Folk Theorems for Present-Biased Players

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Abstract. The folk theorems for infinitely repeated games with discounting presume that the discount rate between two successive periods is constant. Following the literature on quasi-exponential or hyperbolic discounting, I model the repeated interaction between two or more decision makers in a way that allows present-biased discounting where the discount factor between two successive periods increases with the waiting time until the periods are reached. I generalize Fudenberg and Maskin's (1986) and Abreu, Dutta and Smith's (1994) folk theorems for repeated games with discounting so that they apply when discounting is present-biased. Patience is then represented either by the discount factor between the next and the current period or, alternatively, by the sum of the discount factors for all future periods.

Keywords: folk theorem, present-biased, discounting, hyperbolic.

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1. Introduction

A phenomenon that has been observed in laboratory experiments is that people are less willing to postpone pleasure from today to tomorrow than from a day far into the future to the day after that (Eisenhauer and Ventura (2006), Loewenstein and Prelec (1992), Thaler (1981)). Such behavior is consistent with increasing patience where the discount factor between two successive periods increases with the waiting time until the periods are reached. Individuals who discount in this way are said to be "present-biased" since their aversion to a postponement is stronger if a reward is postponed from the present.

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Present-biasedness has been a known phenomenon for economists since Strotz (1956), Pollak (1968) and Phelps and Pollak (1968) initiated the study of how present-biasedness affects individuals', generations' or governments' consumption-savings problems. The discount factor that a present-biased decision maker uses between two adjacent periods depends on how far into the future the periods are, so in effect such a decision maker uses different preference orderings to make decisions at different points in time. In this literature, that was further developed by Peleg and Yaari (1973), Goldman (1980) and Laibson (1997, 1998), a decision maker is therefore represented by a sequence of multiple selves. The central issue is to which extent the current self is hurt by present-biasedness when there is no way to commit to a plan of actions. More recent studies by Krusell and Smith (2003) and Vieille and Weibull (2009) focus on the multiplicity of equilibria in the game between the multiple selves of a present-biased decision maker; and Dasgupta and Maskin (2005) and Wärneryd (2007) provide theoretical explanations of how present-biasedness could be the result of evolutionary forces.

While much is known about how present-biasedness affects the consumption-savings problem, comparatively little is known about what happens when two or more present-biased decision makers interact repeatedly with each other. To model the situation where present-biased decision makers play a given stage game repeatedly as a game, we have to consider each decision maker at each point in time as a distinct player. The result is a well defined game between infinitely many multiple selves, with one player to act in each player role in each period. The subgame perfect equilibria of this game are strategy profiles such that all decision makers act optimally after all histories, taking the actions of other decision makers and their own future actions as given. If we use subgame perfection as our formal rule to predict how the game will be played we thus make two implicit assumptions. First, that decision makers are sophisticated and recognize that they have different preferences at different points in time. Second, that decision makers do not have access to a commitment device and therefore cannot control their own future actions.

Once the model is in place it is possible to ask the same questions as in an exponentially discounted repeated game, and our focus will be on the relationship between patience and the set of equilibrium outcomes. Since Aumann and Shapley's (1976) and Rubinstein's (1979) folk theorems it is a familiar idea that repetition enables many outcomes if players are patient. We will examine to which extent this holds also when discounting is present-biased, with patience represented by the discount factor between the next and the current period, or, alternatively, by the sum of the discount factors for all future periods. The results of the analysis are generalizations of Fudenberg and Maskin's (1986) and Abreu, Dutta and Smith's (1994) folk theorems that apply to present-biased discount functions. Folk theorems for quasi-exponential or hyperbolic discounting follow as corollaries. Unlike in previous work with present-biased players discussed below, a folk theorem with mixed actions is established.

The repeated interaction of present-biased decision makers has been studied before by Streich and Levy (2007), Prokopovych (2005), and Chade, Prokopovych and Smith (2008). Streich and Levy provide a thorough discussion of the empirical relevance of present-biased discounting and proceed to analyze the repeated game in which the stage game is a prisoner's dilemma and discounting is quasi-exponential in the Laibson-Phelps-Pollak beta-delta form. Prokopovych and Chade et al. also assume that present-biasedness takes the form of quasi-exponential discounting and use recursive techniques to characterize equilibrium payoffs and explore the costs of present-biasedness. When discussing the relationship between patience and equilibrium payoffs, Chade et al. note that with quasi-exponential discounting Fudenberg and Maskin's (1986) pure-action folk theorem for repeated games without a public correlation device works as usual. This is precisely what Proposition 2 below implies when discounting is quasi-exponential. Prokopovych (2005) uses decomposability arguments to prove a more general pure-action folk theorem for quasi-exponential discounting. Prokopovych's folk theorem is not a special case of any folk theorem in the present paper since it shows that if discounting is quasi-exponential, then intertemporal averaging can be used to dispense with public signals.

Section 2 contains an informal presentation of the model and discusses what situations it is intended to capture. The model is formally described in section 3. Section 4 discusses two ways to represent patience. Folk theorems for present-biased players are developed in section 5 and 6. Related but still open problems are presented in section 7, and section 8 concludes. All proofs are given in the appendix.

2. Preliminaries

We consider a situation where n decision makers repeat a simultaneous-move stage game infinitely many times, and where the decision makers can observe each others' actions after each period. The n decision makers have the same discount function f.[1] Payoffs that are received t periods into the future are discounted by the factor $f(t)$, independently of which period the current period is. Time preferences are then such that when two alternative sequences of current and future payoffs are compared, the current date is irrelevant. The decision makers do not grow old and change the way they think about intertemporal trade-offs.

The only assumptions we will make about f are that each value $f(t)$ lies in the interval $[0,1]$ and that f is summable. One relevant class of such functions are those positive discount functions f for which the ratio $f(t+1)/f(t)$ is nondecreasing. Such

[1] Section 7.1 describes what changes and what does not change when there is one discount function for each decision maker. The conclusion will be that the results continue to hold but give a less complete description of what can happen in this case.

discount functions will be called *present-biased*. To use the ratio $f(t+1)/f(t)$ to define present-biasedness in this way accords with the discussion in the introduction because $f(t+1)/f(t)$ is the discount factor between the period that lies $t+1$ periods into the future and the period that lies t periods into the future. The ratio $f(t+1)/f(t)$ also plays a key role in Saez-Marti and Weibull's (2005) study of the relationship between discounting of instantaneous utilities and pure altruism toward future selves or future generations. They show that present-biased discounting of instantaneous utilities corresponds to pure altruism toward future generations.

The canonical example of a discount function is the exponential function $f(t) = \delta^t$ with $\delta \in (0, 1)$. This discount function is evidently such that $f(t+1)/f(t)$ is constant at δ for all t, so exponential discounting is the borderline case of present-biased discounting where the ratio $f(t+1)/f(t)$ is constant. Two other examples of present-biased discount functions are the quasi-exponential discount function $f(t) = \beta\delta^t$ with $\beta, \delta \in (0, 1)$ and the hyperbolic discount function $f(t) = (1 + \alpha t)^{-\gamma/\alpha}$ with $0 < \alpha < \gamma$. With quasi-exponential discounting the ratio $f(t+1)/f(t)$ jumps up once and thereafter remains constant, while the hyperbolic discount function is such that the ratio $f(t+1)/f(t)$ is strictly increasing.

When the ratio $f(t+1)/f(t)$ is strictly increasing, the time preferences of the decision makers are inconsistent in the sense that a plan of actions that is optimal as viewed from one period may be suboptimal when viewed from another period. We will assume that the decision makers lack the ability to commit to a plan of actions so that the period t action is controlled in period t only. Since standard game theory requires that each player has a unique preference relation over outcomes, the set of players in the game that models the repeated interaction will therefore be $\{1, \ldots, n\} \times \mathbb{N}$. Player (i, t) is the "current self" of decision maker i in period t. The repeated interaction is then modelled as a game with a countably infinite set of players where each player (i, t) acts only once, after observing the history of play leading up to period t.

We will analyze the set of subgame perfect equilibria in this game. Since each player (i, t) only controls one action, a strategy profile is a subgame perfect equilibrium if there are no profitable one-shot deviations. This is true also in normal δ-discounted repeated games with just a collection $\{1, \ldots, n\}$ of players because in such repeated games the one-shot deviation principle holds. Therefore the splitting of decision maker i into a sequence $((i, t))_{t=0}^{\infty}$ of i-players becomes irrelevant if we set $f(t) = \delta^t$. For this particular discount function a strategy profile is subgame perfect in the game with a sequence of i-players if and only if it is subgame perfect in the repeated game with just one player i. In this sense, the model presented below nests the standard model of an exponentially discounted repeated game with perfect monitoring.

