An Extension of Good-Deal Asset Price Bounds*

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Abstract

In a two-period setup we develop a generalization of good-deal bounds that allows to include in the problem the implications of asset pricing models. Our basis is the distance behind Hansen and Jagannathan's measure of model misspecification since a volatility constraint on the stochastic discount factor is a particular case of a restriction on this distance. We also present an alternative approach which mostly retains the economic interpretation underlying the above extension and it has a very useful property since the resulting bounds can be computed by simply solving a linear program.

1 Introduction

A new approach in the field of asset pricing theory has recently been developed by Cochrane and Saá-Requejo (2000) and Bernardo and Ledoit (2000). These two seminal papers introduce a novel way to deal with the valuation

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of uncertain payoffs which lies between the rigidities of model-based pricing and the looseness of no-arbitrage pricing.

The interest of this new approach can easily be illustrated with a simple example. Suppose we have a stock and a bond as basis assets and our goal is to price a European call option on the stock with strike price 100 and one year to expiration. The one-year interest rate is 5% and the annual volatility of the compounded rate of return of the stock is 14.8%. Figure 1 plots in the horizontal axis the initial stock price and in the vertical axis the corresponding call price.

By assuming absence of arbitrage, the information contained in the prices of this initial set can be used to establish intervals where the price of the call option must lie. Unfortunately, the available basis payoffs, like in most real-life situations, only span a small subset of all possible contingent claims. Thus, the pricing implications of this assumption only give loose bounds. After all, absence of arbitrage is a very weak assumption. It only implies the existence of at least one agent whose utility function has no satiation point.

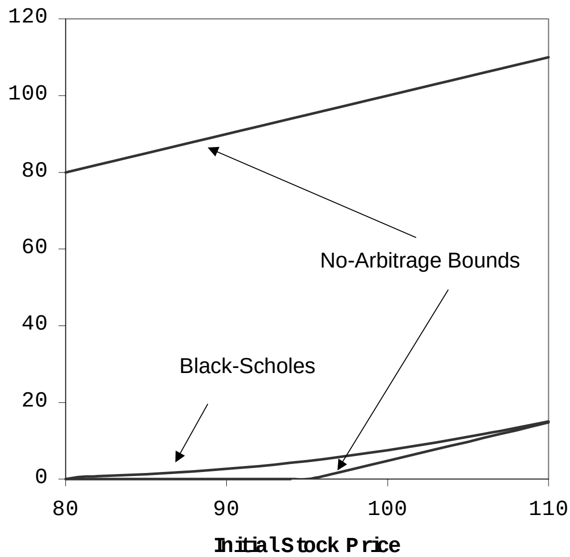

Figure 1. No-arbitrage bounds and Black-Scholes price of a European call option. The option has one year to expiration and a strike price of 100. The one-year risk-free rate of return is 5% and the standard deviation of the continuously compounded rate of return on the stock is 14.09%.

Alternatively, one could embrace the Black-Scholes model and obtain exact prices for the call option. However, the pricing implications of the latter approach are highly sensitive to specification errors and we all know that asset pricing models are always misspecified when applied to real data.

As we can see these two approaches seem a little too extreme. Thus, it is of great interest to develop methods to limit the size of the no-arbitrage intervals without fully assuming a particular asset pricing model. In modern financial jargon, the goal is to tighten the set of admissible stochastic discount factors (hereafter ASDF) without postulating a unique one.

Bernardo and Ledoit (2000) define the gain-loss ratio. They look at excess (zero-price) payoffs and they consider the quotient of the price of their positive part and negative part under a given benchmark stochastic discount factor (hereafter SDF). Then the price interval of a new payoff (the call option in the above example) is set in a way such that the resulting maximum gain-loss ratio does not exceed a given ceiling. That is, they discard those ASDF which imply a too high gain-loss ratio thereby, tightening this initial set and sharpening the no-arbitrage bounds.

