the weaker condition of mutual, rather than Hu's common, p-belief of opponent rationality and of opponents' types belonging to given type sets.

The remainder of the paper is organized as follows. Section 2 contains the game theoretic and epistemic definitions used. Section 3 gives the characterizations of variants of CURB sets. Proofs of the propositions are provided in the appendix.

2 The model

2.1 Game theoretic definitions

Consider a finite normal-form game $\langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$, where $N = \{1,\ldots,n\}$ is the non-empty and finite set of players. Each player $i \in N$ has a non-empty, finite set of pure strategies S_i and a payoff function $u_i : S \to \mathbb{R}$ defined on the set $S := S_1 \times \cdots \times S_n$ of pure-strategy profiles. For any player i, let $S_{-i} := \times_{j\neq i} S_j$. It is over this set of *other* players' pure-strategy combinations that player i will form his or her probabilistic beliefs. These beliefs may, but need not be product measures over the other player's pure-strategy sets. We extend the domain of the payoff functions to probability distributions over pure strategies as usual.

For later convenience, we here introduce some notation. For an arbitrary Polish (separable and completely metrizable) space F, let $\mathcal{M}(F)$ denote the set of Borel probability measures on F, endowed with the topology of weak convergence. For each player $i \in N$, pure strategy $s_i \in S_i$, and probabilistic belief $\sigma_{-i} \in \mathcal{M}(S_{-i})$, write

$$u_i(s_i, \sigma_{-i}) := \sum_{s_{-i}\in S_{-i}} \sigma_{-i}(s_{-i}) u_i(s_i, s_{-i}).$$

Define i's *best-reply correspondence* $\beta_i : \mathcal{M}(S_{-i}) \to 2^{S_i}$ as follows: For all $\sigma_{-i} \in \mathcal{M}(S_{-i})$,

$$\beta_i(\sigma_{-i}) := \{s_i \in S_i \mid u_i(s_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i})\ \forall s'_i \in S_i\}.$$

Let $\mathcal{S} := \{X \in 2^S \mid \varnothing \neq X = X_1 \times \cdots \times X_n\}$ denote the collection of non-empty Cartesian products of subsets of the players' strategy sets. For $X \in \mathcal{S}$ we abuse notation slightly by writing, for each $i \in N$, $\beta_i(\mathcal{M}(X_{-i}))$ as $\beta_i(X_{-i})$. Let $\beta(X) := \beta_1(X_{-1}) \times \cdots \times \beta_n(X_{-n})$. Each constituent set $\beta_i(X_{-i}) \subseteq S_i$ in this Cartesian product is the set of best replies of player i to all probabilistic beliefs over the others' strategy choices $X_{-i} \subseteq S_{-i}$.

Following Basu and Weibull (1991), a set $X \in \mathcal{S}$ is:

closed under rational behavior (CURB) if $\beta(X) \subseteq X$;

fixed under rational behavior (FURB) if $\beta(X) = X$;

minimal CURB (MC) if it is CURB and does not properly contain another one: $\beta(X) \subseteq X$ and there is no $X' \in \mathcal{S}$ with $X' \subset X$ and $\beta(X') \subseteq X'$.

Basu and Weibull (1991) call a FURB set a 'tight' CURB set. The reversed inclusion, $X \subseteq \beta(X)$, is sometimes referred to as the 'best response property' (Pearce, 1984,

p. 1033). It is shown in Basu and Weibull (1991, Prop. 1 and 2) that an MC set exists, that all MC sets are FURB, and that the product set of rationalizable strategies is the game's largest FURB set. While Basu and Weibull (1991) require that players believe that others' strategy choices are statistically independent, $\sigma_{-i} \in \times_{j \neq i} \mathcal{M}(S_j)$, we here allow players to believe that others' strategy choices are correlated, $\sigma_{-i} \in \mathcal{M}(S_{-i})$.[1] Thus, in games with more than two players, the present definition of CURB is somewhat more demanding than that in Basu and Weibull (1991), in the sense that we require closedness under a wider space of beliefs. Hence, the present definition may, in some games with more than two players, lead to larger MC sets.[2]