3. The Model

3.1. Stage Game. There is a stage game $G = \langle N, A, u \rangle$, where $N = \{1, \ldots, n\}$ is the finite set of *players*, $A = \times_{i \in N} A_i$ is the set of *action profiles*, and $u : A \to \mathbb{R}^n$ is the *combined stage-game payoff function.* The set A is a compact subset of a Euclidean space and u is continuous. We denote by A_{-i} the set $\times_{j \neq i} A_j$. Given $a_{-i} \in A_{-i}$ and $a_i \in A_i$, we write (a_i, a_{-i}) for the action profile $(a_i)_{i \in N}$.

A *mixed action* for player i is a probability distribution over A_i. We will consider mixed actions only for the case when A is finite, in which case the set of player i's mixed actions is denoted ΔA_i and the sets ΔA and ΔA_{-i} are defined by $\Delta A = \times_{i \in N} \Delta A_i$ and $\Delta A_{-i} = \times_{j \neq i} \Delta A_j$. Abusing notation, we write $u_i(\alpha)$ for i's expected payoff under $\alpha \in \Delta A$.

The convex hull of $u(A)$ is denoted $\mathcal{F}$. The vectors v in $\mathcal{F}$ are the *feasible payoff vectors.* The *pure-action minmax payoff* for player i is denoted $\underline{v}_i^P$, and the *mixed-action minmax payoff* for player i is denoted $\underline{v}_i$:

$$\begin{aligned} \underline{v}_i^P &= \min_{a_{-i} \in A_{-i}} \max_{a_i \in A_i} u_i(a_i, a_{-i}) \\ \underline{v}_i &= \min_{\alpha_{-i} \in \Delta A_{-i}} \max_{a_i \in A_i} u_i(a_i, \alpha_{-i}) \end{aligned}$$

The sets $\mathcal{F}^P$ and $\mathcal{F}^*$ are defined by

$$\begin{aligned} \mathcal{F}^P &= \{v \in \mathcal{F} : v_i > \underline{v}_i^P \text{ for all } i \in N\}, \\ \mathcal{F}^* &= \{v \in \mathcal{F} : v_i > \underline{v}_i \text{ for all } i \in N\}. \end{aligned}$$

Thus $\mathcal{F}^P$ consists of all feasible payoff vectors that are strictly individually rational relative to the pure-action minmax payoffs, and $\mathcal{F}^*$ consists of all feasible payoff vectors that are strictly individually rational relative to the mixed-action minmax payoffs. The action profile $a \in A$ is *strictly individually rational* if $u_i(a) > \underline{v}_i^P$ for all $i \in N$.

3.2. Repeated Interaction. The stage game G is repeated infinitely many times. For each player role $i \in N$ of the stage game there is a sequence $((i,t))_{t=0}^{\infty}$ of $i-$players. Player (i,t) acts in player role i in period t, and only in period t. A strategy for player (i,t) is a description of how player (i,t) plans to act in every situation in which he can be called upon to act, and thus specifies one action for each history of play leading up to period t. That is, a strategy for player (i,t) is a function $\sigma_{it} : A^t \to A_i$, where A^t is the singleton set that contains the empty history if $t = 0$.

We include 0 in the set of natural numbers so that $\mathbb{N} = \{0, 1, \ldots\}$.

Definition 1. *A* ***discount function*** *is a function $f : \mathbb{N} \to [0,1]$ with $f(0) = 1$ such that $\sum_{t=1}^{\infty} f(t) < +\infty$. This function space is denoted $\mathcal{D}$. A positive discount function*

*f is **present-biased** if the ratio*

$$\frac{f(t+1)}{f(t)}$$

is nondecreasing in t for all $t \in \mathbb{N}$. The subset of $\mathcal{D}$ which consists of all present-biased discount functions is denoted $\mathcal{D}^*$.

The set A^∞ consists of all sequences $(a^t)_{t=0}^\infty$ with $a^t \in A$ for all $t \in \mathbb{N}$. Elements (a^t) of A^∞ will be referred to as *outcomes*. Given a function $f \in \mathcal{D}$, time preferences are defined as follows: An outcome $\boldsymbol{a} \in A^\infty$ gives player (i, τ) the payoff $U_{i\tau}(\boldsymbol{a})$, where the function $U_{i\tau} : A^\infty \to \mathbb{R}$ is defined by

$$U_{i\tau}(\boldsymbol{a}) = u_i(a^\tau) + \sum_{t=1}^{\infty} f(t) u_i(a^{\tau+t}).$$

Preferences are then such that player (i, τ) is altruistic towards future $i-$players, with $f(t)$ being the weight put on the generation that is t periods ahead. However, player (i, τ) does not care about the utility of future generations of $j-$players if $j \neq i$.[2]

The notation $\Gamma(G, f)$ will be used for the game where the set of players is $N \times \mathbb{N}$, player (i, τ)'s strategy set consists of all functions from A^τ to A_i, and player (i, τ)'s preferences over A^∞ are given by $U_{i\tau}$.

Remark 1. *If $f \in \mathcal{D}^*$, then f is strictly decreasing. To see this, suppose that f is a positive function with domain $\mathbb{N}$ such that the ratio $f(t+1)/f(t)$ is nondecreasing, but f is not strictly decreasing. Then there is some τ such that $f(\tau+1) \geq f(\tau)$, and furthermore $f(t+1) \geq f(t)$ for all $t > \tau$. Thus $f(t) \geq f(\tau) > 0$ for all $t > \tau$, which implies $\sum_{t=1}^\infty f(t) = +\infty$ and hence $f \notin \mathcal{D}^*$.*

3.3. Public Correlation Device. The notation $\Gamma_{PC}(G, f)$ will be used for the game that results when period t begins with a realization $\omega^t \in [0, 1]$ of a uniformly distributed public random variable that is observed by all period t players $((i, t))_{i \in N}$ before they choose their actions. In this game, a period t history is an element of $A^t \times [0, 1]^t$, and a strategy σ_{it} for player (i, t) specifies $\alpha_i \in \Delta A_i$ or $a_i \in A_i$ as a Borel function of the period t history and the period t realization of the public signal. If A is infinite, then σ_{it} is required to map into A_i.

A strategy profile σ induces a probability distribution over A for the realization of the period t action profile a^t. The payoff for player (i, τ) for a strategy profile σ is $\mathbb{E}_\sigma \sum_{t=0}^\infty f(t) u_i(a^{\tau+t})$, where $\mathbb{E}_\sigma$ denotes the expectation taken with respect to the the probabilities induced by σ.

[2] If $j \neq i$, then we interpret player (i, t) and player (j, t) as the period t selves of two present-biased decision makers. Another possibility is that $((i, t))_{t=0}^\infty$ and $((j, t))_{t=0}^\infty$ are two dynasties of decision makers such that each generation of a dynasty cares about future generations of its own dynasty, but ignores the welfare of members of the other dynasty.

3.4. Subgame Perfection. In both $\Gamma(G, f)$ and $\Gamma_{PC}(G, f)$, a subgame that is identical to the game itself starts after each period t history. In $\Gamma_{PC}(G, f)$ a subgame also starts after each realization of the public signal. A strategy profile σ is a subgame perfect equilibrium of $\Gamma(G, f)$ if and only if the following is true for all $i \in N$, all $t \in \mathbb{N}$ and all histories $h \in A^t$ of play up to period t : Given that the history of play has been h, and that all other current and future players will play as suggested by σ, it is optimal for player (i, t) to also use the action that σ suggests. In $\Gamma_{PC}(G, f)$, a strategy profile σ is a subgame perfect equilibrium if and only if each player (i, t) finds the action that σ suggests optimal after each history $h \in A^t \times [0, 1]^t$ of play up to period t and each realization $\omega^t \in [0, 1]$ of the public signal in period t.

4. Two Ways to Represent Patience

The folk theorem states that sufficiently patient players can get any payoff vector $v \in \mathcal{F}^*$ in a subgame perfect equilibrium if they are sufficiently patient. With exponential discounting, $f(t) = \delta^t$, patience is naturally measured by the parameter δ. To construct folk theorems that can be used in a wider class of discount functions we first have to figure out how to represent patience. One possibility is to say that decision makers are patient if $\sum_{t=1}^{\infty} f(t)$, the weight placed on all future periods, is large. Another possibility is to use $f(1)$ to measure patience, with the interpretation that decision makers are patient if $f(1)$ is large. If $f(t) = \delta^t$, then both $\sum_{t=1}^{\infty} f(t)$ and $f(1)$ are monotonically increasing in δ and both ways to measure patience are equivalent to measuring patience by the parameter δ.