Cochrane and Saá-Requejo (2000) concentrate on restricting the volatility of the set of ADSF and they exploit its connections with the maximum Sharpe ratio of the economy. Therefore, they rule out those prices for the payoff to be priced which will be implied by ASDF's that generate a risk-premium with respect to volatility which is too high, that is, they rule out *too good-deals*. By discarding those ASDF's whose standard deviation is greater than an appropriate ceiling, this initial set is tightened and therefore, the no-arbitrage bounds are also sharpened. This way of tackling the derivation of asset price bounds has its roots on the distance that underlies Hansen and Jagannathan's measure of model misspecification. Good-deal bounds result by discarding those ASDF which imply a value of the distance that exceeds a given ceiling when the risk-neutral SDF is chosen as a benchmark.

This paper concentrates on the latter contribution in the context of a two-period model (no intermediate trading is allowed) and its goal is twofold. First, we show how to easily extend good-deal bounds to the choice of any given benchmark SDF. Second, we introduce an alternative method which mostly retains the economic interpretation associated with Hansen and Jagannathan's distance and it can also be applied to any benchmark SDF. Its major advantage is the simplicity of the computation of the bounds, since it only entails solving a linear program.

The remainder of the paper is organized as follows. Section 2 deals with notation issues and preliminaries. Sections 3 shows how to generalize good-deal bounds and 4 introduces an alternative method. Section 5 concludes.

2 Preliminaries

2.1 The setup

Consider a two-period economy where N basis assets are traded today at a known price given by a vector $\mathbf{p}$ and they deliver a continuous random payoff denoted by a vector $\mathbf{x}$. No frictions are allowed and hence any linear combination of the basis payoffs constitutes an attainable payoff x whose set will be denoted by X. Let $Z \subset X$ be the subset of zero-price payoffs and $L \supset X$ the space of payoffs in the span of all contingent claims.[1] For any payoff $x \in L$ consider its positive and negative part $x^+ \equiv \max(x, 0)$ and $x^- \equiv \max(-x, 0)$, respectively.

The LOP is assumed to hold which implies the existence of random variables m which satisfy

$$\mathbf{p} = E(m\mathbf{x}). \tag{2.1}$$

Denote by M the set of all random variables m (ASDF's) satisfying (2.1). These random variables give the price of any payoff $x \in L$ through their *pricing extension* defined as

$$\pi_m(x) \equiv E(mx)$$

which obviously assigns the same price for any $m \in M$ if $x \in X$. Define also the sets

$$R_m = \{x \in L : \pi_m(x) = 1\}$$

and

$$Z_m = \{x \in L : \pi_m(x) = 0\}$$

for any $m \in M$. Thus, Z_m and R_m are the sets of returns and excess returns under the pricing function $\pi_m(\bullet)$, respectively.

[1] The space L we consider in each case depends on the particular formulation of the problem.

We will also assume there exists a riskless asset with risk-free rate of return R^f and absence of arbitrage opportunities. With this latter assumption it is well-known that there will be at least one strictly positive $m \in M$. Denote their set by M^+.

The pricing implications of a given model are summarized by its implied SDF. Denote by m^* a strictly positive *benchmark* SDF and for any $x \in L$ consider also its implied pricing function $\pi^*(x) \equiv E(m^* x)$. Let also $E^*(\cdot)$ denote the expectation under the risk-neutral probability measure implied by m^* and thus, we have that

$$\pi^*(x) = E^*(x) E(m^*).$$

2.2 Asset Price Bounds

One can always view the derivation of asset price bounds as the result of a four-stage process. First, a particular benchmark SDF is chosen. Second, a distance[2] between each ASDF and the benchmark is defined. Third, an appropriate ceiling on the maximum value of this distance is set and those ASDF whose distance from the benchmark is above the ceiling are discarded. Finally, by maximizing (minimizing) the price of the new payoff among those SDF's that belong to this latter set the upper (lower) bound is obtained.

In the first two steps, stronger economic assumptions which go beyond absence of arbitrage should be introduced by the choice of both the distance and the benchmark. The value of the ceiling will place the derivation of the bounds somewhere between no-arbitrage pricing and model-based pricing; the larger the distance the wider the bounds. Obviously, this ceiling must be binding, otherwise the set of ASDF will not be tightened and we will be again where we started, at the no-arbitrage bounds. This value may as well be determined by a-priori beliefs on how good representation of the real world the benchmark model is. Also, the problem to be solved in the fourth stage must be feasible which requires a large enough value of the ceiling so that we do not discard every ASDF.