2.2 Epistemic definitions

For each $i \in N$, denote by T_i player i's non-empty Polish space of types. The *state space* is defined by

$$\Omega := S \times T \,,$$

where $T := T_1 \times \cdots \times T_n$. For each player $i \in N$, write $\Omega_i := S_i \times T_i$ and $\Omega_{-i} := S_{-i} \times T_{-i}$, where $T_{-i} := \times_{j \neq i} T_j$. To each type $t_i \in T_i$ of every player i, there corresponds a Borel probability measure $\mu_i(t_i) \in \mathcal{M}(\Omega_{-i})$ over Ω_{-i}. For each player i, we thus have the player's pure-strategy set S_i, type space T_i and a mapping $\mu_i : T_i \to \mathcal{M}(\Omega_{-i})$ that to each of i's types t_i assigns a probabilistic belief, $\mu_i(t_i)$, over the others' strategy choices and types. The structure $(S_1, \ldots, S_n, T_1, \ldots, T_n, \mu_1, \ldots, \mu_n)$ is called an *S-based (interactive) probability structure*. Assume that for each $i \in N$:

- μ_i is onto: all Borel probability measures on Ω_{-i} are represented in T_i. A probability structure with this property is called *complete*.
- T_i is compact.
- μ_i is continuous.

An adaptation of the proof of Brandenburger, Friedenberg, and Keisler (2008, Proposition 7.2) establishes the existence of such a complete probability structure.[3]

[1] Our results carry over — with minor modifications in the proofs — to the case of independent strategies.

[2] We also note that a pure strategy is a best reply to some belief $\sigma_{-i} \in \mathcal{M}(S_{-i})$ if and only if it is not strictly dominated (by any pure or mixed strategy). This follows from Lemma 3 in Pearce (1984), which, in turn, is closely related to Lemma 3.2.1 in van Damme (1983).

[3] The exact result we use is Proposition 6.1 in an earlier working paper version (Brandenburger, Friedenberg, and Keisler, 2004). The existence of a complete probability structure can also be

For each $i \in N$, denote by $\mathbf{s}_i(\omega)$ and $\mathbf{t}_i(\omega)$ i's strategy and type in state $\omega \in \Omega$. In other words, $\mathbf{s}_i : \Omega \to S_i$ is the projection of the state space to i's strategy set, assigning to each state $\omega \in \Omega$ the strategy $s_i = \mathbf{s}_i(\omega)$ that i uses in that state. Likewise, $\mathbf{t}_i : \Omega \to T_i$ is the projection of the state space to i's type space. For each player $i \in N$ and positive probability $p \in (0, 1]$, the *p-belief operator* B_i^p maps each event (Borel-measurable subset of the state space) $E \subseteq \Omega$ to the set of states where player i's type attaches at least probability p to E:

$$B_i^p(E) := \{\omega \in \Omega \mid \mu_i(\mathbf{t}_i(\omega))(E^{\omega_i}) \geq p\},$$

where $E^{\omega_i} := \{\omega_{-i} \in \Omega_{-i} \mid (\omega_i, \omega_{-i}) \in E\}$. This is the same belief operator as in Hu (2007). One may interpret $B_i^p(E)$ as the event 'player i believes E with probability at least p'. For all $p \in (0, 1]$, B_i^p satisfies $B_i^p(\varnothing) = \varnothing$, $B_i^p(\Omega) = \Omega$, $B_i^p(E') \subseteq B_i^p(E'')$ if $E' \subseteq E''$ (monotonicity), and $B_i^p(E) = E$ if $E = \operatorname{proj}_{\Omega_i} E \times \Omega_{-i}$. The last property means that each player i always p-believes his own strategy-type pair, for any positive probability p. Since also $B_i^p(E) = \operatorname{proj}_{\Omega_i} B_i^p(E) \times \Omega_{-i}$ for all events $E \subseteq \Omega$, each operator B_i^p satisfies both positive ($B_i^p(E) \subseteq B_i^p(B_i^p(E))$) and negative introspection ($\neg B_i^p(E) \subseteq B_i^p(\neg B_i^p(E))$). For all $p \in (0, 1]$, B_i^p violates the truth axiom, meaning that the requirement that $B_i^p(E) \subseteq E$ need not be satisfied for all $E \subseteq \Omega$. In the special case $p = 1$, we have $B_i^p(E') \cap B_i^p(E'') \subseteq B_i^p(E' \cap E'')$ for all E', $E'' \subseteq \Omega$.

