# A Smooth Transition ARCH Model for Asset Returns*

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**Abstract**

In the classical ARCH model of Engle [1982] the conditional variance is a linear function of lagged squared residuals. In this paper I introduce nonlinearity, by adding a term that consists of a constant parameter multiplied by a transition function. Two different transition functions are considered, a logistic and an exponential. Furthermore, following Bollerslev [1986], I extend the model by introducing lagged conditional variances in the conditional variance equation. This specification reduces the number of parameters in the model, which proves to be important for successful estimation. The paper also describes a number of specification tests, that can determine if the smooth transition GARCH model can be the data generating process of a times series. The techniques proposed are illustrated on data from four stock index series.

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# 1 Introduction

In this paper I introduce a new class of ARCH/GARCH models. This new class allows for non-linearity in the equation for the conditional variance. Two forms of non-linearity are considered. First, asymmetry regarding the sign of the error term is considered. This specification allows positive and negative shocks of equal size to have different degrees of impact on the conditional variance. Second, non-linearity regarding the size of error terms is considered. The dynamics of the conditional variance will then differ depending on whether the market is relatively volatile or not. Even though models that allow for different forms of non-linearity have been presented before, these new models are believed to have novel features, that are advantageous in a modeling situation. One major advantage is that specification tests can easily be developed from this model class. These specification tests substantially simplify the procedure of finding a suitable model for representing a financial time series.

In his seminal work, Engle [1982] introduces the ARCH (Autoregressive Conditional Heteroskedasticity) model, in which the conditional variance is a linear function of lagged squared residuals, analogous to an MA model for the conditional mean. Bollerslev [1986] introduces the Generalized ARCH (GARCH) model and extends Engle's MA specification into an ARMA model by introducing lagged conditional variances in the conditional variance equation. With this representation, the number of parameters in the model can be reduced considerably. Note, however, that a stationary GARCH model can always be rewritten as an ARCH model with an infinite number of lags.

Among the first to introduce non-linearities in the ARCH framework were Engle and Bollerslev [1986]. They propose a model where the dynamics of the conditional variance change with the magnitude of squared residuals. The transition between different conditional variance states is controlled be a normal cumulative distribution function.

Higgins and Bera [1992] introduce the Non-linear ARCH (NARCH) model, which encompasses various functional forms for the conditional variance. Their model therefore provides a framework for testing the linear ARCH model against different non-linear alternatives. In their article, the authors derive a Lagrange multiplier statistic for such a test. This test is further developed and analyzed in Bera and Higgins [1992].

In Nelson's [1991] Exponential GARCH (EGARCH) model, the natural logarithm of the conditional variance is modeled as an ARMA process. This solves some problems concerning parameter restrictions in the GARCH model. Furthermore, Nelson introduces a term that makes the conditional variance depend on the sign of lagged residuals. This is motivated by the empirical observation that in some time series there is a correlation between the current conditional variance and lagged returns. Models with this feature are often denoted "asymmetric" or "leverage" volatility models.

Another asymmetric model is the GJR model, proposed by Glosten, Jagannathan, and Runkle [1993]. In the GJR model, the standard GARCH model is extended by letting the parameter for the squared residual have one value when the residual is positive, and another when the residual is negative.

Zakoïan [1994] introduces the Threshold ARCH (TARCH) model. In this model the functional form is the same as in the GJR model, but instead of modeling the conditional variance, Zakoïan models the conditional standard deviation. The TARCH model is developed further in Rabemananjara and Zakoïan [1993].

Ding, Granger, and Engle [1993] present the Asymmetric Power ARCH, a model characterized by a large degree of flexibility. In fact, ARCH, GARCH, NARCH, GJR and TARCH are included in the model as special cases.

A recent asymmetric model is the volatility switching (SV) model presented by Fornari and Mele [1996b]. In the SV model, the GARCH equation is augmented by a term that captures *mean reversion* in conditional variance. Mean reversion refers to the observation that when the conditional variance is high and the residual is smaller than expected, the conditional variance will tend to decrease, and when the conditional variance is low and the residual is larger than expected, the conditional variance often increases.

The model class presented in this paper is inspired to a large extent by the Smooth Transition Autoregressive (STAR) model of Luukkonen, Saikkonen, and Teräsvirta [1988]. In the STAR model, the conditional mean is a non-linear function of lagged realizations of the series introduced via a transition function. Two commonly used transition functions are the logistic (LSTAR) and the exponential (ESTAR) ( see Teräsvirta [1994]). In the non-linear ARCH model presented in this paper, the conditional variance is a non-linear function of lagged residuals. As in the STAR models, the non-linearity is, introduced via either a logistic or an exponential transition function. This gives rise two to different models: the logistic and the exponential smooth transition ARCH model.

In the logistic smooth transition ARCH model, the conditional variance will have dynamics similar to those of the GJR model. The GJR model will obtain as a limiting case of the logistic model. In fact, the GJR model will result if the logistic function is replaced by the Heaviside function. The extra flexibility in the model presented in this paper is accomplished with the introduction of one more parameter.

In the exponential smooth transition ARCH model, the dynamics of the conditional variance are independent of the sign of lagged residuals. Instead, the magnitude of lagged squared residuals control the conditional variance. This specification is similar to that proposed in Engle and Bollerslev [1986] (eq. 36). However, the transition function in Engle and Bollerslev's model is the normal cumulative distribution function. In my model it is the exponential function which means that specification tests are easier to derive.

A much debated subject in the ARCH/GARCH literature is the distributional assumptions for the innovations (see e.g. Teräsvirta [1996]). In the ARCH model of Engle [1982], residuals are assumed to be normally distributed. Bollerslev [1987] introduces Student-t distributed innovations with his GARCH-t model. In Nelson [1991], innovations are assumed to be drawn from a Generalized Error Distribution. However, this paper will not focus on these issues. For simplicity, innovations will be assumed to be normally distributed.

The empirical analysis of stock index series should not be considered as a complete investigation of the possible data generating processes; it is included only as an illustrative example. A empirical comparison of some of the asymmetric models mentioned above is given in Fornari and Mele [1996a].

In Section 2 the model is described, and parameter restrictions are given to guarantee stationarity of the return series and non-negativity of the conditional variance. Section 3 considers specification tests. In Section 4 parameter estimation is briefly discussed. Empirical examples of both the specification tests and estimation are given in Section 5, and Section 6 presents the conclusions.

## 2 The Model

The return of an asset is assumed to be generated by the process

$$r_t = \varepsilon_t. \tag{1}$$

The error term, or the residual, is assumed to have the following form

$$\varepsilon_t = z_t h_t^{1/2}, \tag{2}$$

with $z_t$ ~nid(0,1), and $h_t$ being the conditional variance at time $t$. In the ARCH($q$) model of Engle [1982], the conditional variance is given by the process

$$h_t = \gamma + \sum_{j=1}^{q} \alpha_j \varepsilon_{t-j}^2, \tag{3}$$

where $\gamma$ and $\alpha_j$ $(j = 1, .., q)$ are non-negative constants, with at least one $\alpha_j > 0$. The return process will be stationarity if $\sum_{j=1}^{q} \alpha_j < 1$, in which case the unconditional variance is $\gamma/(1 - \sum_{j=1}^{q} \alpha_j)$, Milhøj [1985].

In the standard ARCH model (3), the conditional variance is a linear function of lagged squared error terms. In this study, this linearity condition is relaxed, and the consequences of such an extension of the ARCH model are investigated. As will be clear below, I focus on a fairly narrow class of models, which is believed have attractive features.