We will only let $f(1)$ represent patience when the decision makers are present-biased. This rules out the possibility that for example a discount function f with $f(1) = 0.99$ but $f(t) = 0$ for all $t > 1$ is considered patient. To determine the relation between the two measures of patience for $f \in \mathcal{D}^*$, we can use that for such f we have that $f(t) \geq f(1)^t$ for all t. It follows that $\sum_{t=1}^{\infty} f(t) \geq \frac{f(1)}{1-f(1)}$ for all $f \in \mathcal{D}^*$. So for such f the sum $\sum_{t=1}^{\infty} f(t)$ is necessarily large if $f(1)$ is large. To see that the reverse implication does not hold, consider the quasi-exponential discount function $f(t) = \beta\delta^t$ with $\beta, \delta \in (0, 1)$. For this function $f(1)$ is large when $\beta\delta$ is large, which requires that β is large. By contrast, $\sum_{t=1}^{\infty} f(t) = \beta\frac{\delta}{1-\delta}$ explodes when δ approaches 1 for any fixed β.

In more intuitive terms, $f(1)$ is large when the decision makers are insensitive to postponements from the current period to the next. For $\sum f(t)$ to be large it is sufficient that the decision makers are insensitive to postponements from periods far into the future so that $f(t + 1)/f(t)$ is large for large t.

Remark 2. *All propositions below have immediate consequences for quasi-exponential and hyperbolic discounting. Folk theorems for quasi-exponential discounting are implied by using that if* $f(t) = \beta\delta^t$ *with* $\beta, \delta \in (0, 1)$, *then* $\lim_{(\beta,\delta)\to(1,1)} f(1) = 1$, *and* *for any fixed* β *we have that* $\lim_{\delta \to 1} \sum f(t) = +\infty$. *Folk theorems for hyperbolic discounting are implied by using that if* $f(t) = (1 + \alpha t)^{-\gamma/\alpha}$ *with* $0 < \alpha < \gamma$, *then* $\lim_{(\alpha,\gamma)\to(0,0)} f(1) = 1$, *and for any fixed* α *we have that* $\lim_{\gamma \to \alpha} \sum f(t) = +\infty$.

5. Two-Player Stage Games

In two-player stage games there is an action pair where each player minmaxes the other player. Fudenberg and Maskin (1986) use this action pair to develop a two-player folk theorem for exponential discounting, and we will follow the same route.

Heuristically, suppose that two present-biased decision makers initially agree to repeat some strictly individually rational action pair $a \in A$. If one of the decision makers deviate from this suggested path of play, then a minmax phase starts where the mutual minmax action pair is played for T periods, after which the decision makers go back to playing a. Deviations during the minmax phase are punished by restarting the minmax phase.

This strategy profile is subgame perfect if f is such that $\sum_{t=1}^{T} f(t)$ and $f(T)$ are both sufficiently large. That $\sum_{t=1}^{T} f(t)$ is large implies that the decision makers do not want to deviate from a and start the minmax phase because being minmaxed during the next T periods is painful. That $f(T)$ is large implies that the decision makers want to conform during the minmax phase because the desire to escape the minmax phase T periods into the future is stronger than the desire to myopically best reply in the current period. The problem is that to make $\sum_{t=1}^{T} f(t)$ large requires that the length of the minmax phase T is long, and this is in conflict with having $f(T)$ large. For all $f \in \mathcal{D}^*$ with $f(1)$ sufficiently large we can find a T with the desired properties. Therefore Fudenberg and Maskin's (1986) two-player folk theorem generalizes as follows:

Proposition 1. *Suppose* $a \in A$ *is strictly individually rational in the two-player stage game* G. *Then there exists* $\lambda \in (0, 1)$ *such that for all* $f \in \mathcal{D}^*$ *with* $f(1) > \lambda$ *there is a subgame perfect equilibrium of* $\Gamma(G, f)$ *in which* a *is played in each period.*

For finite stage games with just a few actions for each player this proposition will have few implications. In for example a prisoner's dilemma the only strictly individually rational action pair is the action pair where both players cooperate.

The appendix proves Proposition 1 by way of proving a more general result that can be useful if there are many strictly individually rational action pairs. This more general result is that if $(a^t)_{t=0}^{\infty}$ is such that $u_i(a^t) \geq \underline{v}_i^P + \varepsilon$ for all $t \in \mathbb{N}$, $i = 1, 2$ and some $\varepsilon > 0$, then there exists $\lambda \in (0, 1)$ such that for all $f \in \mathcal{D}^*$ with $f(1) > \lambda$ there is a subgame perfect equilibrium that generates the outcome path $(a^t)_{t=0}^{\infty}$. For sufficiently patient decision makers, the outcome (a^t) can be supported by letting a deviation from a^t in period t start a minmax phase after which the decision makers return to the outcome (a^t). The reason is that $u_i(a^t)$ is bounded away from the minmax payoff $\underline{v}_i^P$ by the positive number ε.

Suppose that we add a public correlation device to the picture. Then the decision makers can use the public signals to play a probability distribution over A, and any feasible payoff vector $v \in \mathcal{F}$ can be the expected payoff vector in a given period. With public signals Proposition 1 transforms:

Corollary 1. *Let G be a two-player stage game. For all $v \in \mathcal{F}^P$ there exists $\lambda \in (0,1)$ such that for all $f \in \mathcal{D}^*$ with $f(1) > \lambda$ there is a subgame perfect equilibrium of $\Gamma_{PC}(G, f)$ in which v is the expected payoff vector in each period.*

If for example G should happen to be a prisoner's dilemma, then Corollary 1 applies to any feasible payoff vector that dominates the payoffs that the players get when both defect. It is crucial here that $\mathcal{F}^P$ consists of *strictly* individually rational payoff vectors. Chade, Prokopovych and Smith (2008) give an example of a weakly individually rational payoff vector that is not a subgame perfect payoff for $f \in \mathcal{D}^*$ with $f(1)$ arbitrarily close to 1, but leave open the question if the same thing can happen for a strictly individually rational payoff vector. Corollary 1 answers this question negatively.

6. n-PLAYER STAGE GAMES

6.1. Pure Actions. If the stage game has more than two players, then there may not exist an action profile where the players minmax each other since an action of player 1 that punishes player 2 may be good for player 3. With the mutual minmax strategy profile we studied in section 5, the incentives to punish a deviator are created by a threat. To prove the folk theorem for games with more than two players Fudenberg and Maskin (1986) develop the idea that another way to create incentives to punish a deviator is to reward punishers for punishing another decision maker.

Consider a strategy profile σ where the decision makers initially repeat some strictly individually rational action profile $a \in A$. If decision maker i deviates from this suggested path of play, then a sequence of action profiles that is specific for decision maker i starts. This i'th punishment path consists of two phases: first decision maker i is minmaxed during T periods, and then an action profile $a(i) \in A$ which is bad for player i but good for players $j \neq i$ is played. Deviations from a punishment path are punished in the same way as deviations from a: if decision maker i deviates from a punishment path, then the i'th punishment path starts or restarts.

For decision maker $i \neq j$, the incentive to conform to σ when play is on the j'th punishment path is that in the long run play then ends up in the action profile $a(j)$ and not the action profile $a(i)$. In this sense, σ rewards decision maker i for minmaxing decision maker j. We will say that the stage game allows player-specific punishments from a if it is possible to find action profiles $a(1), \ldots, a(n)$ with the desired properties:

Definition 2. *The stage game G* ***allows player-specific punishments from*** *$a \in A$ if there is a collection $(a(i))_{i \in N}$ of strictly individually rational action profiles such that $u_i(a) \geq u_i(a(i))$ and $u_i(a(j)) > u_i(a(i))$ for all $i \in N$ and all $j \neq i$.*

When player-specific punishments can be constructed is well known. Later we will discuss this and apply a result from Abreu, Dutta and Smith (1994) on this topic.