Define i's *choice correspondence* $C_i : T_i \to 2^{S_i}$ as follows: For each of i's types $t_i \in T_i$,

$$C_i(t_i) := \beta_i(\operatorname{marg}_{S_{-i}} \mu_i(t_i))$$

consists of i's best replies when player i is of type t_i. Let $\mathcal{T}$ denote the collection of non-empty Cartesian products of subsets of the players' type spaces:

$$\mathcal{T} := \{Y \in 2^T \mid \varnothing \neq Y = Y_1 \times \cdots \times Y_n\}.$$

For any such set $Y \in \mathcal{T}$ and player $i \in N$, write $C_i(Y_i) := \bigcup_{t_i \in Y_i} C_i(t_i)$ and $C(Y) := C_1(Y_1) \times \cdots \times C_n(Y_n)$. In other words, these are the choices and choice profiles associated with Y. If $Y \in \mathcal{T}$ and $i \in N$, write

$$[Y_i] := \{\omega \in \Omega \mid \mathbf{t}_i(\omega) \in Y_i\}.$$

This is the event that player i is of a type in the subset Y_i. Likewise, write $[Y] := \bigcap_{i \in N} [Y_i]$ for the event that the type profile is in Y. Finally, for each player $i \in N$,

established by constructing a universal state space (cf. Mertens and Zamir, 1985).

write $[R_i]$ for the event that player i uses a best reply:

$$[R_i] := \{\omega \in \Omega \mid \mathbf{s}_i(\omega) \in C_i(\mathbf{t}_i(\omega))\}.$$

One may interpret the event $[R_i]$ as 'i is rational'.

3 Epistemic robustness

This section contains epistemic characterizations of CURB and FURB sets. Proposition 1 below stresses the robustness both to alternative best replies and to non-best replies of CURB sets. A set of choice profiles associated with a Cartesian product Y of type sets with the robustness properties that it allows for *any* mutual belief according to which it is *sufficiently likely* that the others have types from Y and behave rationally is a CURB set. Conversely, any CURB set includes a set of choice profiles associated with a Cartesian product Y of type sets with these robustness properties.

Denote, for each $i \in N$ and $X_i \subseteq S_i$ the pre-image (upper inverse) of X_i under player i's best response correspondence by

$$\beta_i^{-1}(X_i) := \{\sigma_{-i} \in \mathcal{M}(S_{-i}) \mid \beta_i(\sigma_{-i}) \subseteq X_i\}.$$

Proposition 1 *Let $X \in \mathcal{S}$.*

(a) If there exist $\underline{p} \in (0,1]$ and $Y \in \mathcal{T}$ such that $C(Y) = X$ and, for each $i \in N$ and each $p \in [\underline{p}, 1]$,

$$B_i^p \left(\bigcap\nolimits_{j \neq i} ([R_j] \cap [Y_j]) \right) \subseteq [Y_i], \tag{3}$$

then X is a CURB set.

(b) If $X \in \mathcal{S}$ is a CURB set, then there exist $\underline{p} \in (0,1)$ and $Y \in \mathcal{T}$ such that $C(Y) = \times_{i \in N} \beta_i(\beta_i^{-1}(X_i))$ and, for each $i \in N$ and each $p \in [\underline{p}, 1]$, inclusion (3) holds.

We note that (a) applies to $\underline{p} = 1$, in which case the hypothesis is simply that $Y \in \mathcal{T}$ is such that $C(Y) = X$ and (3) holds for $p = 1$. In the appendix we also prove the claim that if $p \in (0,1]$ and $Y \in T$ are such that $C(Y) = X$ and (3) holds for all $i \in N$, then X is a p-best response set in the sense of Tercieux (2006).

The following result shows that also FURB sets can be characterized by these robustness properties.