Assume that $\{\alpha_j\}_{j=1}^{q}$ in (3) are not constants, but functions of lagged error terms, according to $\alpha_j = \alpha_j(\varepsilon_{t-j})$. This implies the following equation for the conditional variance

$$h_t = \gamma + \sum_{j=1}^{q} \alpha_j(\varepsilon_{t-j}) \varepsilon_{t-j}^2,$$

The functional form proposed for $\alpha_j(\varepsilon_{t-j})$ is

$$\alpha_j(\varepsilon_{t-j}) = \alpha_{1j} + \alpha_{2j} F(\varepsilon_{t-j}),$$

where $\alpha_{1j}$ and $\alpha_{2j}$ are constants, and $F(.)$ is a transition function. The proposed smooth transition ARCH model will therefore have the form

$$h_t = \gamma + \sum_{j=1}^{q} \left[\alpha_{1j} + \alpha_{2j} F(\varepsilon_{t-j})\right] \varepsilon_{t-j}^2. \tag{4}$$

Below, two specific transition functions will be considered, one logistic and one exponential. Both these functions have simple expressions for the derivatives with respect to the innovations $\varepsilon_{t-j}$. This will prove to be advantageous when specification tests are derived. The logistic function considered has the form

$$F(\varepsilon_{t-j}) = (1 + \exp[-\theta\varepsilon_{t-j}])^{-1} - \frac{1}{2}, \ \theta > 0, \tag{5}$$

and the exponential function is

$$F(\varepsilon_{t-j}) = 1 - \exp\left[-\theta\varepsilon_{t-j}^2\right], \ \theta > 0. \tag{6}$$

The two functions will generate quite different dynamics for the conditional variance. The logistic function (5) will generate a return process where the dynamics of the conditional variance differ depending on the sign of innovations. A related non-linear model is the model of Glosten, Jagannathan, and Runkle [1993]. In the GJR model, the conditional variance follows one process when the innovations are positive and another process when the innovations are negative. In this model, however, the transition between states is smooth. For $\varepsilon_{t-j} \to -\infty$ the transition function will be equal to $-1/2$, and when $\varepsilon_{t-j} \to +\infty$ the transition function will be equal to $1/2$.

The exponential function (6) is symmetric with respect to the sign of the error term. This transition function will generate data for which the dynamics of the conditional variance depend on the magnitude of the innovations. For $|\varepsilon_{t-j}| \to \infty$ the transition function will be equal to unity, and when $\varepsilon_{t-j} = 0$ the transition function is equal to zero.

To derive conditions for non-negativity of the conditional variance and stationarity of the return series, results of Milhøj [1985] and Tjøstheim [1986] are used and it is noted that for (5), $-1/2 \leq F(.) \leq 1/2$, and for (6), $0 \leq F(.) \leq 1$. Sufficient conditions for strictly positive conditional variance in the logistic smooth transition ARCH model are

$$\begin{aligned} \gamma &> 0 \\ \alpha_{1j} &\geq 0 \quad (j = 1, .., q) \\ \alpha_{1j} &\geq \frac{1}{2}|\alpha_{2j}| \quad (j = 1, .., q). \end{aligned}$$

For stationarity of the return process it is required

$$\sum_{j=1}^{q}[\alpha_{1j} - \frac{1}{2}|\alpha_{2j}| + \max(\alpha_{2j}, 0)] < 1.$$

Sufficient conditions for strictly positive conditional variance in the exponential smooth transition ARCH model are

$$\begin{aligned} \gamma &> 0 \\ \alpha_{1j} &\geq 0 \quad (j = 1, .., q) \\ \alpha_{1j} + \alpha_{2j} &\geq 0 \quad (j = 1, .., q). \end{aligned}$$

For stationarity of the return process it is required

$$\sum_{j=1}^{q}[\alpha_{1j} + \max(\alpha_{2j}, 0)] < 1.$$

For the smooth transition ARCH process to be defined, it is required that at least one $\alpha_{ij} > 0$.

A natural extension of the smooth transition ARCH model is to include lagged conditional variances in the equation for the conditional variance, as is done by Bollerslev [1986] for the ARCH($q$) model. The standard GARCH($q,p$) model of Bollerslev is in our notation equal to

$$h_t = \gamma + \sum_{j=1}^{q} \alpha_{1j} \varepsilon_{t-j}^2 + \sum_{i=1}^{p} \beta_i h_{t-i}, \tag{7}$$

where the inequality conditions $\gamma > 0$, $\alpha_{1j} \geq 0$ $(j = 1,..,q)$, and $\beta_i \geq 0$ $(i = 1,..,p)$, are imposed to ensure that the conditional variance is strictly positive. The return process will be stationary if

$$\sum_{j=1}^{q} \alpha_{1j} + \sum_{i=1}^{p} \beta_i < 1,$$

in which case the unconditional variance is $\gamma/(1 - \sum_{j=1}^{q} \alpha_j - \sum_{i=1}^{q} \beta_i)$, Bollerslev [1986]. The smooth transition GARCH($q,p$) model proposed is given by

$$h_t = \gamma + \sum_{j=1}^{q} \left[\alpha_{1j} + \alpha_{2j} F\left(\varepsilon_{t-j}\right)\right] \varepsilon_{t-j}^2 + \sum_{i=1}^{p} \beta_i h_{t-i}, \tag{8}$$

where $F(.)$ is either of the form (5) or (6). The two resulting models will be termed the logistic and the exponential smooth transition GARCH model. A model similar to the logistic smooth transition GARCH model has been independently proposed by González-Rivera [1996].

The GARCH model can parsimoniously represent a higher order ARCH model. Therefore, specification (8) has the advantage that it will generally require fewer parameters than the smooth transition ARCH model. Using the parameter restrictions of the GARCH model (7), in conjunction with the properties that for (5), $-1/2 \leq F(.) \leq 1/2$, and for (6), $0 \leq F(.) \leq 1$, sufficient parameter restriction in model (8) can be derived. For positive conditional variance in the logistic smooth transition GARCH model it is required that

$$\begin{aligned} \gamma &> 0 \\ \alpha_{1j} &\geq 0 \ (j = 1,..,q) \\ \beta_i &\geq 0 \ (j = 1,..,p) \\ \alpha_{1j} &\geq \frac{1}{2} \left|\alpha_{2j}\right|. \end{aligned}$$

For stationarity of the return process it is required

$$\sum_{j=1}^{q} [\alpha_{1j} - \frac{1}{2} \left|\alpha_{2j}\right| + \max(\alpha_{2j}, 0)] + \sum_{i=1}^{p} \beta_i < 1.$$

For positive conditional variance in the exponential smooth transition GARCH model it is required that

$$\begin{aligned} \gamma &> 0 \\ \alpha_{1j} &\geq 0 \ (j = 1,..,q) \\ \beta_i &\geq 0 \ (j = 1,..,p) \\ \alpha_{1j} + \alpha_{2j} &\geq 0. \end{aligned}$$

For stationarity of the return process it is required

$$\sum_{j=1}^{q}[\alpha_{1j} + \max(\alpha_{2j}, 0)] + \sum_{i=1}^{p}\beta_i < 1.$$

For the smooth transition GARCH process to be defined it is required that at least one $\alpha_{ij} > 0$.

# 3 Specification Tests

The main purpose of this section is to present procedures for testing the null of linear conditional variance against the alternative of non-linear conditional variance. However, before testing such a hypothesis, it is natural to first test the null of constant variance against the alternative of smooth transition ARCH or smooth transition GARCH, as specified in (4) or (8). This section therefore begins with a description of a test of no smooth transition ARCH. The test procedure is similar to Engle's [1982] test of no ARCH. Following that, a test procedure that can discriminate between a linear and a non-linear ARCH model is presented. Finally, tests are derived which are to be used when the smooth transition GARCH model is considered. As will be clear below the test statistics presented will differ, depending on the functional form chosen for the transition function.