The strategy profile σ outlined above is subgame perfect if the decision makers want to avoid starting their punishment path when a or $a(i)$ is supposed to be played; and if a punishment path is started, then the punishers want their reward after the minmax phase more than they want to avoid the costs of minmaxing another decision maker. If the discount function f is such that $\sum_{t=1}^{\infty} f(t)$ is large, then the length of the minmax phase T can be chosen such that $\sum_{t=1}^{T} f(t)$ is large but at the same time $\sum_{t=1}^{T} f(t) / \sum_{t=T+1}^{\infty} f(t)$ is small. That $\sum_{t=1}^{T} f(t)$ is large ensures that the minmax phase is painful so that the decision makers want to avoid starting their punishment path. That $\sum_{t=1}^{T} f(t) / \sum_{t=T+1}^{\infty} f(t)$ is small guarantees that if a punishment path is started, then the punishers want their reward after the minmax phase more than they want to avoid the costs of minmaxing. Therefore Fudenberg and Maskin's (1986) pure-action folk theorem for more than two players generalizes as follows:

Proposition 2. *Suppose that the stage game G allows player-specific punishments from $a \in A$. Then there exists $M \in \mathbb{R}$ such that for all $f \in \mathcal{D}$ with $\sum_{t=1}^{\infty} f(t) > M$ there is a subgame perfect equilibrium of $\Gamma(G, f)$ in which a is played in each period.*

If the stage game is finite, then not many action profiles will allow player-specific punishments, perhaps none will. Therefore, Proposition 2 as stated above is more likely to have significant implications if A_i is an interval for each i. A typical game in which player-specific punishments can be created from any strictly individually rational action profile is the Cournot game with a continuum action space. The initially suggested $a \in A$ is then an action profile which gives each player a positive profit. The action profile $a(i) \in A$ can be a profile where the sum of the players profits is the same as with a, but player i gets a smaller market share and every other player gets a larger market share.

If there are public signals, then it will usually be possible to construct player-specific punishments even if the stage game is finite. Fudenberg and Maskin (1986) show that with public signals, player-specific punishments can be created if a full dimensionality condition holds. Abreu, Dutta and Smith (1994) replace the full dimensionality condition with the weaker and clarifying condition that no two players have equivalent utilities. Mailath and Samuelson (2006) provide an enlightening overview of these and other results. We can apply Abreu et al.'s result to Proposition 2 to produce a corollary for the case when the stage game is played with a public correlation device.

Definition 3. *The stage game G satisfies* ***non-equivalent utilities (NEU)*** *if for all $i \in N$ and all $j \neq i$ there do not exist constants $c, d \in \mathbb{R}$ with $d > 0$ such that $u_i(a) = du_j(a) + c$ for all $a \in A$.*

Corollary 2. *Let G be a stage game that satisfies NEU. For all $v \in \mathcal{F}^P$ there exists $M \in \mathbb{R}$ such that for all $f \in \mathcal{D}$ with $\sum_{t=1}^{\infty} f(t) > M$ there is a subgame perfect equilibrium of $\Gamma_{PC}(G, f)$ in which v is the expected payoff vector in each period.*

To apply Proposition 2 or Corollary 2 to quasi-exponential or hyperbolic discounting, or any other functional form for the discounting, one only needs to know for which values of the parameters that $\sum f(t)$ is large. The results hold for the function space $\mathcal{D}$ rather than the smaller space $\mathcal{D}^*$; that is, present-biasedness is not required.

6.2. Mixed Actions. The propositions above only state that payoffs $v \in \mathcal{F}^P$ are subgame perfect payoffs when decision makers are sufficiently patient. Since $\underline{v}_i$ may be strictly smaller than $\underline{v}_i^P$, the set $\mathcal{F}^P$ may be a proper subset of $\mathcal{F}^*$. So far we have avoided the difficulty associated with mixed actions which is that if a player uses a mixed action then he must be indifferent between all pure actions in its support. Fudenberg and Maskin (1986) show that it is possible to deal with this problem when $\mathcal{F}^*$ has full dimension by adjusting future payoffs differently depending on which action profiles that are realized when the players use mixed actions. Abreu, Dutta and Smith's (1994) folk theorem shows that the weaker condition NEU is sufficient to ensure that any $v \in \mathcal{F}^*$ is an equilibrium payoff for sufficiently patient players. Exponential discounting simplifies things, but it is not crucial for Abreu et al.'s argumentation:

Proposition 3. *Let G be a finite stage game that satisfies NEU. For all $v \in \mathcal{F}^*$ there exists $\lambda \in (0, 1)$ such that for all $f \in \mathcal{D}^*$ with $f(1) > \lambda$ there is a subgame perfect equilibrium of $\Gamma_{PC}(G, f)$ in which v is the expected payoff vector in each period.*

This proposition uses $f(1)$ to represent patience. For the special cases of quasi-exponential and hyperbolic discounting, Lemma 2 in the appendix implies mixed-action folk theorems that do not require that $f(1)$ is large. These folk theorems, stated below, show that a large $f(1)$ should be thought of as a sufficient but not a necessary form of patience for the mixed-action folk theorem.

Claim 1 [Quasi-Exponential Discounting]**.** *Let G be a finite stage game that satisfies NEU, and let $\Gamma_{PC}(G, \beta, \delta)$ denote the game $\Gamma_{PC}(G, f)$ with $f(t) = \beta\delta^t$. For all $v \in \mathcal{F}^*$ and all $\beta \in (0, 1]$ there exists $\underline{\delta} \in (0, 1)$ such that for all $\delta \in (\underline{\delta}, 1)$ there is a subgame perfect equilibrium of $\Gamma_{PC}(G, \beta, \delta)$ in which v is the expected payoff vector in each period.*

Claim 2 [Hyperbolic Discounting]**.** *Let G be a finite stage game that satisfies NEU, and let $\Gamma_{PC}(G, \alpha, \gamma)$ denote the game $\Gamma_{PC}(G, f)$ with $f(t) = (1 + \alpha t)^{-\gamma/\alpha}$. For all $v \in \mathcal{F}^*$ and all $\alpha > 0$ there exists $\bar{\gamma} > \alpha$ such that for all $\gamma \in (\alpha, \bar{\gamma})$ there is a subgame perfect equilibrium of $\Gamma_{PC}(G, \alpha, \gamma)$ in which v is the expected payoff vector in each period.*

7. Open Problems

7.1. Differential Time Preferences. Suppose that instead of one common discount function f there is a collection $(f_i)_{i \in N}$ of discount functions with one discount function f_i for each decision maker i. Proposition 1, Corollary 1 and Proposition 3 then continue to hold if the phrase "$f(1) > \lambda$" is replaced with "$f_i(1) > \lambda$ for all $i \in N$". The strategy profile with player-specific punishments is also subgame perfect if $f_i(1)$ is large for all $i \in N$, or if there is some T such that $\sum_{t=1}^{T} f_i(t)$ is sufficiently large and $\sum_{t=1}^{T} f_i(t) / \sum_{t=1}^{\infty} f_i(t)$ is sufficiently small for all $i \in N$. A precise formulation of this modified Proposition 2 is provided in the appendix with a proof.

The content of Corollary 1 and 2 is weaker with differential time preferences because, as emphasized by Lehrer and Pauzner (1999), the convex hull of $u(A)$ then no longer contains all feasible payoff vectors when there is repeated interaction. By arranging payoffs so that impatient decision makers get high payoffs initially and patient decision makers get high payoffs later it is possible to break out of this convex hull. It is an open question which feasible payoffs outside the convex hull of $u(A)$ that are subgame perfect payoffs for sufficiently patient decision makers when time preferences are both differential and inconsistent.

7.2. Subgame Perfection and Individual Rationality. In repeated games with time-consistent preferences we know that a subgame perfect equilibrium gives each player a payoff that is weakly above his minmax payoff in the stage game. This is because the equilibrium strategy must be at least as good as myopically best replying in each period. In the present model with potentially time-inconsistent preferences the same argument does not apply because player $(i, 0)$ can be punished not only by players acting in player role $j \neq i$, but also by his future selves $((i, t))_{t=1}^{\infty}$. These future selves in turn could find it optimal to carry out the punishment because otherwise they will be punished by their future selves. As the following example from Vieille and Weibull (2009) shows, such chains of punishments can give player $(i, 0)$ a subgame perfect payoff in $\Gamma(G, f)$ that is below $\sum_{t=0}^{\infty} f(t) \underline{v}_i^P$.