Formally, for any $m \in M^+$ denote by $d(m, m^*)$ a distance between m and the benchmark model m^*. Let $\overline{d}$ be an appropriate ceiling on the maximum value of the above distance and discard those $m \in M^+$ such that $d(m, m^*) >$

[2] We abuse the term "distance" in our framework since the symmetry property does not have to hold.

$\overline{d}$. Finally, the lower bound on the price of a payoff $x^c \notin X$ will be given by

$$\underline{C} = \min_{\substack{m \in M^+ \\ d(m,m^*) \leq \overline{d}}} E\left(mx^c\right) \tag{2.2}$$

or, equivalently,

$$\underline{C} = \min_{m \in \overline{M}^+} E\left(mx^c\right), \tag{2.3}$$

where $\overline{M}^+ \equiv \left\{m \in M^+ : d\left(m, m^*\right) \leq \overline{d}\right\}$. The upper bound $\overline{C}$ follows from replacing min with max in the above optimization. Hence, by setting a ceiling on the maximum value of $d\left(m, m^*\right)$ we limit further the set of ASDF, thereby going beyond simple no-arbitrage restrictions. Within this latter set $\overline{M}^+$ we choose those SDF's which result in the lowest and highest price of the payoff x^c. This procedure gives the upper and lower asset price bounds.

It should also be noted that the value of the ceiling must satisfy

$$\min_{m \in M^+} d\left(m, m^*\right) \leq \overline{d} \leq \max_{m \in M^+} d\left(m, m^*\right) \tag{2.4}$$

because if the left-hand inequality does not hold the feasible set in (2.2) is empty and if the right-hand inequality is violated the original no-arbitrage bounds are obtained.

Thus, the difference between the alternative methods to derive asset price bounds we will be discussing only depends on the form of the distance $d\left(m, m^*\right)$. This distance usually has a distinct economic interpretation which is obtained through a duality result.

Let us illustrate the above formulation with an example. Bernardo and Ledoit (2000) present a derivation of asset price bounds based on what they define as the gain-loss ratio. These bounds result from setting

$$d\left(m, m^*\right) = \frac{\sup \frac{m}{m^*}}{\inf \frac{m}{m^*}} \tag{2.5}$$

and the interpretation of (2.5) goes as follows.

Proposition 2.1 *For the distance in (2.5), the following equality holds*

$$d\left(m, m^*\right) = \max_{x \in Z_m} \frac{E^*\left(x^+\right)}{E^*\left(x^-\right)}.$$

Proof. A straightforward application of Theorem 1 in Bernardo and Ledoit (2000) gives the desired result. ■

For some $m \in M^{+}$ and a given $x \in Z_m$ these authors call $E^{*}\left(x^{+}\right)$ the gain, $E^{*}\left(x^{-}\right)$ the loss and $E^{*}\left(x^{+}\right)/E^{*}\left(x^{-}\right)$ the gain-loss ratio. Thus, in this case $d\left(m, m^{*}\right)$ gives the maximum gain-loss ratio for zero-price payoffs in the span of all contingent claims under the pricing extension $\pi_m\left(\bullet\right)$ implied by m.

3 Generalized Good-Deal Bounds

3.1 The Definition

Hansen and Jagannathan (1997) introduce a measure of model misspecification which is based on the distance

$$d\left(m, m^{*}\right) = \left[E\left(m - m^{*}\right)^{2}\right]^{\frac{1}{2}} \tag{3.1}$$

and whose economic meaning is formalized in the proposition below.

Proposition 3.1 *For the distance given in (3.1), the following equality holds*

$$d\left(m, m^{*}\right) = \max_{\substack{x \in L \\ E\left(x^{2}\right)=1}} \left|\pi_m\left(x\right) - \pi^{*}\left(x\right)\right|. \tag{3.2}$$

Proof. See Hansen and Jagannathan (1997). ■

Hence, (3.1) gives the maximum pricing discrepancy between π_m and π^{*} for payoffs in the span of all contingent claims whose second moment are equal to one.[3] An alternative and more intuitive interpretation goes as follows. Suppose there are two different (complete) financial markets where the whole set of contingent claims are traded. In one market, prices are set according to m and in the other one, prices are set according to the benchmark. Arbitrage opportunities do not exist within each market since both m and m^{*} are strictly positive. However, there are cross-market strategies that give infinite riskless benefits as long as there exist pricing discrepancies between π_m and

[3] Within this formulation L stands for L^{2}.