Suppose we have an observed time series $r_{-q+1}, .., r_0, r_1, ..., r_T$. Let $\mathbf{w}_t' = (1, \varepsilon_{t-1}^2, ..., \varepsilon_{t-q}^2, F(\varepsilon_{t-1})\varepsilon_{t-1}^2, ..., F(\varepsilon_{t-q})\varepsilon_{t-q}^2)$, and $\boldsymbol{\alpha}' = (\gamma, \alpha_{11}, ..., \alpha_{1q}, \alpha_{21}, ..., \alpha_{2q})$. Using this notation the conditional variance equation (4) can be written

$$h_t = \mathbf{w}_t'\boldsymbol{\alpha}. \tag{9}$$

To detect ARCH effects, it is necessary to test the null hypothesis that in (9) $H_0 : \alpha_{1j} = \alpha_{2j} = 0$ $(j = 1, .., q)$, against $H_1$ : at least one $\alpha_{ij} \neq 0$. Because (9) is constant under the null, it is natural to apply the Lagrange multiplier principle. The Lagrange multiplier test of no smooth transition ARCH is

$$\frac{1}{2}\left\{\sum_{t=1}^{T}\left[\frac{\varepsilon_t^2}{\widetilde{\sigma}^2} - 1\right]\mathbf{w}_t\right\}'\left\{\sum_{t=1}^{T}\mathbf{w}_t\mathbf{w}_t'\right\}^{-1}\left\{\sum_{t=1}^{T}\left[\frac{\varepsilon_t^2}{\widetilde{\sigma}^2} - 1\right]\mathbf{w}_t\right\}, \tag{10}$$

where $\widetilde{\sigma}^2 = T^{-1}\sum_{t=1}^{T}\varepsilon_t^2$. For a derivation of equation (10), see the appendix.

However, the vector $\mathbf{w}_t$ in (10) is dependent on the transition function, which under the null has an unidentified parameter $\theta$. Therefore, the test statistic is not operational as stated in equation (10). Following Luukkonen, Saikkonen, and Teräsvirta [1988], this problem is solved by making a Taylor expansion of the transition function around zero. The obtained approximation of $F(.)$ is then inserted into the vector $\mathbf{w}_t$.

If the transition function is logistic, as specified in (5), it is possible to approximate $F(x)$ by

$$T_l = F'(0)x = \frac{\theta}{4}x.$$

The vector $\mathbf{w}_t'$ can therefore be approximated by

$$\widehat{\mathbf{w}}_t' = (1, \varepsilon_{t-1}^2, ..., \varepsilon_{t-q}^2, \frac{\theta}{4}\varepsilon_{t-1}^3, ..., \frac{\theta}{4}\varepsilon_{t-q}^3).$$

Substituting $\widehat{\mathbf{w}}_t'$ into (9) yields

$$h_t = \widehat{\mathbf{w}}_t' \boldsymbol{\alpha}. \tag{11}$$

Furthermore, equation (11) can be reformulated and reparameterized as

$$h_t = \widehat{\boldsymbol{\omega}}_t' \widehat{\boldsymbol{\alpha}}, \tag{12}$$

where $\widehat{\boldsymbol{\omega}}_t = (1, \varepsilon_{t-1}^2, ..., \varepsilon_{t-q}^2, \varepsilon_{t-1}^3, ..., \varepsilon_{t-q}^3)$ and $\widehat{\boldsymbol{\alpha}} = (\gamma, \alpha_{11}, ..., \alpha_{1q}, \alpha_{21}\theta/4, ..., \alpha_{21}\theta/4)$. The null hypothesis of no ARCH is reformulated as $H_0 : \alpha_{1j} = \alpha_{2j}\theta/4 = 0$ $(j = 1, .., q)$, against $H_1$ : at least one $\alpha_{1j}$ or one $\alpha_{2j}\theta/4$ is different from zero. This hypothesis can now be tested with the Lagrange multiplier principle. The test statistic is derived using the same techniques as above, and is equal to

$$\frac{1}{2}\left\{\sum_{t=1}^{T}\left[\frac{\varepsilon_t^2}{\widetilde{\sigma}^2}-1\right]\widehat{\boldsymbol{\omega}}_t\right\}'\left\{\sum_{t=1}^{T}\widehat{\boldsymbol{\omega}}_t\widehat{\boldsymbol{\omega}}_t'\right\}^{-1}\left\{\sum_{t=1}^{T}\left[\frac{\varepsilon_t^2}{\widetilde{\sigma}^2}-1\right]\widehat{\boldsymbol{\omega}}_t\right\}. \tag{13}$$

Following the arguments of Engle [1982], it is more convenient however to perform the following test procedure that yields an asymptotically equivalent statistic:

1. Compute the residual sum of squares $SSR_0 = \sum_{t=1}^{T} \varepsilon_t^2$.

2. Regress $\varepsilon_t^2$ on $\widehat{\boldsymbol{\omega}}_t$, and compute the residual sum of squares from the regression, $SSR_3$.

3. Compute the test statistic

$$LM_1 = T \cdot \frac{SSR_0 - SSR_3}{SSR_0}. \tag{14}$$

When $H_0$ is valid, $LM_1$ has an asymptotic $\chi^2$ distribution with $2 \cdot q$ degrees of freedom.[1] The statistic (14) can alternatively be written as $T \cdot R_u^2$, where $R_u^2$ denotes the coefficient of multiple correlation from an OLS estimation of the artificial model

$$\varepsilon_t^2 = \widehat{\boldsymbol{\omega}}_t'\mathbf{a} + \xi_t.$$

If the transition function is exponential, as specified in (6), $F(x)$ can be approximated by

$$T_e = F''(0)x^2 = 2\theta x^2.$$

The vector $\mathbf{w}_t$ can therefore be approximated by

$$\widetilde{\mathbf{w}}_t' = (1, \varepsilon_{t-1}^2, ..., \varepsilon_{t-q}^2, 2\theta\varepsilon_{t-1}^4, ..., 2\theta\varepsilon_{t-q}^4).$$

This suggests the following test procedure:

1. Compute the residual sum of squares $SSR_0 = \sum_{t=1}^{T} \varepsilon_t^2$.

[1] The asymptotic equivalence of (13) and (14) is shown by noting that under normality

$$plim \sum_{t=1}^{T}\left[\frac{\varepsilon_t^2}{\widetilde{\sigma}^2}-1\right]^2 = 2T.$$

2. Regress $\varepsilon_t^2$ on $\widetilde{\boldsymbol{\omega}}_t = (1, \varepsilon_{t-1}^2, ..., \varepsilon_{t-q}^2, \varepsilon_{t-1}^4, ..., \varepsilon_{t-q}^4)$, and compute the residual sum of squares from the regression, $SSR_4$.

3. Compute the test statistic

$$LM_2 = T \cdot \frac{SSR_0 - SSR_4}{SSR_0} = T \cdot R_u^2, \tag{15}$$

which under the null is asymptotically $\chi^2$ distributed with $2 \cdot q$ degrees of freedom.

Again, it can be shown that the two test statistics (10) calculated with $\mathbf{w}_t$ replaced by $\widetilde{\boldsymbol{\omega}}_t$, and (15) are asymptotically equivalent. $R_u^2$ in (15) is the coefficient of multiple correlation from an OLS estimation of the artificial model

$$\varepsilon_t^2 = \widetilde{\boldsymbol{\omega}}_t' \mathbf{a} + \xi_t.$$

Below is a description of a procedure to test the null hypothesis that in (9) $H_0 : \alpha_{2j} = 0\,(j = 1, .., q)$, against $H_1$ : at least one $\alpha_{2j} \neq 0$. A Lagrange multiplier statistic for this test would have the form

$$\frac{1}{2}\left\{\sum_{t=1}^{T} \frac{1}{h_{0t}}\left[\frac{\varepsilon_t^2}{h_{0t}} - 1\right]\mathbf{w}_t\right\}' \left\{\sum_{t=1}^{T} \frac{1}{h_{0t}^2}\mathbf{w}_t\mathbf{w}_t'\right\}^{-1} \times \left\{\sum_{t=1}^{T} \frac{1}{h_{0t}}\left[\frac{\varepsilon_t^2}{h_{0t}} - 1\right]\mathbf{w}_t\right\}, \tag{16}$$

where $h_{0t}$ is the conditional variance under the null. Under the null the conditional variance equation will be given by an ARCH($q$) model. $h_{0t}$ can therefore be obtained by estimating an ARCH($q$) on the data. Following the same arguments as above, (16) can be made operational by replacing the transition function in $\mathbf{w}_t$ by a Taylor expansion of $F(.)$. If the transition function is logistic, the test statistic is equal to

$$\frac{1}{2}\left\{\sum_{t=1}^{T} \frac{1}{h_{0t}}\left[\frac{\varepsilon_t^2}{h_{0t}} - 1\right]\widehat{\boldsymbol{\omega}}_t\right\}' \left\{\sum_{t=1}^{T} \frac{1}{h_{0t}^2}\widehat{\boldsymbol{\omega}}_t\widehat{\boldsymbol{\omega}}_t'\right\}^{-1} \times \left\{\sum_{t=1}^{T} \frac{1}{h_{0t}}\left[\frac{\varepsilon_t^2}{h_{0t}} - 1\right]\widehat{\boldsymbol{\omega}}_t\right\}. \tag{17}$$