Example 1. *Consider the trivial two-player stage game G with $A_1 = \{0, 1\}$, $A_2 = \{1\}$, $u_1(a) = u_2(a) = a_1$. Let $f \in \mathcal{D}$ be such that $f(1) = 1$, and let $(a^t)_{t=0}^{\infty}$ be the outcome with $a_1^0 = 0$ and $a_1^t = 1$ for all $t > 1$. A strategy profile is defined by the following instructions: play the outcome $(a^t)_{t=0}^{\infty}$; if decision maker 1 deviates,* *then restart* $(a^t)_{t=0}^{\infty}$. *This strategy profiles requires that player* $(1,0)$ *plays* 0, *and the punishment for deviating is that otherwise player* $(1,1)$ *plays* 0. *If player* $(1,1)$ *is required to play* 0, *then the incentives are given by the threat that otherwise player* $(1,2)$ *plays* 0, *and so on in an infinite sequence of threats. This strategy profile is a subgame perfect equilibrium because player* $(1,0)$ *gets* $\sum_{t=1}^{\infty} f(t)$ *for conforming and* $f(0) + \sum_{t=2}^{\infty} f(t)$ *for deviating, and these numbers are equal since* $f(0) = f(1) = 1$. *The strategy profile gives player* $(1,0)$ *the payoff* $\sum_{t=1}^{\infty} f(t)$ *which is less than the minmax payoff* $\sum_{t=0}^{\infty} f(t)\underline{v}_1^P = 1 + \sum_{t=1}^{\infty} f(t)$.

Some restriction on which discount functions that are allowed is necessary to ensure that subgame perfection implies individual rationality. Chade, Prokopovych and Smith (2008) show that quasi-exponentiality is enough: with $f(t) = \beta\delta^t$ and $\beta \leq 1$, all subgame perfect equilibria of $\Gamma(G, f)$ give player $(i, 0)$ a payoff that is weakly above $\sum_{t=0}^{\infty} f(t)\underline{v}_i^P$. It is an open question if there is a weaker condition which ensures that subgame perfection implies individual rationality. Vieille and Weibull (2009) provide the solution to this problem for the case of one time-inconsistent decision maker. Their Proposition 3.1 implies that if there is one decision maker, then it is sufficient to require that discount functions are strictly decreasing. If all actions give distinct payoffs there is then a unique subgame perfect equilibrium in which the optimal action is used in each period.

8. CONCLUSION

Fudenberg and Maskin (1986) show that if payoffs are discounted exponentially, then repeated games permit many subgame perfect outcomes if the decision makers who interact with each other are sufficiently patient. It was shown here that the same is true also for present-biased decision makers.

More specifically, if the stage game has only two players, then the folk theorem holds for present-biased decision makers when patience is represented by the discount factor between the next and the current period. If player-specific punishments can be constructed, then the folk theorem holds for present-biased decision makers when patience is represented by the sum of the discount factors for all future periods. As shown by Abreu, Dutta and Smith (1994), such player-specific punishments can be created when there are public signals and the utilities of the stage game are not equal for any two players.

The discount factor between the next and the current period is large when decision makers are insensitive to a one-period postponement from the current period. But, for the sum of all discount factors to be large it is sufficient that decision makers are insensitive to postponements from periods far into the future. The present analysis therefore shows that for the purpose of the folk theorem, present-biased decision makers can sometimes be considered patient even if they are highly sensitive to a one-period postponement from the current period.

9. APPENDIX

All proofs have a similar structure. We first define a number λ or M and then show, by explicitly constructing a subgame perfect equilibrium, that this number has the desired property.

9.1. Proposition 1.

Proposition 1 is implied by the following lemma:

Lemma 1. *Let G be a two-player stage game and suppose $v \in \mathbb{R}^2$ and $(a^t)_{t=0}^{\infty}$ are such that $u_i(a^t) \geq v_i > \underline{v}_i^P$ for all $t \in \mathbb{N}$ and $i = 1, 2$. Then there exists $\lambda \in (0, 1)$ such that for all $f \in \mathcal{D}^*$ with $f(1) > \lambda$ there is a subgame perfect equilibrium of $\Gamma(G, f)$ that generates the outcome path $(a^t)_{t=0}^{\infty}$.*

Let $v \in \mathbb{R}^2$ and $(a^t)_{t=0}^{\infty}$ be such that $u_i(a^t) \geq v_i > \underline{v}_i^P$ for all $t \in \mathbb{N}$ and $i = 1, 2$. Define μ by $\mu = \max_{b,b' \in A, i \in N}(u_i(b) - u_i(b'))$ so that μ is an upper bound on the deviation gain from any action profile. For $i = 1, 2$, let $p_i \in A_i$ be a solution to the problem minmaxing player $j \neq i$:

$$p_i \in \arg\min_{b_i \in A_i} \left(\max_{b_j \in A_j} u_j(b_1, b_2) \right)$$

Let p be the action profile $p = (p_1, p_2)$. Let $\lambda \in (0, 1)$ and $T \in \mathbb{N}$ be such that

$$\begin{aligned} \sum_{t=1}^{T} \lambda^t &> \frac{\mu}{v_i - u_i(p)}, \quad \text{and} \\ \lambda^T &> \frac{\underline{v}_i^P - u_i(p)}{v_i - u_i(p)}, \end{aligned}$$

for $i = 1, 2$. Such λ and T exist since $v_i > \underline{v}_i^P \geq u_i(p)$ for $i = 1, 2$.[3] Fix an arbitrary $f \in \mathcal{D}^*$ with $f(1) > \lambda$. Since $f(t) \geq \lambda^t$ for all $t \in \mathbb{N}$, this f is such that

$$\begin{aligned} \sum_{t=1}^{T} f(t)\,(v_i - u_i(p)) &> \mu, \quad \text{and} && (1) \\ u_i(p) - \underline{v}_i^P + f(T)\,(v_i - u_i(p)) &> 0, && (2) \end{aligned}$$

for $i = 1, 2$.

Let σ be the following automaton: The set of states is $\{(t, s) : t \in \mathbb{N}, s = 0, 1, \ldots, T\}$. The variable t says which period we are in. The variable s measures

[3] First pick T such that $T > \frac{\mu}{v_i - u_i(p)}$. Then the inequalities hold for all λ sufficiently close to 1 since $\lim_{\lambda \to 1} \sum_{t=1}^{T} \lambda^t = T > \frac{\mu}{v_i - u_i(p)}$ and $\lim_{\lambda \to 1} \lambda^T = 1 > \frac{\underline{v}_i^P - u_i(p)}{v_i - u_i(p)}$.

how far into the minmax phase the decision makers are when they are in the minmax phase, and $s = 0$ if the decision makers are not in the minmax phase. The initial state is $(0, 0)$. The output function specifies that a^t is played in state (t, s) if $s = 0$, and that p is played in state (t, s) if $s \neq 0$. The transition rules are:

$(t, 0)$. Go to state $(t + 1, 0)$ if a^t is played. If a^t is not played, go to state $(t + 1, 1)$.

(t, s), $s \notin \{0, T\}$. Go to state $(t + 1, s + 1)$ if p is played. If p is not played, go to state $(t + 1, 1)$.

(t, T). Go to state $(t + 1, 0)$ if p is played. If p is not played, go to state $(t + 1, 1)$.

It remains only to show that σ is a subgame perfect equilibrium. Suppose that we are in state $(\tau, 0)$ in period τ. It is optimal for player (i, τ) to play as suggested by σ if, for all $b \in A$,

$$u_i(a^\tau) + \sum_{t=1}^{\infty} f(t) u_i(a^{\tau+t}) \geq u_i(b) + \sum_{t=1}^{T} f(t) u_i(p) + \sum_{t=T+1}^{\infty} f(t) u_i(a^{\tau+t}).$$

This inequality holds since $u_i(a^{\tau+t}) \geq v_i$ for $t = 1, \ldots, T$, and since f satisfies (1).

Let $s \in \{1, \ldots, T\}$. Suppose that we are in state (τ, s) in period τ. It is optimal for player (i, τ) to play as suggested by σ if

$$\sum_{t=0}^{T-s} f(t) u_i(p) + \sum_{t=T-s+1}^{\infty} f(t) u_i(a^{\tau+t}) \geq \underline{v}_i^P + \sum_{t=1}^{T} f(t) u_i(p) + \sum_{t=T+1}^{\infty} f(t) u_i(a^{\tau+t}). \quad (3)$$

Since $u_i(a^{\tau+t}) > u_i(p)$ for $t = T - s, \ldots, T - 1$, (3) holds if

$$\sum_{t=0}^{T-1} f(t) u_i(p) + \sum_{t=T}^{\infty} f(t) u_i(a^{\tau+t}) \geq \underline{v}_i^P + \sum_{t=1}^{T} f(t) u_i(p) + \sum_{t=T+1}^{\infty} f(t) u_i(a^{\tau+t}).$$

This inequality holds since $u_i(a^{\tau+T}) \geq v_i$, and since f satisfies (2).