π^* $(d(m, m^*) > 0)$. The role of the normalization $E(x^2) = 1$ is simply to guarantee boundedness thereby giving a measure of the size of the above benefits in relative terms; it has no economic meaning beyond that. Hence, a restriction on the value of $d(m, m^*)$ is equivalent to a restriction on the optimal value of cross-market arbitrage strategies for those payoffs in L whose second moment is equal to one.

In other words, a ceiling on the value of (3.1) rules out investment opportunities that are too attractive where the level of attractiveness implied by a given ASDF is measured in terms of the size of the disintegration that creates with respect to the benchmark market given by m^*.

However, further economic intuition can be derived for (3.1) which we present once again in the form of a proposition.

Proposition 3.2 *For the distance define in (3.1) and any benchmark m^* such that $E(m^*) = 1/R^f$, it holds that*

$$\frac{|E^*(x)|}{\sigma(x)} \leq d(m, m^*) R^f$$

for all $x \in Z_m$ and

$$\frac{\left|E^*(x) - R^f\right|}{\sigma(x)} \leq d(m, m^*) R^f$$

for all $x \in R_m$.

Proof. See Appendix. ■

Therefore, a restriction on the above distance (which is equal to the volatility of the difference between the ASDF and the benchmark if $E(m^*) = 1/R^f$) implies a restriction on the *generalized* Sharpe ratio, that is, a restriction on the Sharpe ratio where the expectation in the numerator is taken under the risk-adjusted probability measure that the benchmark implies. It should be noted that for a constant benchmark, Proposition 3.2 gives the well-known result that imposing a bound on the volatility of the SDF m implies a bound on the standard Sharpe ratio. Cochrane and Saá-Requejo (2000) use this restriction to derive good-deal bounds which therefore, can be considered a particular case of the derivation of asset price bounds based on the distance in (3.1). Thus, we call the latter *generalized* good-deal bounds.[4]

We believe that the inclusion of meaningful economic restrictions that go beyond mean-variance considerations together with the appealing economic interpretation of Hansen and Jagannathan's measure of model misspecification largely justify a generalization of good-deal bounds to any given benchmark SDF. Furthermore, this extension does not add any major complexities to the computation of the bounds as we turn to show now.

3.2 Calculating Generalized Good-Deal Bounds

Our exposition will parallel Cochrane and Saá-Requejo's (2000). Problem (2.2) for the above distance can be written as

$$\underline{C} = \min_{m} E\left(mx^{c}\right) \tag{3.3}$$
$$\text{s.t.} \begin{cases} E\left(m\mathbf{x}\right) = \mathbf{p} \\ E\left(m - m^{*}\right)^{2} \leq \overline{d}^{2} \\ m \geq 0 \end{cases}.$$

where the corresponding maximization gives the upper bound.

The problem has two inequality constraints and the solution can be found by checking all possible combinations of binding and nonbinding constraints. When the second constraint is slack, no-arbitrage bounds are obtained so we will concentrate on the two remaining possibilities.

Assume that the second constraint in (3.3) is binding and that the positivity constraint is slack. In this case, our problem becomes

$$\underline{C} = \min_{m} E\left(mx^{c}\right) \tag{3.4}$$
$$\text{s.t.} \begin{cases} E\left(m\mathbf{x}\right) = \mathbf{p} \\ E\left(m - m^{*}\right)^{2} \leq \overline{d}^{2} \end{cases}.$$

where min should be replaced by max to obtain the upper bound. Let $p^{*} = E\left(m^{*}x\right)$ be the price that the benchmark SDF assigns to the vector of basis payoffs and define $q \equiv p - p^{*}$ and $y \equiv m - m^{*}$. Consider also the orthogonal decomposition of the focus payoff

$$x^{c} = \widehat{x}^{c} + \omega$$

[4]Note that our definition of generalized Sharpe ratio and generalized good-deal bounds is different from the one in Černý (2001).

where $\widehat{x}^c \equiv E(x^c x') E(xx')^{-1} x$ and $\omega = x^c - \widehat{x}^c$. The proposition below gives the solution to problem (3.4).