The statistic (17) will under the null be asymptotically distributed as $\chi^2$ with $q$ degrees of freedom. In a similar way to what was shown above, it is possible to derive a regression based test statistic, which is asymptotically equivalent to (17). This statistic would be calculated as $T \cdot R_u^2$ from the regression of

$$\left\{\frac{\varepsilon_t^2}{h_{0t}} - 1\right\}$$

on

$$\left\{\frac{1}{h_{0t}}, \frac{\varepsilon_{t-1}^2}{h_{0t}}, ..., \frac{\varepsilon_{t-q}^2}{h_{0t}}, \frac{\varepsilon_{t-1}^3}{h_{0t}}, ..., \frac{\varepsilon_{t-q}^3}{h_{0t}}\right\}.$$

The statistic for the test of ARCH($q$) against the alternative of exponential smooth transition ARCH($q$) will have a form similar to (17), with $\widehat{\boldsymbol{\omega}}_t$ replaced by $\widetilde{\boldsymbol{\omega}}_t$. The test statistic will be asymptotically distributed $\chi^2$ with $q$ degrees of freedom. An asymptotically equivalent $T \cdot R_u^2$ statistic can also be derived in this case.

Note, that the method of replacing the transition function with a suitable approximation, is not restricted to the case where the transition functions have either the form (5) or (6). Thus, this test procedure can be used for all non-linear ARCH models which have a transition function that can be approximated by a second order Taylor expansion around zero, and for which $F(0) = 0$.

Above, the cases where the transition function is either the logistic or the exponential have been deliberately separated . However, it is possible to test the null of homoskedasticity against the alternative of non-linear ARCH($q$) of both forms simultaneously. This can be done in the regression model of the form

$$\begin{aligned} \varepsilon_t^2 &= a_0 + a_{11}\varepsilon_{t-1}^2 + ... + a_{1q}\varepsilon_{t-q}^2 + a_{21}\varepsilon_{t-1}^3 + ... + a_{2q}\varepsilon_{t-q}^3 + \\ &\quad + a_{31}\varepsilon_{t-1}^4 + ... + a_{3q}\varepsilon_{t-q}^4 + \xi_t. \end{aligned} \tag{18}$$

A $LM$ type test statistic for the hypothesis is calculated as $T \cdot R_u^2$, where $R_u^2$ is calculated from the model (18). The test statistic will be asymptotically distributed $\chi^2$ with $3 \cdot q$ degrees of freedom. Furthermore, it is straightforward to derive a $LM$ statistic for testing the null of logistic smooth transition ARCH($q$) against the alternative of both a logistic and an exponential smooth transition ARCH($q$) model. Likewise, it is possible to test the null of exponential smooth transition non-linear ARCH against the alternative of both a logistic and an exponential transition function. However, both these tests require that the series of conditional variance under the null is estimated. Since it is far more time consuming to estimate a smooth transition ARCH($q$) model than a standard ARCH($q$) model, the tests will be more complicated to perform.

This section will close by describing specification procedures designed to test for smooth transition GARCH, rather than for smooth transition ARCH. Bollerslev [1986] notes that under the null of no heteroskedasticity there is no general Lagrange multiplier test for GARCH($p, q$). This is due to the fact that the Hessian is singular if both $p$>0 and $q$>0. In the smooth transition GARCH model this will also occur. To test for smooth transition GARCH, the same test procedures as described for the ARCH model are proposed, with a fairly large number of lags. This is motivated by the fact that the smooth transition GARCH model can be rewritten as a smooth transition ARCH model with an infinite number of lags.

However, it is straightforward to derive a $LM$ test for the null of GARCH($p, q$) against the alternative of smooth transition GARCH($p, q$). The derivation of such a statistic is performed using techniques similar to those presented in the appendix. For simplicity the description is only of tests of the null of GARCH(1,1) against the alternative of smooth transition GARCH(1,1). Thus, the model considered under the null is

$$h_t = \gamma + [\alpha_{11} + \alpha_{21} F(\varepsilon_{t-1})] \varepsilon_{t-1}^2 + \beta_1 h_{t-1},$$

and the hypothesis to be tested is

$$\begin{aligned} H_0 &: \alpha_{21} = 0 \\ H_1 &: \alpha_{21} \neq 0. \end{aligned}$$

Given that transition function is the logistic (5), and that $\varepsilon_t$ is distributed conditionally normal, a

Lagrange multiplier test statistics for the hypothesis is

$$
\begin{aligned}
LM_3 &= \frac{1}{2}\left\{\sum_{t=1}^{T}\frac{1}{h_{0t}}\left[\frac{\varepsilon_t^2}{h_{0t}}-1\right]\frac{\partial h_t}{\partial\widehat{\boldsymbol{\alpha}}}\right\}'\left\{\sum_{t=1}^{T}\left[\frac{1}{h_{0t}}\frac{\partial h_t}{\partial\widehat{\boldsymbol{\alpha}}}\right]\left[\frac{1}{h_{0t}}\frac{\partial h_t}{\partial\widehat{\boldsymbol{\alpha}}}\right]'\right\}^{-1} \\
&\times\left\{\sum_{t=1}^{T}\frac{1}{h_{0t}}\left[\frac{\varepsilon_t^2}{h_{0t}}-1\right]\frac{\partial h_t}{\partial\widehat{\boldsymbol{\alpha}}}\right\}, \qquad (19)
\end{aligned}
$$

where

$$
\frac{\partial h_t}{\partial\widehat{\boldsymbol{\alpha}}'} = \left[\sum_{i=1}^{t-1}\widehat{\beta}_1^{i-1}, \sum_{i=1}^{t-1}\widehat{\beta}_1^{i-1}\varepsilon_{t-i}^2, \sum_{i=1}^{t-1}\widehat{\beta}_1^{i-1}\varepsilon_{t-i}^3, \sum_{i=1}^{t-1}\widehat{\beta}_1^{i-1}h_{0t-i}\right],
$$

$h_{0t}$ is the conditional variance under the null of GARCH(1,1), $\widehat{\boldsymbol{\alpha}}'$ is the vector of parameters $(\gamma, \alpha_{11}, \alpha_{21}\theta/4, \beta_1)$, and $\widehat{\beta}_1$ is the estimated parameter $\beta_1$ in the GARCH(1,1) model. The statistic $LM_3$ will under the null of GARCH(1,1) be asymptotically distributed $\chi^2$ with one degree of freedom. An asymptotically equivalent statistic would be calculated as $T \cdot R_u^2$ from the regression of

$$
\left\{\frac{\varepsilon_t^2}{h_{0t}}-1\right\}
$$

on

$$
\left\{\frac{\sum_{i=1}^{t-1}\widehat{\beta}_1^{i-1}}{h_{0t}}, \frac{\sum_{i=1}^{t-1}\widehat{\beta}_1^{i-1}\varepsilon_{t-i}^2}{h_{0t}}, \frac{\sum_{i=1}^{t-1}\widehat{\beta}_1^{i-1}h_{0t-i}}{h_{0t}}, \frac{\sum_{i=1}^{t-1}\widehat{\beta}_1^{i-1}\varepsilon_{t-i}^3}{h_{0t}}\right\}.
$$