Proposition 2 *$X \in \mathcal{S}$ is a* FURB *set if and only if there exists a $\underline{p} \in (0,1)$ such that for each $p \in [\underline{p}, 1]$ there exists a $Y^p \in \mathcal{T}$ with $C(Y^p) = X$ and, for each $i \in N$,*

$$B_i^p \left(\bigcap\nolimits_{j \neq i} \big([R_j] \cap [Y_j^p] \big) \right) = [Y_i^p] \,. \tag{4}$$

Observe that the set Y^p above is chosen in such a way that also higher order beliefs conform with the players choosing in X. As an important corollary, Proposition 2 characterizes the set of rationalizable strategy profiles (Bernheim, 1984; Pearce, 1984), since this is the game's largest FURB set (Basu and Weibull, 1991), without involving any explicit assumption of common belief of rationality; only mutual p-belief of rationality and type sets are assumed. Thus, Proposition 2 generalizes the main result of Zambrano (2008) to p-belief for p sufficiently close to 1. Proposition 2 also applies to MC sets, as these sets are FURB.

To illustrate our final result, consider the Nash equilibrium x^* in game (2) in the introduction. This equilibrium corresponds to a type profile (t_1, t_2) where t_1 assigns probability 1/4 to (l, t_2) and probability 3/4 to (c, t_2), and where t_2 assigns probability 2/3 to (u, t_1) and probability 1/3 to (m, t_1). We have that $C(\{t_1, t_2\}) = \{u, m\} \times \{l, c\}$, while S is the smallest CURB set that includes $C(\{t_1, t_2\})$. The following result shows that $C(\{t_1, t_2\})$ is not epistemically robust since it does not coincide with the smallest CURB set that includes it. By contrast, for the type profile (t_1', t_2') where t_1' assigns probability 1 to (r, t_2') and t_2' assigns probability 1 to (d, t_1') we have that $C(\{t_1', t_2'\}) = \{(d, r)\}$ coincides with the smallest CURB set that includes it. Thus, the strict equilibrium (d, r) to which (t_1', t_2') corresponds is epistemically robust.

Proposition 3 *Let $Y \in \mathcal{T}$ and let $X \in \mathcal{S}$ be the smallest* CURB *set satisfying $C(Y) \subseteq X$. Then $X = C\left(\bigcup_{k \in \mathbb{N}} Y(k) \right)$, where $Y(0) := Y$, and for each $k \in \mathbb{N}$ and $i \in N$,*

$$[Y_i(k)] := [Y_i(k-1)] \cup B_i^1 \left(\bigcap\nolimits_{j \neq i} \big([R_j] \cap [Y_j(k-1)] \big) \right) . \tag{5}$$

Proposition 3 presumes that for each set $X \in \mathcal{S}$, there is a unique smallest CURB set $X' \in S$ with $X \subseteq X'$ (that is, X' is a subset of all other CURB sets X'', if any, with $X \subseteq X''$). This presumption is met in all finite games, since the collection of CURB sets containing a given set $X \in \mathcal{S}$ is non-empty and finite, and the intersection of two CURB sets containing X is again a CURB set containing X.

Proposition 3 checks robustness to alternative best replies by including all beliefs over the opponents' best replies, and any beliefs over opponents' types that has such beliefs over their opponents, and so on. We may, for example, start with any Nash equilibrium and assume that, at some type profile $t \in T$, there is common 1-belief of the event that all players believe that all the others play according to this equilibrium. However, these equilibrium beliefs are not robust, unless the equilibrium is strict. Otherwise, if all beliefs over the opponents' best replies are included, and any beliefs over opponents' types that has such beliefs over their opponents are included, and so on, then the resulting Cartesian product of type sets correspond to the smallest CURB set that contains the Nash equilibrium that was our point of departure.

Appendix

Proof of Proposition 1. *Part (a).* By assumption, there is a $Y \in \mathcal{T}$ with $C(Y) = X$ such that for each $i \in N$, $B_i^1\left(\bigcap_{j\neq i}([R_j] \cap [Y_j])\right) \subseteq [Y_i]$.

Fix $i \in N$, and consider any $\sigma_{-i} \in \mathcal{M}(X_{-i})$. Since $C(Y) = X$, it follows that, for each $s_{-i} \in S_{-i}$ with $\sigma_{-i}(s_{-i}) > 0$, there exists $t_{-i} \in Y_{-i}$ such that, for all $j \neq i$, $s_j \in C_j(t_j)$. Hence, since the probability structure is complete, there exists a

$$\omega \in B_i^1\left(\bigcap\nolimits_{j\neq i}([R_j] \cap [Y_j])\right) \subseteq [Y_i]$$

with $\mathrm{marg}_{S_{-i}}\, \mu_i(\mathbf{t}_i(\omega)) = \sigma_{-i}$. So

$$\beta_i(X_{-i}) := \beta_i(\mathcal{M}(X_{-i})) \subseteq \bigcup\nolimits_{t_i \in Y_i} \beta_i(\mathrm{marg}_{S_{-i}}\, \mu_i(t_i)) := C_i(Y_i) = X_i\,.$$

Since this holds for all $i \in N$, X is a CURB set.