Proposition 3.3 *The discount factor that generates the lower bound is*

$$\underline{m} = x^* - \underline{v} + m^*$$

and the bound is

$$\underline{C} = E(x^* x^c) - \underline{v} E\left(\omega^2\right) + E(m^* x^c)$$

where

$$x^* = q' E(xx') x$$

and

$$\underline{v} = \sqrt{\frac{\overline{d}^2 - E(x^{*2})}{E(\omega^2)}}.$$

The upper bound is given by $\overline{v} = -\underline{v}$.

Proof. See Appendix. ■

Note that the size of the bounds has an expression which is identical to equation (12) in Cochrane and Saá-Requejo (2000) with the exception that x^* is now the projection of y onto the space of payoffs X. Hence, the bounds are tighter if the value $\overline{d}$ is smaller, if the size of the residual $\sqrt{E(\omega^2)}$ is smaller or, equivalently, if the approximate hedge is better.

Assume now that both constraints in (3.3) bind. We introduce Lagrange multipliers and the problem is

$$\underline{C} = \min_{m>0} \max_{\lambda,\delta>0} E(mx^c) + \lambda' [E(yx) - p] + \frac{\delta}{2}\left[E\left[(m - m^*)^2\right] - \overline{d}^2\right] \quad (3.5)$$

and its first-order conditions give

$$m = \left[-\frac{x^c + \lambda' x - \delta m^*}{\delta}\right]^+. \quad (3.6)$$

As in Cochrane and Sáa-Requejo we interchange min and max in (3.5) and use (3.6) to obtain after simplifying

$$\underline{C} = \max_{\lambda, \delta > 0} E\left\{ -\frac{\delta}{2}\left[-\frac{x^c + \lambda' x - \delta m^*}{\delta} \right]^{+2} \right\} - \lambda' p + \frac{\delta}{2}\left[E\left(m^{*2}\right) - \overline{\delta}^2 \right].$$

This problem is solved by searching numerically over (λ, δ). Once again, the upper bound is found by replacing max with min but this time it must hold that $\delta < 0$.

4 L^1-Generalized Good-Deal Bounds

We turn now to present an alternative derivation which can be regarded as the L^1-norm equivalent of Generalized Good-Deal Bounds. This alternative approach translates the economic restrictions behind Generalized Good-Deal Bounds into their corresponding L^1 formulations. As a by-product the computation of the resulting bounds enormously simplifies since it only requires solving a linear program. The bounds follow from setting

$$d\left(m, m^*\right) = \sup\left| \frac{m}{m^*} - 1 \right|. \tag{4.1}$$

Proposition 4.1 *For the distance in (4.1), the following equality holds*

$$d\left(m, m^*\right) = \max_{\substack{x \in L \\ E(m^*|x|)=1}} \left| \pi_m\left(x\right) - \pi^*\left(x\right) \right|.$$

Proof. See Appendix. ■

Thus, both (3.1) and (4.1) produce an interpretation that can be read in terms of maximum pricing discrepancies. Also, we can regard them both as the optimal value of arbitrage strategies across two complete markets where prices are given by $\pi_m(\bullet)$ and $\pi^*(\bullet)$ in each case. However, they differ on the target set of payoffs in the space of all contingent claims implied in the necessary normalization. (3.1) looks at those payoffs whose second norm equals one. (4.1) normalizes by restricting the maximization to payoffs whose absolute value has unit price under the postulated model or, equivalently, expectation equal to one under the risk-neutral measure implied by the model.

In other words, the attractiveness of the investment opportunities implied by a given ASDF is measured in terms of the size of the disintegration that creates with respect to the benchmark market given by m^*.

Moreover, further economic insights can also be obtained for (4.1) through a result which parallels Proposition 3.2.