When the transition function is the exponential (6), the Lagrange multiplier test statistics will be

$$
\begin{aligned}
LM_4 &= \frac{1}{2}\left\{\sum_{t=1}^{T}\frac{1}{h_{0t}}\left[\frac{\varepsilon_t^2}{h_{0t}}-1\right]\frac{\partial h_t}{\partial\widetilde{\boldsymbol{\alpha}}}\right\}'\left\{\sum_{t=1}^{T}\left[\frac{1}{h_{0t}}\frac{\partial h_t}{\partial\widetilde{\boldsymbol{\alpha}}}\right]\left[\frac{1}{h_{0t}}\frac{\partial h_t}{\partial\widetilde{\boldsymbol{\alpha}}}\right]'\right\}^{-1} \\
&\times\left\{\sum_{t=1}^{T}\frac{1}{h_{0t}}\left[\frac{\varepsilon_t^2}{h_{0t}}-1\right]\frac{\partial h_t}{\partial\widetilde{\boldsymbol{\alpha}}}\right\}, \qquad (20)
\end{aligned}
$$

where

$$
\frac{\partial h_t}{\partial\widetilde{\boldsymbol{\alpha}}'} = \left[\sum_{i=1}^{t-1}\widehat{\beta}_1^{i-1}, \sum_{i=1}^{t-1}\widehat{\beta}_1^{i-1}\varepsilon_{t-i}^2, \sum_{i=1}^{t-1}\widehat{\beta}_1^{i-1}\varepsilon_{t-i}^4, \sum_{i=1}^{t-1}\widehat{\beta}_1^{i-1}h_{0t-i}\right],
$$

$h_{0t}$ is the conditional variance under the null of GARCH(1,1), $\widetilde{\boldsymbol{\alpha}}'$ is the vector of parameters $(\gamma, \alpha_{11}, \alpha_{21}2\theta, \beta_1)$, and $\widehat{\beta}_1$ is the estimated parameter $\beta_1$ in the GARCH(1,1) model. When $H_0$ is valid $LM_4$ has an asymptotic $\chi^2$ distribution with one degree of freedom. The asymptotically equivalent statistic is $T \cdot R_u^2$ from the regression of

$$
\left\{\frac{\varepsilon_t^2}{h_{0t}}-1\right\}
$$

on

$$
\left\{\frac{\sum_{i=1}^{t-1}\widehat{\beta}^{i-1}}{h_{0t}}, \frac{\sum_{i=1}^{t-1}\widehat{\beta}^{i-1}\varepsilon_{t-i}^2}{h_{0t}}, \frac{\sum_{i=1}^{t-1}\widehat{\beta}^{i-1}h_{0t-i}}{h_{0t}}, \frac{\sum_{i=1}^{t-1}\widehat{\beta}^{i-1}\varepsilon_{t-i}^4}{h_{0t}}\right\}.
$$

# 4 Estimation

If the conditional mean follows equation (1), and given that the innovations $z_t$ are Gaussian, the parameters of the models are estimated by maximizing the likelihood function $\sum_{t=1}^{T} l_t$, where $l_t$ is given by (22). This is done using standard numerical methods. Since the magnitude of the parameters is quite different in this model, it is recommended that the parameters be scaled, so that the diagonal elements of the Hessian are roughly equal. It is also advisable to set the starting value of the parameter $\theta$ in a region where the transition function will not just take on the extreme values of the function.

If the conditional mean follows a regression model, the parameters of the conditional variance process should be estimated simultaneously with the parameters of the conditional mean model. However, the procedure can be simplified when the exponential model is considered. Engle [1982] shows that when the conditional variance is symmetric with respect to the innovations, as it is in the exponential smooth transition ARCH model, the two models can be estimated separately. Thus, first the parameters of the conditional mean model are estimated, and then estimate the parameters of the ARCH/GARCH model are estimated on the estimated residuals from the conditional mean model.

# 5 Empirical example

For the empirical analysis, observations used were daily return series from four different equity indexes: the Copenhagen Stock Exchange general index (CGI), the Financial Times all share index (FT-all), the Milan Stock Exchange general index (MGI), and the Stockholm equity index (OMX).[2] The period investigated is January 3, 1991 to July 15, 1996. The number of observations per series is approximately 1400. Returns are calculated as $ln(P_t/P_{t-1})$, where $P_t$ is the index level at the end of day $t$.

The series are first examined for autocorrelation using a test developed by Richardson and Smith [1994]. The test, in the form used here, is a robust version of a standard Box and Pierce [1970] procedure. The statistic is calculated as

$$RS(k) = T \sum_{i=1}^{k} \frac{\widehat{\rho}_i^2}{1 + c_i}, \tag{21}$$

where $\widehat{\rho}_i$ is the estimated autocorrelation between the returns at time $t$ and $t - i$. The terms $c_i$ is an adjustment factor for heteroskedasticity, given by

$$c_i = \frac{cov[\overline{r}_t^2, \overline{r}_{t-i}^2]}{var[r_t]^2},$$

where $\overline{r}_t$ is the demeaned return at time $t$. Under the null of no autocorrelation, the statistic is distributed asymptotically $\chi^2$ with $k$ degrees of freedom.

Richardson and Smith's test (21) was calculated with $k$ equal to five. No autocorrelation was detected for FT-all and MGI, on five percent significance level, while autocorrelation was indicated for CGI and OMX. For CGI and OMX AR(1) models were fitted. To ensure that these models capture the detected

[2] These series are not chosen randomly from the population of equity index series. Rather, they have been selected because they fulfill the purpose of showing how the specification tests work and how estimation results may turn out. However, it was no difficult to find series that suited this purpose.

autocorrelation, the test (21) were applied again, calculated on estimated residuals from the AR(1) models. Following the recommendations of Box and Pierce [1970] and Ljung and Box [1978], the value of the statistic in this case is compared to a $\chi^2$ distribution with $k-1$ degrees of freedom. No further autocorrelation was detected for CGI and OMX, on five percent significance level. It is therefore concluded that for FT-all and MGI a suitable mean equation is

$$r_t = \varphi_0 + \varepsilon_t,$$

and for CGI and OMX the mean specification chosen is

$$r_t = \varphi_0 + \varphi_1 r_{t-1} + \varepsilon_t.$$

After having considered the conditional mean specification, tests are performed regarding the conditional variance. These tests are calculated on estimated residuals from the conditional mean models.[3] Results from these specification tests are showed in Table 1. Column two reports results for Engle's [1982] test of no ARCH, calculated on ten lagged squared residuals. The reported p-values show that it is possible to reject the null of constant variance against heteroskedasticity in the form of linear ARCH for all four series, on five percent significance level. Columns three and four show similar results, with respect to the smooth transition ARCH model. The last two columns report the results from the tests of GARCH(1,1) against smooth transition GARCH(1,1). According to column five, for CGI and MGI it is possible to reject the null of GARCH(1,1) against the alternative of logistic smooth transition GARCH(1,1), on five percent significance level. However, for FT-all and OMX we can reject the null against the alternative of logistic smooth transition GARCH(1,1). Column six shows that only for CGI is it possible to reject the null of GARCH(1,1) against the alternative of exponential smooth transition GARCH(1,1). Thus, it is concluded that for CGI the exponential smooth transition GARCH(1,1) model might be the data generating process of the conditional variance. For FT-all and OMX the conditional variance could have been generated from the logistic model.

For FT-all and OMX the logistic smooth transition GARCH(1,1) model was estimated. The conditional mean specification was estimated simultaneously. The exponential smooth transition GARCH(1,1) model is estimated on CGI. In this case the conditional mean is estimated separately. Since no higher order GARCH effects are detected for MGI, no further estimations are performed on the series. The results from the three estimations are compared to the results from estimations where the standard GARCH(1,1) model is fitted to data. The innovations in all models are assumed to be normally distributed.