9.2. Corollary 1. The corollary is proven in the same way as Lemma 1. Given any $v \in \mathcal{F}^P$, there exist $\lambda \in (0, 1)$ and $T \in \mathbb{N}$ such that (1) and (2) hold for all $f \in \mathcal{D}^*$ with $f(1) > \lambda$. This λ has the desired property. The decision makers can use the public signals to get the expected payoff vector v and let deviations start a minmax phase of length T after which play returns to v.

9.3. Proposition 2. Let $a \in A$ be such that there is a collection $(a(i))_{i \in N}$ of strictly individually rational action profiles such that $u_i(a) \geq u_i(a(i))$ and $u_i(a(j)) > u_i(a(i))$ for all $i \in N$ and all $j \neq i$. Define μ by $\mu = \max_{b,b' \in A, i \in N}(u_i(b) - u_i(b'))$. Put

$$\begin{aligned} m_1 &= \min_{i \in N} \left(u_i(a(i)) - \underline{v}_i^P \right), \\ m_2 &= \min_{i \in N, j \in N \setminus \{i\}} \left(u_i(a(j)) - u_i(a(i)) \right), \\ M &= \left(\frac{\mu}{m_1} + 2 \right) \frac{\mu}{m_2} + \frac{\mu}{m_1} + 1. \end{aligned}$$

The action profiles $(a(i))_{i \in N}$ are such that $m_1 > 0$ and $m_2 > 0$. We will see that this choice of M works. Fix an arbitrary $f \in \mathcal{D}$ with the property that $\sum_{t=1}^{\infty} f(t) > M$. Let T be a positive integer such that

$$\frac{\mu}{m_1} \leq \sum_{t=1}^{T} f(t) \leq \frac{\mu}{m_1} + 1.$$

Using the definition of m_1 it follows that

$$\sum_{t=1}^{T} f(t) \left(u_i(a(i)) - \underline{v}_i^P \right) \geq \mu \tag{4}$$

for all $i \in N$. We also have that

$$\begin{aligned} \sum_{t=T+1}^{\infty} f(t) m_2 &= m_2 \left(\sum_{t=1}^{\infty} f(t) - \sum_{t=1}^{T} f(t) \right) \\ &\geq m_2 \left(\left(\frac{\mu}{m_1} + 2 \right) \frac{\mu}{m_2} + \frac{\mu}{m_1} + 1 - \left(\frac{\mu}{m_1} + 1 \right) \right) \\ &= \mu + \mu \left(\frac{\mu}{m_1} + 1 \right) \\ &\geq \mu + \mu \sum_{t=1}^{T} f(t). \end{aligned}$$

Using the definition of m_2 it follows that

$$\sum_{t=T+1}^{\infty} f(t)(u_i(a(j)) - u_i(a(i))) \geq \mu \sum_{t=0}^{T} f(t) \tag{5}$$

for all $i \in N$ and all $j \neq i$.

For $i \in N$, let $p(i)$ be an action profile where player i is minmaxed. That is, $p(i) \in A$ is such that $u_i(p(i)) = \underline{v}_i^P$ and $p_i(i) \in A_i$ is a best reply to $p_{-i}(i) \in A_{-i}$. Put $a(0) = a$ and let σ be the following automaton: The set of states is $\{a(i) : i \in N \cup \{0\}\} \cup \{(p(i), s) : i \in N, s = 1, \ldots, T\}$. The initial state is $a(0)$. The output function specifies that $a(i)$ is played in state $a(i)$, and that $p(i)$ is played in state $(p(i), s)$. The transition rules are:

$a(j)$. Remain in state $a(j)$ unless a single player deviates from $a(j)$. If a single player that acts in player role i deviates from $a(j)$, go to state $(p(i), 1)$.

$(p(j), s)$, $s < T$. Go to state $(p(j), s+1)$ unless a single player deviates from $p(j)$. If a single player that acts in player role i deviates from $p(j)$, go to state $(p(i), 1)$.

$(p(j), T)$. Go to state $a(j)$ unless a single player deviates from $p(j)$. If a single player that acts in player role i deviates from $p(j)$, go to state $(p(i), 1)$.

It remains only to show that σ is a subgame perfect equilibrium. Suppose that in period τ we are in state $a(j)$ for some $j \in N \cup \{0\}$. It is optimal for player (i, τ) to play as suggested by σ if, for all $b \in A$,

$$u_i(a(j)) + \sum_{t=1}^{\infty} f(t) u_i(a(j)) \geq u_i(b) + \sum_{t=1}^{T} f(t) \underline{v}_i^P + \sum_{t=T+1}^{\infty} f(t) u_i(a(i)). \tag{6}$$

Since $u_i(a(j)) \geq u_i(a(i))$, (6) holds if

$$\sum_{t=1}^{T} f(t) \left(u_i(a(i)) - \underline{v}_i^P \right) \geq u_i(b) - u_i(a(j)).$$

This inequality holds since f satisfies (4).

Suppose that in period τ we are in state $(p(j), s)$ for some $j \in N$ and some $s \in \{1, \ldots, T\}$. If $i = j$, then it is optimal for player (i, τ) to play as suggested by σ since a deviation cannot increase the payoff in the current period or any future period. If $i \neq j$, then it is optimal for player (i, τ) to play as suggested by σ if, for all $b \in A$,

$$\sum_{t=0}^{T-s} f(t) u_i(p(j)) + \sum_{t=T-s+1}^{\infty} f(t) u_i(a(j)) \geq u_i(b) + \sum_{t=1}^{T} f(t) \underline{v}_i^P + \sum_{t=T+1}^{\infty} f(t) u_i(a(i)). \tag{7}$$

By definition of μ, (7) holds if

$$\sum_{t=T+1}^{\infty} f(t) \left(u_i(a(j)) - u_i(a(i)) \right) \geq \mu \sum_{t=0}^{T} f(t).$$

This is precisely the inequality (5) which f does satisfy.

9.4. Corollary 2. It follows from the analysis in Abreu, Dutta and Smith (1994) that if G satisfies NEU, and if $v \in \mathcal{F}^P$, then there is a collection $(v(i))_{i\in N}$ of payoff vectors from $\mathcal{F}$ such that $v_i > v_i(i) > \underline{v}_i^P$ and $v_i(j) > v_i(i)$ for all $i \in N$ and all $j \neq i$. This result is stated as proposition 3.5.1 in Mailath and Samuelson (2006). Using these payoff vectors, the corollary is proven in the same way as Proposition 2 but v_i replaces $u_i(a)$ and $v_i(j)$ replaces $u_i(a(j))$. The decision makers now use the public signals to get the expected payoff vector v initially, and $v(i)$ if the i'th punishment path is started.

9.5. Proposition 3. The key to proving Proposition 3 is to prove the following:

Lemma 2. *Let G be a finite stage game that satisfies NEU. For all $v \in \mathcal{F}^*$ there exist m and $(M_T)_{T=1}^{\infty}$ such that for all $f \in \mathcal{D}$ with $\sum_{t=1}^{T} f(t) > m$ and $\sum_{t=1}^{\infty} f(t) > M_T$ for some $T \in \{1, 2, \ldots\}$ there is a subgame perfect equilibrium of $\Gamma_{PC}(G, f)$ in which v is the expected payoff vector in each period.*

Assume, for now, that this lemma is true. Let m and $(M_T)_{T=1}^{\infty}$ be the numbers whose existence Lemma 2 guarantees. Let T be a positive integer such that $T > m$, and let $\lambda \in (0, 1)$ be such that $\sum_{t=1}^{T} \lambda^t > m$ and $\sum_{t=1}^{\infty} \lambda^t > M_T$. Then $\sum_{t=1}^{T} f(t) > m$ and $\sum_{t=1}^{\infty} f(t) > M_T$ for all $f \in \mathcal{D}^*$ with $f(1) > \lambda$. So, by Lemma 2, for all $f \in \mathcal{D}^*$ with $f(1) > \lambda$ there is a subgame perfect equilibrium of $\Gamma_{PC}(G, f)$ with the payoffs v. Thus Proposition 3 is implied by Lemma 2. To prove the lemma we first define m and $(M_T)_{T=1}^{\infty}$ and then follow the proof in Abreu, Dutta and Smith (1994), henceforth ADS, as closely as is possible in the current setting.