Part (b). Assume that $X \in \mathcal{S}$ is a CURB set, i.e., X satisfies $\beta(X) \subseteq X$. Define $Y \in \mathcal{T}$ by taking, for each $i \in N$, $Y_i := \{t_i \in T_i \mid C_i(t_i) \subseteq X_i\}$. Since the probability structure is complete, it follows that $C_i(Y_i) = \beta_i(\beta_i^{-1}(X_i))$. For notational convenience, write $X_i' = \beta_i(\beta_i^{-1}(X_i))$ and $X' = \times_{i\in N} X_i'$. Since the game is finite, there is, for each player $i \in N$, a $\underline{p}_i \in (0, 1)$ such that $\beta_i(\sigma_{-i}) \subseteq \beta_i(X_{-i}')$ for all $\sigma_{-i} \in \mathcal{M}(S_{-i})$ with $\sigma_{-i}(X_{-i}') \geq \underline{p}_i$. Let $\underline{p} = \max\{\underline{p}_1, \ldots, \underline{p}_n\}$.

We first show that $\beta(X') \subseteq X'$. By definition, $X' \subseteq X$, so for each $i \in N$: $\mathcal{M}(X_{-i}') \subseteq \mathcal{M}(X_{-i})$. Moreover, as $\beta(X) \subseteq X$ and, for each $i \in N$, $\beta_i(X_i) := \beta_i(\mathcal{M}(X_{-i}))$, it follows that $\mathcal{M}(X_{-i}) \subseteq \beta_i^{-1}(X_i)$. Hence, for each $i \in N$,

$$\beta_i(X_i') := \beta_i(\mathcal{M}(X_{-i}')) \subseteq \beta_i(\mathcal{M}(X_{-i})) \subseteq \beta_i(\beta_i^{-1}(X_i)) = X_i'\,.$$

For all $p \in [\underline{p}, 1]$ and $i \in N$, we have that

$$\begin{aligned}
& B_i^p\Big(\bigcap\nolimits_{j\neq i}([R_j] \cap [Y_j])\Big) \\
= \;& B_i^p\Big(\bigcap\nolimits_{j\neq i}\{\omega \in \Omega \mid \mathbf{s}_j(\omega) \in C_j(\mathbf{t}_j(\omega)) \subseteq X'_j\}\Big) \\
\subseteq \;& \big\{\omega \in \Omega \mid \mu_i(\mathbf{t}_i(\omega))\{\omega_{-i} \in \Omega_{-i} \mid \text{for all } j \neq i,\ \mathbf{s}_j(\omega) \in X'_j\} \geq p\big\} \\
\subseteq \;& \{\omega \in \Omega \mid \operatorname{marg}_{S_{-i}} \mu_i(\mathbf{t}_i(\omega))(X'_{-i}) \geq p\} \\
\subseteq \;& \{\omega \in \Omega \mid C_i(\mathbf{t}_i(\omega)) \subseteq \beta_i(X'_{-i})\} \\
\subseteq \;& \{\omega \in \Omega \mid C_i(\mathbf{t}_i(\omega)) \subseteq X'_{-i}\} \;=\; [Y_i],
\end{aligned}$$

using $\beta(X') \subseteq X'$. ■

For $X \in \mathcal{S}$ and $p \in (0, 1]$, write, for each $i \in N$,

$$\begin{aligned}
\beta_i^p(X_{-i}) := \{s_i \in S_i \mid \exists \sigma_{-i} \in \mathcal{M}(S_{-i}) \text{ with } \sigma_{-i}(X_{-i}) \geq p \\
\text{such that } u_i(s_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i})\ \forall s'_i \in S_i\}\,.
\end{aligned}$$

Let $\beta^p(X) := \beta_1^p(X_{-1}) \times \cdots \times \beta_n^p(X_{-n})$. Following Tercieux (2006), a set $X \in \mathcal{S}$ is a *p-best response set* if $\beta^p(X) \subseteq X$.