Parameter estimates for the models are shown in Table 2. The two last rows give the values on the log-likelihood function and the value on Akaike's information criterion (AIC). All estimated parameters fall into the regions where, given the sufficient conditions in Section 2, it is known that the return processes are stationary and that the conditional variance is non-negative. Since specification tests have been used

[3] Bera, Higgins and Lee [1992] thoroughly analyze the problem of testing for ARCH when the conditional mean is given by an AR model. They suggest that the Lagrange multiplier statistic should be adjusted for possible autoregression. The simplistic test procedure here is motivated by the observation that even though autocorrelation is present in financial time series, any AR model can only explain a very small fraction of observed returns.

to establish the possible data generating processes of the series, standard errors of the estimates have intentionally been left unreported. If the residuals are assumed to be distributed conditionally normal, the standard errors can be computed from the inverse of the Hessian. If normality is not fulfilled, the estimation procedure used is a quasi-maximum likelihood method. In such cases, standard errors can be computed with a method presented by Bollerslev and Wooldridge [1992].

The first two columns of Table 2 report estimation results for CGI. According to AIC, the exponential smooth transition GARCH(1,1) model constitutes an improvement over the standard GARCH(1,1) model in describing the dynamics of the conditional variance. The estimated parameter value for $\theta$ gives a transition function which moves smoothly between zero and one, as can be seen from Figure 2. Figure 3 shows how the conditional variance reacts to different residual values, based on the estimated parameters. From the figure it can be observed that the *news impact curve* for exponential smooth transition GARCH model nearly coincides with that of the GARCH model when residuals are in the interval -1.5 to +1.5 percent.[4] However, for larger absolute returns, the two curves show a different pattern. In the smooth transition GARCH(1,1) case, the relative influence of large absolute returns will be lower then in the GARCH(1,1) case. Only 2.5 percent of all absolute returns are larger than 1.5 percent.

For FT-all the logistic model maximizes the likelihood function and minimizes AIC. As can be seen in Figure 1, the estimated value for $\theta$ will give a transition function that moves very slowly between its extreme values -1/2 and +1/2. In the region where most returns are present, between $-1.5$ and $+1.5$ percent, the transition function seems to be almost linear with respect to the residual value. Figure 4 shows how the conditional variance reacts to different residual values, based on the estimated parameters. As expected, the news impact curve for the logistic model is asymmetric around $\varepsilon_t = 0$. The reaction to negative residual values is much larger than the reaction to positive residuals of the same magnitude. Thus, a leverage effect seems to be present in the return series.

The last two columns of Table 2 report estimation results for OMX. Even in this case, the logistic model maximizes the likelihood function and minimizes the AIC. From Figure 1 it can be observed that the estimated parameter value for $\theta$ gives a transition function which is much more sensitive to different residual values, than what was found for FT-all. When absolute residuals are larger than 1.5 percent, the transition function will always be at its extreme values. 16 percent of all absolute returns in the sample are larger than 1.5 percent. Figure 5 shows the news impact curve for the two models estimated on OMX. As was the case for FT-all, the news impact curve show leverage effects. Thus, large positive returns will increase the conditional variance less than large negative returns.

[4] The news impact curve, as a way of illustrating the level of asymmetry in a heteroskedatic model, was introduced by Engle and Ng [1993].

# 6 Summary and Conclusion

This paper has presented a new class of ARCH model, the smooth transition ARCH model. In these models, the conditional variance is a non-linear function of lagged squared residuals. The non-linearity is introduced by a transition function. Two specific transition function are considered, the logistic and the exponential. These two functions will each give the conditional variance quite different dynamics. The logistic function allows for asymmetric behavior of the conditional variance with respect to the signs of residuals. In the exponential smooth transition model, the dynamics of the conditional variance will differ depending on the absolute size of lagged residuals.

Following the work of Bollerslev [1986] the smooth transition ARCH model is extended to a smooth transition GARCH model. In this model, the conditional variance is a function of both lagged residuals and lagged conditional variances. This formulation is likely to limit the number of parameters needed for a successful estimation of the model.

In Section 3 a number of specification tests for the smooth transition ARCH/GARCH model were presented. Since the estimation of the model requires iterative procedures, these tests are most valuable in a practical situation.

Section 5 contained a short empirical example, where four equity index returns series were estimated. Data is daily and the sample period is from January 1991 to July 1996. Using the specification tests it can be concluded that two of the series could have had the logistic smooth transition GARCH(1,1) model as their data generating process. One conditional variance series might have been generated from an exponential smooth transition GARCH(1,1) model. The models were estimated on the three series that indicated higher order GARCH effects, and these estimates are compared to standard GARCH(1,1) estimates. According to AIC, the smooth transition GARCH(1,1) model constitutes an improvement over the GARCH model. The estimated parameters for the logistic model indicate that large negative residuals increase the conditional variance more than positive residuals. Thus, for these two series the conditional volatility seems to increase in bear markets. The estimation results for the exponential model show that large absolute residuals are given too high an influence in the standard GARCH(1,1) model.

One question that has not been answered in this paper is whether this model can outperform other non-linear ARCH models presented in the literature. This is obviously a question that calls for further research. Another question that needs to be examined more carefully is the parameter restrictions required for stationarity. In Section 2, sufficient conditions for stationarity are provided, but these conditions could be too restrictive. Furthermore, small sample properties of the specification tests need to be analyzed.

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# Appendix

**1. Derivation of LM statistic (10)**

The Lagrange multiplier statistic has the general form

$$LM = T\overline{q}_T(\boldsymbol{\alpha}_0)' I(\boldsymbol{\alpha}_0)^{-1}\overline{q}_T(\boldsymbol{\alpha}_0),$$

where $\boldsymbol{\alpha}_0$ is the vector of parameters under the null. $\overline{q}_T(\boldsymbol{\alpha})$ is the average score and $I(\boldsymbol{\alpha})$ is the information matrix. If the innovations are assumed to be Gaussian, the log likelihood of one observation is equal to

$$l_t = -\frac{1}{2}\ln 2\pi - \frac{1}{2}\ln h_t - \frac{1}{2}\frac{\varepsilon_t^2}{h_t}. \tag{22}$$

It is then straightforward to show that the average score is equal to

$$\overline{q}_T(\boldsymbol{\alpha}) = \frac{1}{T}\sum_{t=1}^{T}\frac{1}{2h_t}\left[\frac{\varepsilon_t^2}{h_t} - 1\right]\mathbf{w}_t.$$

The information matrix is the negative expectation of the Hessian averaged over all observations

$$I(\boldsymbol{\alpha}) = -E\left[\frac{1}{T}\sum_{t=1}^{T}\frac{\partial^2 l_t}{\partial\boldsymbol{\alpha}\partial\boldsymbol{\alpha}'}\right].$$

The Hessian for one observation can be shown to be equal to

$$\frac{\partial^2 l_t}{\partial\boldsymbol{\alpha}\partial\boldsymbol{\alpha}'} = -\frac{1}{2h_t^2}\left[\frac{\varepsilon_t^2}{h_t}\right]\mathbf{w}_t\mathbf{w}_t' - \frac{1}{2}\frac{1}{h_t^2}\left[\frac{\varepsilon_t^2}{h_t} - 1\right]\mathbf{w}_t\mathbf{w}_t',$$

which implies that the information matrix becomes

$$I(\boldsymbol{\alpha}) = \sum_{t=1}^{T}\frac{1}{2T}E\left[\frac{1}{h_t^2}\mathbf{w}_t\mathbf{w}_t'\right].$$

The information matrix is consistently estimated by

$$\widehat{I}(\boldsymbol{\alpha}) = \frac{1}{2T}\sum_{t=1}^{T}\left[\frac{1}{h_t^2}\mathbf{w}_t\mathbf{w}_t'\right].$$

Now consider the average score under the null, when the conditional variance is constant. Let $\sigma^2$ denote the constant conditional variance under the null. The average score can then be written

$$\overline{q}_T(\boldsymbol{\alpha}_0) = \frac{1}{2T\sigma^2}\sum_{t=1}^{T}\left[\frac{\varepsilon_t^2}{\sigma^2} - 1\right]\mathbf{w}_t,$$

and the consistently estimated information matrix under the null is equal to

$$\widehat{I}(\boldsymbol{\alpha}_0) = \frac{1}{2T\left(\sigma^2\right)^2}\sum_{t=1}^{T}\mathbf{w}_t\mathbf{w}_t'.$$