DEFINING m AND $(M_T)_{T=1}^{\infty}$. Let $v \in \mathcal{F}^*$. ADS establish that there exists a collection $(x^i)_{i\in N}$ of payoff vectors from $\mathcal{F}$ such that $v_i > x_i^i > \underline{v}_i$ and $x_i^j > x_i^i$ for all $i \in N$ and all $j \neq i$. For all $i \in N$, let $N(i)$ be the subset of N that consists of all $j \in N \backslash \{i\}$ for which there exists a payoff vector $c^{ij} \in \mathcal{F}$ such that

$$\begin{aligned} c_j^{ij} &\neq x_j^i,\ c_i^{ij} = x_i^i, \quad \text{and} \\ c_k^{ij} &\geq x_k^k \quad \text{for all } k \in N. \end{aligned}$$

Define μ by $\mu = \max_{a,b\in A, i\in N}(u_i(a) - u_i(b))$. Let m be a real number such that

$$m > \frac{\mu}{x_i^i - \underline{v}_i}$$

for all $i \in N$. Fix an arbitrary positive integer T, and let $q \in (0, 1)$ be such that

$$m \cdot q^{T-1} > \frac{\mu}{x_i^i - \underline{v}_i} \tag{8}$$

for all $i \in N$. Let M_T be such that

$$\frac{n \cdot |A| \cdot \mu}{M_T \cdot \min_{i\in N, j\in N(i)} \left|c_j^{ij} - x_j^i\right|} < 1 - q. \tag{9}$$

Such a number M_T exists since A is a finite set, $q < 1$, and $\min_{i\in N, j\in N(i)} \left|c_j^{ij} - x_j^i\right| > 0$.

CONSTRUCTION OF STRATEGY PROFILE. Fix an arbitrary $f \in \mathcal{D}$ with the property that for some $T \in \{1, 2, \ldots\}$ we have that $\sum_{t=1}^{T} f(t) > m$ and $\sum_{t=1}^{\infty} f(t) > M_T$. Let $q \in (0, 1)$ be such that (8) and (9) holds. For all $i \in N$, let m^i be a profile of mixed actions where player i is minmaxed. That is, $m^i \in \Delta A$ is such that $u_i(m^i) = \underline{v}_i$ and $m_i^i \in \Delta A_i$ is a best reply to $m_{-i}^i \in \Delta A_{-i}$. For all $i \in N$ and all $j \in N(i)$, let $p^{ij} : A_j \to [0, +\infty)$ be a function such that $p^{ij}(a_j) = 0$ for some $a_j \in A_j$, and for all $a_j, b_j \in A_j$ we have that

$$u_j(a_j, m_{-j}^i) - u_j(b_j, m_{-j}^i) = (p^{ij}(b_j) - p^{ij}(a_j))(c_j^{ij} - x_j^i) \sum_{t=1}^{\infty} f(t). \tag{10}$$

Such a function p^{ij} exist since $c_j^{ij} \neq x_j^i$. The function p^{ij} is such that

$$\left|p^{ij}(b_j) - p^{ij}(a_j)\right| = \left|\frac{u_j(a_j, m_{-j}^i) - u_j(b_j, m_{-j}^i)}{(c_j^{ij} - x_j^i)\sum_{t=1}^{\infty} f(t)}\right| < \frac{\mu}{M_T \cdot \min_{k\in N, l\in N(k)} \left|c_l^{kl} - x_l^k\right|} < \frac{1-q}{n \cdot |A|}, \tag{11}$$

where the last inequality uses (9). Since $p^{ij}(a_j) = 0$ for some $a_j \in A_j$, it follows from (11) that $p^{ij}(a_j) < \frac{1-q}{n}$ for all $a_j \in A_j$. Therefore the functions $(p^{ij})_{i\in N, j\in N(i)}$ are such that for all $i \in N$ and all $a \in A$ we have that

$$\sum_{j\in N(i)} p^{ij}(a_j) < 1 - q.$$

This will ensure that the strategy profile σ described below only uses well defined probabilities.

Let σ be a strategy profile with the following properties: The set of "states" is

$$\{v\} \cup \left\{x^i : i \in N\right\} \cup \left\{c^{ij} : i \in N, j \in N(i)\right\} \cup \left\{m^i : i \in N\right\}.$$

The initial state is v. The decision makers "play v" in state v, which means that in state v the decision makers use the public signals to play a probability distribution over A which gives the expected payoff v. After any public signal, a pure action $a \in A$ should be played to ensure that defections can be detected. Analogously, the decision makers play x^i in state x^i, and play c^{ij} in state c^{ij}. In state m^i, the mixed action $m^i \in \Delta A$ is played. The transition rules are:

v. The next periods state is v unless a single player deviates. If a single player acting in player role i deviates, play goes to state m^i.

m^i. Suppose that the observed action profile is $a \in A$. For $j \in N(i)$, play goes to state c^{ij} with probability $p^{ij}(a_j)$. Play goes to state x^i with probability $(1 - q - \sum_{j \in N(i)} p^{ij}(a_j))$. With probability q, play remains in state m^i.

x^i. The next periods state is x^i unless a single player deviates. If a single player acting in player role k deviates, play goes to state m^k.

c^{ij}. The next periods state is c^{ij} unless a single player deviates. If a single player acting in player role k deviates, play goes to state m^k.

VERIFICATION OF EQUILIBRIUM. We first verify that it is optimal for player (i, τ) to play as suggested by σ in state x^i, even if there is a deviation which gives the maximal deviation gain μ. Thereafter we check that if $j \in N(i)$, then it is optimal for player (j, τ) to play as suggested by σ in state m^i. Finally we consider the remaining states.

Suppose that we are in state x^i in period τ. A deviation by player (i, τ) gives at most the current gain μ. Thereafter, player (i, τ) is punished with the minmax payoff $\underline{v}_i$ for as long as play remains in state m^i. After that, player (i, τ) gets the payoff x^i_i again. This last assertion follows from $c^{ij}_i = x^i_i$ for all $j \in N(i)$. The probability that state m^i will last at least T periods is q^{T-1}. Because of this, it is optimal for player (i, τ) to conform if

$$q^{T-1} \sum_{t=1}^{T} f(t) \left(x^i_i - \underline{v}_i \right) \geq \mu.$$

This inequality holds since $x^i_i - \underline{v}_i > 0$, $\sum_{t=1}^{T} f(t) > m$, and m and q satisfy (8).

Let $i \in N$, $j \in N(i)$ and suppose that we are in state m^i in period τ. Consider the problem that player (j, τ) faces. Player (j, τ) can only affect the probability of the states c^{ij} and x^i. Using a_j in the current period has three effects: it gives the current payoff $u_j(a_j, m^i_{-j})$, the probability $p^{ij}(a_j)$ that the next periods state is c^{ij}, and affects the probability that the next periods state is x^i through the term $-p^{ij}(a_j)$. Hence, the payoff for player (j, τ) for the action a_j is

$$u_j(a_j, m^i_{-j}) + p^{ij}(a_j) \sum_{t=1}^{\infty} f(t) \left(c^{ij}_j - x^i_j \right) + C,$$

where C is some constant whose value player (j, τ) cannot affect. It follows that player (j, τ) is indifferent between two actions a_j and b_j from A_j precisely when

$$u_j(a_j, m^i_{-j}) + p^{ij}(a_j) \sum_{t=1}^{\infty} f(t)(c^{ij}_j - x^i_j) = u_j(b_j, m^i_{-j}) + p^{ij}(b_j) \sum_{t=1}^{\infty} f(t)(c^{ij}_j - x^i_j).$$

This inequality holds because p^{ij} was chosen such that (10) holds. Thus player j is indifferent between all actions from A_j, so in particular player j is content with using the mixed action m^i_j.

Let $i \in N$, $j \in N(i)$, $k \in N \backslash \{i\}$ and $l \in N(k)$. Suppose that in period τ we are in one of the states v, x^k, c^{kl}, c^{ij}. If player (i,τ) conforms, then he gets one of the payoffs $v_i, x^k_i, c^{kl}_i, c^{ij}_i$ in all future periods. We have that v_i, x^k_i, c^{kl}_i and c^{ij}_i all are weakly greater than x^i_i, and also that a deviation by player (i,τ) always starts the same punishment path. Therefore, since it is optimal for player (i,τ) to conform in state x^i even if there is a deviation which gives the maximal gain μ, it is optimal for player (i,τ) to conform also in the states v, x^k, c^{kl}, c^{ij}. The only remaining case to consider is when $j \notin N(i)$ and we are in state m^i in period τ. ADS show that if $j \notin N(i)$, then m^i_j is a best reply to m^i_{-j}. Thus, since player (j,τ) cannot affect the probability distribution over states in the next period, it is optimal for player (j,τ) to play as suggested σ.