Claim: *Let $X \in \mathcal{S}$ and $p \in (0, 1]$. If $Y \in \mathcal{T}$ is such that $C(Y) = X$ and* (3) *holds for each $i \in N$, then X is a p-best response set.*

Proof. By assumption, there is a $Y \in \mathcal{T}$ with $C(Y) = X$ such that for each $i \in N$, $B_i^p\left(\bigcap_{j\neq i}([R_j] \cap [Y_j])\right) \subseteq [Y_i]$.

Fix $i \in N$ and consider any $\sigma_{-i} \in \mathcal{M}(S_{-i})$ with $\sigma_{-i}(X_{-i}) \geq p$. Since $C(Y) = X$, it follows that, for each $s_{-i} \in X_{-i}$, there exists $t_{-i} \in Y_{-i}$ such that $s_j \in C_j(t_j)$ for all $j \neq i$. Hence, since the probability structure is complete, there exists a

$$\omega \in B_i^p\left(\bigcap\nolimits_{j\neq i}([R_j] \cap [Y_j])\right) \subseteq [Y_i]$$

with $\operatorname{marg}_{S_{-i}} \mu_i(\mathbf{t}_i(\omega)) = \sigma_{-i}$. So, by definition of $\beta_i^p(X_{-i})$:

$$\beta_i^p(X_{-i}) \subseteq \bigcup\nolimits_{t_i \in Y_i} \beta_i(\operatorname{marg}_{S_{-i}} \mu_i(t_i)) := C_i(Y_i) = X_i\,.$$

Since this holds for all $i \in N$, X is a p-best response set. ■

Proof of Proposition 2. *(If)* By assumption, there is a $Y \in \mathcal{T}$ with $C(Y) = X$ such that for all $i \in N$, $B_i^1\left(\bigcap_{j\neq i}([R_j] \cap [Y_j])\right) = [Y_i]$.

Fix $i \in N$. Since $C(Y) = X$, and the probability structure is complete, there exists a

$$\omega \in B_i^1\left(\bigcap_{j\neq i}\big([R_j]\cap[Y_j]\big)\right) = [Y_i]$$

with $\operatorname{marg}_{S_{-i}} \mu_i(\mathbf{t}_i(\omega)) = \sigma_{-i}$ if and only if $\sigma_{-i} \in \mathcal{M}(X_{-i})$. So

$$\beta_i(X_{-i}) := \beta_i(\mathcal{M}(X_{-i})) = \bigcup_{t_i \in Y_i} \beta_i(\operatorname{marg}_{S_{-i}} \mu_i(t_i)) := C_i(Y_i) = X_i\,.$$

Since this holds for all $i \in N$, X is a FURB set.

(Only if) Assume that $X \in \mathcal{S}$ satisfies $X = \beta(X)$. Since the game is finite, there exists, for each player $i \in N$, a $\underline{p}_i \in (0,1)$ such that $\beta_i(\sigma_{-i}) \subseteq \beta_i(X_{-i})$ if $\sigma_{-i}(X_{-i}) \geq \underline{p}_i$. Let $\underline{p} = \max\{\underline{p}_1, \ldots, \underline{p}_n\}$.

For each $p \in [\underline{p}, 1]$, construct the sequence of Cartesian products of type subsets $\langle Y^p(k)\rangle_k$ as follows: For each $i \in N$, let $Y_i^p(0) = \{t_i \in T_i \mid C_i(t_i) \subseteq X_i\}$. The correspondence $C_i : T_i \rightrightarrows S_i$ is upper hemi-continuous. Thus $Y_i^p(0) \subseteq T_i$ is closed, and, since T_i is compact, so is $Y_i^p(0)$. There exists a closed set $Y_i^p(1) \subseteq T_i$ such that

$$[Y_i^p(1)] = B_i^p\left(\bigcap_{j\neq i}\big([R_j]\cap[Y_j^p(0)]\big)\right)\,.$$