Proposition 4.2 *For the distance define in (4.1) and any benchmark m^* such that $E(m^*) = 1/R^f$, it holds that*

$$\frac{|E^*(x)|}{E^*(|x|)} \leq d(m, m^*)$$

for all $x \in Z_m$ and

$$\frac{\left|E^*(x) - R^f\right|}{E^*(|x|)} \leq d(m, m^*)$$

for all $x \in R_m$.

Proof. See Appendix. ■

Thus, in the same way as a restriction on (3.1) implies a restriction on the generalized Sharpe ratio, a bound on (4.1) imposes a constraint on the value of the ratio

$$\frac{|E^*(x)|}{E^*(|x|)} = \frac{|E^*(x)|}{E^*(x^+) + E(x^-)} \tag{4.2}$$

for excess returns. In particular, for a constant (risk-neutral) benchmark, the restriction is imposed on

$$\frac{|E(x)|}{E(|x|)}$$

which can be seen as the L^1-norm equivalent of the Sharpe ratio. Furthermore, note that (4.2) is also close in spirit to the gain-loss ratio.

An important feature of L^1-generalized asset price bounds is that their computation is extraordinarily easy: a linear program does it. To see this, for the distance in (4.1) rewrite the inequality constraint of problem (2.2) to obtain

$$-\overline{d} \leq \frac{m}{m^*} - 1 \leq \overline{d}$$

which together with the pricing equation in (2.1) and the objective $E\left(mx^c\right)$ give a linear optimization program in m.[5]

Let us go back to our initial example and consider again the case of two basis assets: a stock and a bond. Suppose also that we set the benchmark equal to the SDF implied by the Black-Scholes model. The parameters are the same as above. The stock's continuously compounded rate of return has an annual volatility equal to 14.8% and the one-year risk-free rate is equal to 5%. We want to compute asset price bounds on a European call option on the stock with one year to expiration and a strike price equal to one hundred. For initial prices of the stock between 80 and 110 we compute the upper and lower bound for values of the ceiling between 0 and .9 which

[5] One might suggest as the basis of L^1-generalized good-deal bounds the exact L^1 equivalent of the least-squares distance, namely

$$d\left(m, m^*\right) = E\left(\left|m - m^*\right|\right).$$

However, it can be proved through the corresponding duality result that in this case

$$d\left(m, m^*\right) = \max_{\substack{x \in L \\ |x|=1}} \left|\pi_m\left(x\right) - \pi^*\left(x\right)\right|$$

which has a very poor economic interpretation.

Alternatively, one could try to obtain the exact L^1 equivalent of the expression in (3.2), that is,

$$\max_{\substack{x \in L \\ E(|x|)=1}} \left|\pi_m\left(x\right) - \pi^*\left(x\right)\right|.$$

It can easily be shown that this expression is associated with the distance

$$d\left(m, m^*\right) = \sup\left|\mu - \mu^*\right|$$

where μ and μ^* are the risk-adjusted probability measures implied by m and m^*, respectively. This distance is in our opinion totally uninteresting.

In any case, note that the bounds obtained in the above two cases can also be computed by solving a linear program.

are given in Figure 2. The bounds lie between the Black-Scholes price and the no-arbitrage bounds. As the ceiling increases we move towards the no-arbitrage bounds. As the ceiling decreases we move towards the Black-Scholes price. A value of zero on the ceiling gives the Black-Scholes price, that is, the smaller set of ASDF's contains only the one implied by the benchmark. By introducing this economically meaningful restrictions no-arbitrage bounds are clearly sharpened. Also, as one should expect, larger bounds are obtained for near-the-money values of the stock price.