The Lagrange multiplier test of no smooth transition ARCH can therefore be written

$$T\left\{\frac{1}{2T\widetilde{\sigma}^2}\sum_{t=1}^{T}\left[\frac{\varepsilon_t^2}{\widetilde{\sigma}^2} - 1\right]\mathbf{w}_t\right\}'\left\{\frac{1}{2T\left(\widetilde{\sigma}^2\right)^2}\sum_{t=1}^{T}\mathbf{w}_\mathbf{t}\mathbf{w}_t'\right\}^{-1}$$
$$\times\left\{\frac{1}{2T\widetilde{\sigma}^2}\sum_{t=1}^{T}\left[\frac{\varepsilon_t^2}{\widetilde{\sigma}^2} - 1\right]\mathbf{w}_t\right\}, \tag{23}$$

where $\widetilde{\sigma}^2 = T^{-1}\sum_{t=1}^{T}\varepsilon_t^2$. Formula (23) can be simplified to

$$\frac{1}{2}\left\{\sum_{t=1}^{T}\left[\frac{\varepsilon_t^2}{\widetilde{\sigma}^2}-1\right]\mathbf{w}_t\right\}'\left\{\sum_{t=1}^{T}\mathbf{w_t}\mathbf{w}_t'\right\}^{-1}\left\{\sum_{t=1}^{T}\left[\frac{\varepsilon_t^2}{\widetilde{\sigma}^2}-1\right]\mathbf{w}_t\right\},$$

which corresponds to formula (10). ■

**Table 1. Results from specification tests**

The table reports p-values for specification tests performed on the four series of estimated residuals. For the Financial Times all share index and for the Milan general index, the residuals are calculated as returns minus mean return. For the Copenhagen general index and for the Stockholm OMX index, residuals are from an AR(1) model. The investigated period is January 3, 1991 to July 15, 1996. The column labeled No ARCH gives the results for Engle's [1982] test of no ARCH, calculated on ten lagged squared residuals. The statistic has an approximate $\chi^2(10)$ distribution under the null. The column labeled $LM_1$ reports the results from the test of no ARCH, against the alternative of smooth transition ARCH with a logistic transition function, as specified in equation (14), calculated on ten lagged residuals. The column labeled $LM_2$ reports the results from the test of no ARCH, against the alternative of smooth transition ARCH with an exponential transition function, as specified in equation (15), calculated on ten lagged residuals. The column labeled $LM_3$ reports the results from the test of GARCH(1,1), against the alternative of smooth transition GARCH(1,1) with a logistic transition function, as specified in equation (19). The column labeled $LM_4$ reports the results from the test of GARCH(1,1), against the alternative of smooth transition GARCH(1,1) with an exponential transition function, as specified in equation (20).

| Index | No ARCH | $LM_1$ | $LM_2$ | $LM_3$ | $LM_4$ |
|---|---|---|---|---|---|
| Copenhagen | $1.85 \cdot 10^{-17}$ | $1.84 \cdot 10^{-20}$ | $3.42 \cdot 10^{-23}$ | 0.7762 | 0.0156 |
| FT-all share | $3.28 \cdot 10^{-21}$ | $4.82 \cdot 10^{-27}$ | $4.35 \cdot 10^{-33}$ | 0.0461 | 0.6547 |
| Milan | $9.52 \cdot 10^{-61}$ | $4.43 \cdot 10^{-83}$ | $1.24 \cdot 10^{-84}$ | 0.3142 | 0.8617 |
| OMX | $9.49 \cdot 10^{-18}$ | $2.54 \cdot 10^{-17}$ | $5.12 \cdot 10^{-19}$ | 0.0123 | 0.4348 |

**Table 2. Results from estimations**

The table shows results from estimations of the model

$$r_t = \varphi_0 + \varphi_1 r_{t-1} + \varepsilon_t$$

with $\varepsilon_t \sim N(0, h_t)$. Three models for $h_t$ are estimated: the logistic smooth transition GARCH(1,1) model

$$h_t = \gamma + \left(\alpha_{11} + \alpha_{21}\left[(1+\exp\left[-\theta\varepsilon_{t-1}\right])^{-1} - \frac{1}{2}\right]\right)\varepsilon_{t-1}^2 + \beta_1 h_{t-1},$$

the exponential smooth transition GARCH(1,1) model

$$h_t = \gamma + \left(\alpha_{11} + \alpha_{21}\left[1-\exp\left[-\theta\varepsilon_{t-1}^2\right]\right]\right)\varepsilon_{t-1}^2 + \beta_1 h_{t-1},$$

and the standard GARCH(1,1) model

$$h_t = \gamma + \alpha_{11}\varepsilon_{t-1}^2 + \beta_1 h_{t-1}.$$

The three estimated series are daily returns for the Copenhagen Stock Exchange general index, the Financial Times all-share index , and the Stockholm equity index (OMX). The investigated period is January 3, 1991 to July 15, 1996. For FT-all no autocorrelation in returns was detected, and therefore $\varphi_1$ is excluded in the estimation.The row labeled LL gives the value of the log-likelihood function.

| | Copenhagen | | FT-all share index | | OMX | |
|---|---|---|---|---|---|---|
| | EST-GARCH | GARCH | LST-GARCH | GARCH | LST-GARCH | GARCH |
| $\varphi_0$ | $1.67{\cdot}10^{-4}$ | $1.67{\cdot}10^{-4}$ | $3.97{\cdot}10^{-4}$ | $4.32{\cdot}10^{-4}$ | $4.19{\cdot}10^{-4}$ | $6.34{\cdot}10^{-4}$ |
| $\varphi_1$ | 0.264 | 0.264 | - | - | 0.126 | 0.126 |
| $\gamma$ | $7.74{\cdot}10^{-6}$ | $9.21{\cdot}10^{-6}$ | $3.82{\cdot}10^{-7}$ | $4.46{\cdot}10^{-7}$ | $4.61{\cdot}10^{-6}$ | $6.03{\cdot}10^{-6}$ |
| $\alpha_{11}$ | 0.215 | 0.158 | 0.037 | 0.036 | 0.086 | 0.086 |
| $\alpha_{21}$ | -0.166 | - | -0.050 | - | -0.094 | - |
| $\beta_1$ | 0.612 | 0.580 | 0.956 | 0.955 | 0.884 | 0.870 |
| $\theta$ | 2128 | - | 43.9 | - | 430 | - |
| LL | 5223.5 | 5219.7 | 4880.1 | 4878.1 | 4290.7 | 4281.7 |
| AIC | -10437.0 | -10433.5 | -9748.3 | -9748.1 | -8567.5 | -8553.4 |

**Figure 1. Logistic transition function for the Financial Times all share index and the OMX index**

The figure shows the value of the logistic transition function

$$F(\varepsilon_{t-j}) = (1 + \exp[-\theta\varepsilon_{t-j}])^{-1} - \frac{1}{2},\ \theta > 0$$

for different values on the residual $\varepsilon_{t-j}$. The transition function for FT-all is created with the estimated parameter value $\theta = 43.9$. The transition function for OMX is created with the estimated parameter value $\theta = 430$.