It follows that $Y_i^p(1) \subseteq Y_i^p(0)$. Since $Y_i^p(0)$ is compact, so is $Y_i^p(1)$. By induction,

$$[Y_i^p(k)] = B_i^p\left(\bigcap_{j\neq i}\big([R_j]\cap[Y_j^p(k-1)]\big)\right)\,. \tag{6}$$

defines, for each player i, a decreasing chain $\langle Y_i^p(k)\rangle_k$ of compact and non-empty subsets: $Y_i^p(k+1) \subseteq Y_i^p(k)$ for all k. By the finite-intersection property, $Y_i^p := \bigcap_{k\in\mathbb{N}} Y_i^p(k)$ is a non-empty and compact subset of T_i. For each k, let $Y^p(k) = \times_{i\in N} Y_i^p(k)$ and let $Y^p := \bigcap_{k\in\mathbb{N}} Y^p(k)$. Again, these are non-empty and compact sets.

Next, $C(Y^p(0)) = \beta(X)$, since the probability structure is complete. Since X is FURB, we thus have $C(Y^p(0)) = X$. For each $i \in N$,

$$[Y_i^p(1)] \subseteq \{\omega \in \Omega \mid \operatorname{marg}_{S_{-i}} \mu_i(\mathbf{t}_i(\omega))(X_{-i}) \geq p\}\,,$$

implying that $C_i(Y_i^p(1)) \subseteq \beta_i(X_{-i}) = X_{-i}$ by the construction of $\underline{p}$. Moreover, since the probability structure is complete, for each $i \in N$ and $\sigma_{-i} \in \mathcal{M}(X_{-i})$, there exists $\omega \in [Y_i^p(1)] = B_i^p\big(\bigcap_{j\neq i}([R_j]\cap[Y_j^p(0)])\big)$ with $\operatorname{marg}_{S_{-i}}\mu_i(\mathbf{t}_i(\omega)) = \sigma_{-i}$, implying that $C_i(Y_i^p(1)) \supseteq \beta_i(X_{-i}) = X_{-i}$. Hence, $C_i(Y_i^p(1)) = \beta_i(X_{-i}) = X_i$. By induction, it holds for all $k \in N$ that $C(Y^p(k)) = \beta(X) = X$. Since $\langle Y_i^p(k)\rangle_k$ is a decreasing chain, we also have that $C(Y^p) \subseteq X$. The converse inclusion follows by upper hemi-continuity of the correspondence C. To see this, suppose that $x^o \in X$ but $x^o \notin C(Y^p)$. Since $x^o \in X$, $x^o \in C(Y^p(k))$ for all k. By the Axiom of Choice: for each k there exists a $y_k \in Y^p(k)$ such that $(y_k, x^o) \in graph\,(C)$. By the Bolzano-Weierstrass Theorem, we can extract a convergent subsequence for which $y_k \to y^o$, where $y^o \in Y^p$, since Y^p is closed. Moreover, since the correspondence C is closed-valued and u.h.c., with S compact (it is in fact finite), $graph\,(C) \subseteq T \times S$ is closed, and thus $(y^o, x^o) \in graph\,(C)$, contradicting the hypothesis that $x^o \notin C(Y^p)$. This establishes the claim that $C(Y^p) \subseteq X$.

It remains to prove that equation (4) holds for each $i \in N$. Fix $i \in N$, and let

$$E_k = \bigcap\nolimits_{j \neq i} \big([R_j] \cap [Y_j^p(k)]\big) \qquad \text{and} \qquad E = \bigcap\nolimits_{j \neq i} \big([R_j] \cap [Y_j^p]\big)\,.$$

Since, for each $j \in N$, $\langle Y_j^p(k) \rangle_k$ is a decreasing chain with limit Y_j^p, it follows that $\langle E_k \rangle_k$ is a decreasing chain with limit E.