9.6. Claim 1. Let m and $(M_T)_{T=1}^{\infty}$ be the numbers whose existence Lemma 2 guarantees. Fix an arbitrary $\beta \in (0,1]$, and let T be a positive integer such that $\beta T > m$. Then, for all δ sufficiently close to 1, we have that $\sum_{t=1}^{T} \beta\delta^t > m$ and $\sum_{t=1}^{\infty} \beta\delta^t > M_T$. So Claim 1 is implied by Lemma 2.

9.7. Claim 2. Let m and $(M_T)_{T=1}^{\infty}$ be the numbers whose existence Lemma 2 guarantees. Fix an arbitrary $\alpha > 0$. Let $\gamma_1 > \alpha$ and T be such that $\sum_{t=1}^{T}(1 + \alpha t)^{-\gamma_1/\alpha} > m$. Such γ_1 and T exist since $\lim_{\gamma\to\alpha} \sum_{t=1}^{\infty}(1 + \alpha t)^{-\gamma/\alpha} = +\infty$. Let $\bar{\gamma} \in (\alpha, \gamma_1)$ be such that $\sum_{t=1}^{\infty}(1 + \alpha t)^{-\bar{\gamma}/\alpha} > M_T$. Such $\bar{\gamma}$ exists since $\lim_{\gamma\to\alpha} \sum_{t=1}^{\infty}(1 + \alpha t)^{-\gamma/\alpha} = +\infty$. Then, for all $\gamma \in (\alpha, \bar{\gamma})$, we have that $\sum_{t=1}^{T}(1 + \alpha t)^{-\gamma/\alpha} > m$ and $\sum_{t=1}^{\infty}(1 + \alpha t)^{-\gamma/\alpha} > M_T$. So Claim 2 is implied by Lemma 2

9.8. Differential Time Preferences. Suppose that there is a collection $(f_i)_{i\in N}$ of discount functions such that decision maker i uses f_i to discount future payoffs. That is, player (i,τ) acts to maximize $\sum_{t=0}^{\infty} f_i(t)u_i(a^{\tau+t})$.

Claim 3. *Suppose that the stage game G allows player-specific punishments from the action profile $a \in A$. Then there exists $m \in \mathbb{R}$ and a sequence $(M_T)_{T=1}^{\infty}$ of real numbers with the following property: For all collections $(f_i)_{i\in N}$ of discount functions from $\mathcal{D}$ such that for some $T \in \{1, 2, \ldots\}$ we have that $\sum_{t=1}^{T} f_i(t) > m$ and $\sum_{t=1}^{\infty} f_i(t) > M_T$ for all $i \in N$ there is a subgame perfect equilibrium of $\Gamma(G,(f_i))$ in which a is played in each period.*

Proof. Let μ, m_1 and m_2 be defined as in the proof of Proposition 2. Let m be such that $m > \frac{\mu}{m_1}$, and put $M_T = \frac{\mu}{m_2}(T+1)+T$ for all positive integers T. Fix a collection

$(f_i)_{i\in N}$ of discount functions such that $\sum_{t=1}^{T} f_i(t) > m$ and $\sum_{t=1}^{\infty} f_i(t) > M_T$ for some $T \in \{1, 2, \ldots\}$ and all $i \in N$. Then

$$\sum_{t=1}^{T} f_i(t)\left(u_i(a(i)) - \underline{v}_i^P\right) \geq \mu, \quad \text{and}$$

$$\sum_{t=T+1}^{\infty} f_i(t)(u_i(a(j)) - u_i(a(i))) \geq \mu \sum_{t=0}^{T} f_i(t),$$

for all $i \in N$ and all $j \neq i$. Hence, with this choice of T, the strategy profile σ constructed in the proof of Proposition 2 is subgame perfect in $\Gamma(G, (f_i))$. □

Claim 4. *Suppose that the stage game G allows player-specific punishments from the action profile $a \in A$. Then there exists $\lambda \in (0, 1)$ such that for all collections $(f_i)_{i\in N}$ of discount functions from $\mathcal{D}^*$ with $f_i(1) > \lambda$ for all $i \in N$ there is a subgame perfect equilibrium of $\Gamma(G, (f_i))$ in which a is played in each period.*

Proof. Let m and $(M_T)_{T=1}^{\infty}$ be the numbers whose existence Claim 3 guarantees. Let T be a positive integer such that $T > m$. Let $\lambda \in (0, 1)$ be such that $\sum_{t=1}^{T} \lambda^t > m$ and $\sum_{t=1}^{\infty} \lambda^t > M_T$. Then $\sum_{t=1}^{T} f(t) > m$ and $\sum_{t=1}^{\infty} f(t) > M_T$ for all $f \in \mathcal{D}^*$ with $f(1) > \lambda$, so Claim 4 is implied by Claim 3. □

REFERENCES

[1] Abreu, Dilip. Prajik K. Dutta, and Lones Smith. 1994. “The Folk Theorem for Repeated Games: A Neu Condition.” *Econometrica*, 62(4): 939-948.

[2] Aumann, Robert J., and Lloyd S. Shapley. 1976. “Long Term Competition: A Game Theoretic Analysis.” mimeo, Hebrew University, Jerusalem.

[3] Chade, Hector, Pavlo Prokopovych, and Lones Smith. 2008. “Repeated games with present-biased preferences.” *Journal of Economic Theory*, 139(1): 157-175.

[4] Dasgupta, Partha, and Eric Maskin. 2005. “Uncertainty and Hyperbolic Discounting.” *American Economic Review*, 95(4): 1290-1299.

[5] Eisenhauer, Joseph G., and Luigi Ventura. 2006. “The prevalence of hyperbolic discounting: some European evidence.” *Applied Economics*, 38(11): 1223-1234

[6] Fudenberg, Drew, and Eric Maskin. 1986. “The Folk Theorem in Repeated Games with Discounting or with Incomplete Information.” *Econometrica,* 54(3): 533-554.

[7] Krusell, Per, and Anthony A. Smith Jr. 2003. "Consumption-Savings Decisions with Quasi-Geometric Discounting." *Econometrica*, 71(1): 365-375.

[8] Laibson, David. 1997. "Golden Eggs and Hyperbolic Discounting." *Quarterly Journal of Economics*, 112(2): 443-477.

[9] Laibson, David. 1998. "Life-cycle Consumption and Hyperbolic Discount Functions." *European Economic Review*, 42: 861-871.

[10] Lehrer, Ehud, and Andy Pauzner. 1999. "Repeated Games with Differential Time Preferences." *Econometrica*, 67(2): 393-412.

[11] Loewenstein, George, and Drazen Prelec. 1992. "Anomalies in Intertemporal Choice and an Interpretation." *Quarterly Journal of Economics*, 107(2): 443-477.

[12] Goldman, Steven M. 1980. "Consistent Plans." *Review of Economic Studies*, 47(3): 533-537.

[13] Mailath, George J., and Larry Samuelson. 2006. *Repeated Games and Reputations: Long-Run Relationships.* Oxford University Press.

[14] Peleg, Bezalel, and Menahem E. Yaari. 1973. "On the Existence of a Consistent Course of Action when Tastes are Changing." *Review of Economic Studies*, 40(3): 391-401.

[15] Phelps, Edmund S., and Robert A. Pollak. 1968. "On Second-Best National Saving and Game-Equilibrium Growth." *Review of Economic Studies,* 35(2): 195–199.

[16] Pollak, Robert A. 1968. "Consistent Planning." *Review of Economic Studies,* 35(2): 201-208.

[17] Prokopovych, Pavlo V. 2005. *Essays on Repeated Games.* Thesis, (PhD). The University of Michigan.

[18] Rubinstein, Ariel 1979. "Equilibrium in Supergames with the Overtaking Criterion." *Journal of Economic Theory,* 21(1): 1-9.

[19] Saez-Marti, Maria, and Jörgen W. Weibull. 2005. "Discounting and altrusim to future decision-makers." *Journal of Economic Theory*, 122(2): 254-266.

[20] Streich, Philip, and Jack Levy. 2007. "Time Horizons, Discounting, and Intertemporal Choice." *Journal of Conflict Resolution*, 51(2): 199-226.

[21] Strotz, Robert H. 1956. "Myopia and Inconsistency in Dynamic Utility Maximization." *Review of Economic Studies*, 23: 165-180.

[22] Thaler, Richard. 1981. "Some Empirical Evidence on Dynamic Inconsistency" *Economics Letters*, 8(3): 201-207.

[23] Vieille, Nicolas, and Jörgen W. Weibull. 2009. "Multiple Solutions under Quasi-Exponential Discounting." *Economic Theory,* 39(3): 513-526.

[24] Wärneryd, Karl. 2007. "Sexual Reproduction and Time-Inconsistent Preferences." *Economics Letters,* 95(1): 14–16.