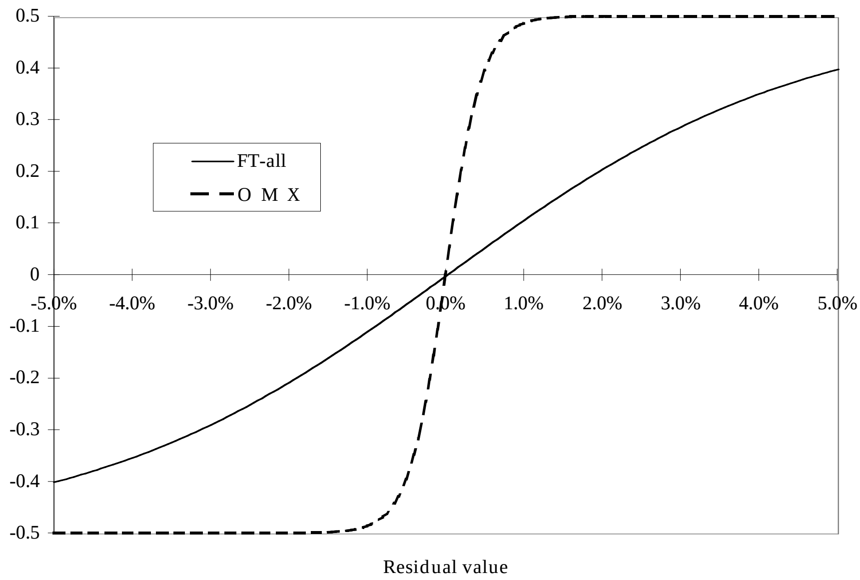

**Figure 2. Exponential transition function for the Copenhagen Stock Exchange general index**

The figure shows the value of the exponential transition function

$$F(\varepsilon_{t-j}) = 1 - \exp\left[-\theta\varepsilon_{t-j}^2\right], \ \theta > 0$$

for different values on the residual $\varepsilon_{t-j}$. The transition function is created with the estimated parameter value $\theta = 2128$.

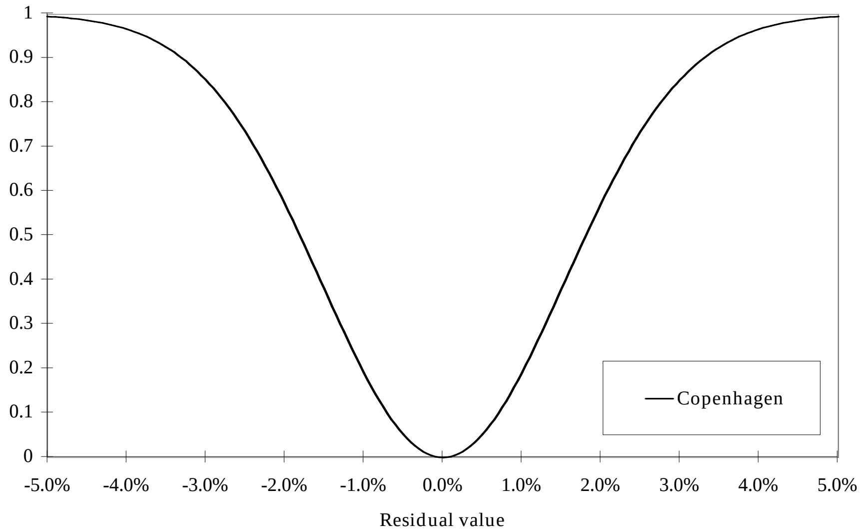

**Figure 3. News impact curves for the Copenhagen Stock Exchange general index**

The figure shows how the conditional variance reacts to different values on lagged residuals. The curves are created from the estimated parameter values given in Table 2. The ESTGARCH(1,1) curve is created with the formula

$$h_t = \gamma + \left(\alpha_{11} + \alpha_{21}\left[1 - \exp\left[-\theta\varepsilon_{t-1}^2\right]\right]\right)\varepsilon_{t-1}^2 + \beta_1 h_{t-1},$$

and the GARCH(1,1) curve with the formula

$$h_t = \gamma + \alpha_{11}\varepsilon_{t-1}^2 + \beta_1 h_{t-1}.$$

The initial conditional variance in all curves is equal to the unconditional variance for the GARCH(1,1) model, $3.52{\cdot}10^{-5}$.

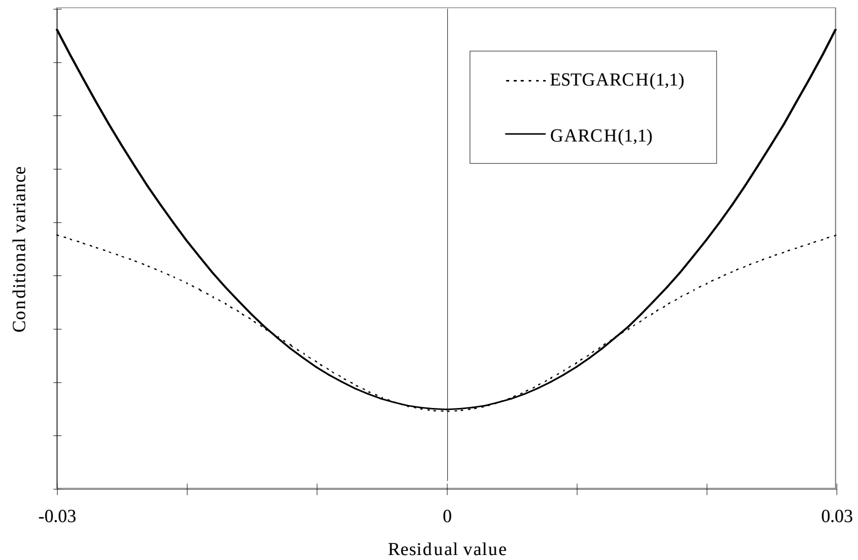

**Figure 4. News impact curves for the Financial Times all share index**

The figure shows how the conditional variance reacts to different values on lagged residuals. The curves are created from the estimated parameter values given in Table 2. The LSTGARCH(1,1) curve is created with the formula

$$h_t = \gamma + \left(\alpha_{11} + \alpha_{21}\left[(1 + \exp[-\theta\varepsilon_{t-1}])^{-1} - \frac{1}{2}\right]\right)\varepsilon_{t-1}^2 + \beta_1 h_{t-1},$$

and the GARCH(1,1) curve with the formula

$$h_t = \gamma + \alpha_{11}\varepsilon_{t-1}^2 + \beta_1 h_{t-1}.$$

The initial conditional variance in all curves is equal to the unconditional variance for the GARCH(1,1) model, $5.13 \cdot 10^{-5}$.

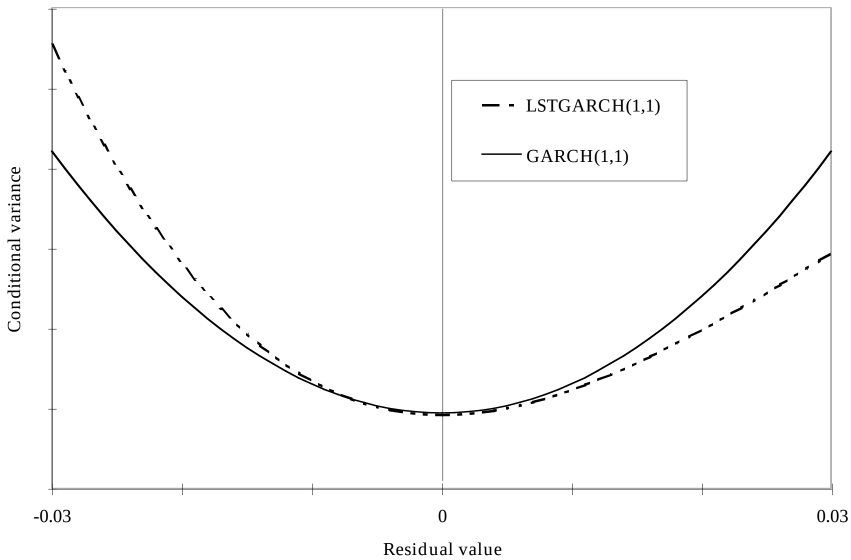

**Figure 5. News impact curves for the OMX index**

The figure shows how the conditional variance reacts to different values on lagged residuals. The curves are created from the estimated parameter values given in Table 2. The LSTGARCH(1,1) curve is created with the formula

$$h_t = \gamma + \left(\alpha_{11} + \alpha_{21}\left[(1+\exp\left[-\theta\varepsilon_{t-1}\right])^{-1} - \frac{1}{2}\right]\right)\varepsilon_{t-1}^2 + \beta_1 h_{t-1},$$

and the GARCH(1,1) curve with the formula

$$h_t = \gamma + \alpha_{11}\varepsilon_{t-1}^2 + \beta_1 h_{t-1}.$$

The initial conditional variance in all curves is equal to the unconditional variance for the GARCH(1,1) model, $1.42 \cdot 10^{-4}$.