To show $B_i^p(E) \subseteq [Y_i^p]$, note that by (6) and monotonicity of B_i^p, we have, for each $k \in \mathbb{N}$, that

$$B_i^p(E) \subseteq B_i^p(E_{k-1}) = [Y_i^p(k)]\,.$$

As the inclusion holds for all $k \in \mathbb{N}$:

$$B_i^p(E) \subseteq \bigcap\nolimits_{k \in \mathbb{N}} [Y_i^p(k)] = [Y_i^p]\,.$$

To show $B_i^p(E) \supseteq [Y_i^p]$, assume that $\omega \in [Y_i^p]$.[4] This implies that $\omega \in [Y_i^p(k)]$ for all k, and, using (6): $\omega \in B_i^p(E_k)$ for all k. Since $E_k = \Omega_i \times \mathrm{proj}_{\Omega_{-i}} E_k$, we have that $E_k^{\omega_i} = \mathrm{proj}_{\Omega_{-i}} E_k$. It follows that

$$\mu_i(\mathbf{t}_i(\omega))(\mathrm{proj}_{\Omega_{-i}} E_k) \geq p \quad \text{for all } k\,.$$

Thus, since $\langle E_k \rangle_k$ is a decreasing chain with limit E,

$$\mu_i(\mathbf{t}_i(\omega))(\mathrm{proj}_{\Omega_{-i}} E) \geq p\,.$$

Since $E = \Omega_i \times \mathrm{proj}_{\Omega_{-i}} E$, we have that $E^{\omega_i} = \mathrm{proj}_{\Omega_{-i}} E$. Hence, the inequality implies that $\omega \in B_i^p(E)$. ■

Proof of Proposition 3. Assume that $X \in \mathcal{S}$ is the minimal CURB set containing $C(Y)$: (i) $C(Y) \subseteq X$ and $\beta(X) \subseteq X$ and (ii) there exists no $X' \in \mathcal{S}$ with

[4] We thank Itai Arieli for suggesting this proof of the reversed inclusion, shorter than our original proof. A proof of both inclusions can also be based on property (8) of Monderer and Samet (1989).

$C(Y) \subseteq X'$ and $\beta(X') \subseteq X' \subset X$. Consider the sequence of Cartesian products of type subsets $\langle Y(k)\rangle_k$ defined recursively in (5), for some $Y(0) \in \mathcal{T}$ satisfying $C(Y(0)) \subseteq X$.

We first show, by induction, that $C(Y(k)) \subseteq X$ for all $k \in \mathbb{N}$. By assumption, $Y(0) \in \mathcal{T}$ satisfies this condition. Assume that $C(Y(k-1)) \subseteq X$ for some $k \in \mathbb{N}$, and fix $i \in N$. Then, $\forall j \neq i$, $\beta_j(\text{marg}_{S_{-j}}\mu_j(\mathbf{t}_j(\omega))) \subseteq X_j$ if $\omega \in [Y_j(k-1)]$ and $\mathbf{s}_j(\omega) \in X_j$ if, in addition, $\omega \in [R_j]$. Hence, if $\omega \in B_i^1\big(\bigcap_{j\neq i}\big([R_j] \cap [Y_j(k-1)]\big)\big)$, then $\text{marg}_{S_{-i}}\mu_i(\mathbf{t}_i(\omega)) \in \mathcal{M}(X_{-i})$ and $C_i(\mathbf{t}_i(\omega)) \subseteq \beta_i(X_{-i}) \subseteq X_{-i}$. Since this holds for all $i \in N$, we have $C(Y(k)) \subseteq X$. This completes the induction.

Secondly, since the sequence $\langle Y(k)\rangle_k$ is non-decreasing and $C(\cdot)$ is monotonic w.r.t. set inclusion, and the game is finite, there exist a $k' \in \mathbb{N}$ and some $X' \subseteq X$ such that $C(Y(k)) = X'$ for all $k \geq k'$. Let $k > k'$ and consider any player $i \in N$. Since the probability structure is complete, there exists, for each $\sigma_{-i} \in \mathcal{M}(X'_{-i})$ a state $\omega \in [Y_i(k)]$ with $\text{marg}_{S_{-i}}\mu_i(\mathbf{t}_i(\omega)) = \sigma_{-i}$, implying that $\beta_i(X'_{-i}) \subseteq C_i(Y_i(k)) = X'_i$. Since this holds for all $i \in N$, $\beta(X') \subseteq X'$. Therefore, if $X' \subset X$ would hold, then this would contradict that there exists no $X' \in \mathcal{S}$ with $C(Y) \subseteq X'$ such that $\beta(X') \subseteq X' \subset X$. Hence, $X = C\left(\bigcup_{k\in\mathbb{N}} Y(k)\right)$. ■

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