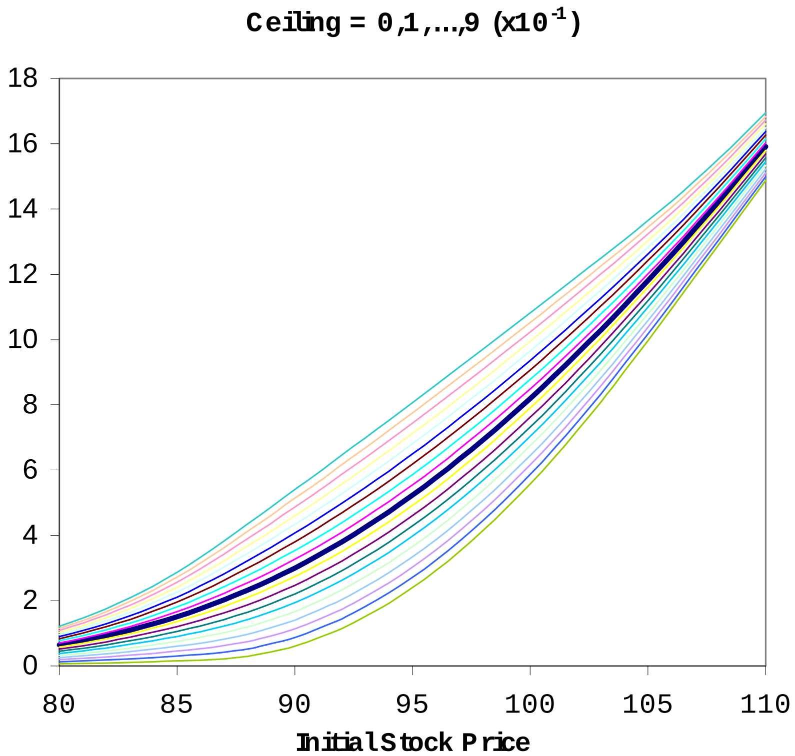

Figure 2. Price bounds on a European call option with Black-Scholes as a benchmark. The option has one year to expiration and a strike price of 100. The one-year risk-free rate of return is 5% and the standard deviation of the continuously compounded rate of return on the stock is 14.09%. The thick line represents the Black-Scholes price (the value of the ceiling is zero) and the two outer lines give the lower and upper bounds for a value of the ceiling equal to .9.

5 Conclusions

This paper introduces an extension of good-deal bounds which allows to insert in the derivation the restrictions that a particular asset pricing model implies. By treating a limit on the volatility of the discount factor as a particular case of a restriction on Hansen and Jagannathan's distance when the benchmark is a constant SDF, we manage to formalize a generalization of good-deal bounds to any benchmark model. A restriction on this distance is equivalent to a restriction on the volatility of the difference between the ASDF and the benchmark (when both objects have the same mean). This restriction is already pointed out by Cochrane and Saá-Requejo as a possible way to develop the above generalization. We also show that in a context with no intermediate trading, the computation of the resulting bounds does not involve any additional complexities.

Furthermore, we also suggest an L^1 alternative derivation of asset price bounds which has clear similarities with the above approach. It also has a clear advantage: its simple calculation. We believe that this latter property is also important for other reasons. The idea of implementing the valuation of asset payoffs with an approach that lies in between no-arbitrage pricing and model-based pricing has also potential applications to other fields like portfolio theory and performance evaluation. However, this applications may be difficult to implement both theoretically and practically. Therefore, a simple method seems like a good idea as a starting point.

The extension of the above arguments to a continuous time setup has not been addressed and it may be a subject for future research. We consider this step technically possible but not all trivial.

Appendix

Proof of Proposition 3.2. First, note that since

$$E(m^*) = E(m) = 1/R^f \tag{A.1}$$

it holds that

$$d(m, m^*) = \sigma(m - m^*). \tag{A.2}$$

We have that if $x \in Z_m$, then

$$E[(m - m^*)x] = -E(m^* x). \tag{A.3}$$

By using (A.1), the definition of covariance and the expectation under the risk-adjusted measure implied by m^*, (A.3) may be rewritten as

$$cov\left(m - m^*, x\right) = -\frac{E^*\left(x\right)}{R^f}$$

which implies by (A.2) and well-known arguments that

$$d\left(m, m^*\right)\sigma\left(x\right) \geq \frac{\left|E^*\left(x\right)\right|}{R^f}.$$

An identical derivation gives the desired result for $x \in R_m$.

Proof of Proposition 3.3. The proof uses very similar arguments to the ones in the proof of Proposition 1 in Cochrane and Saá-Requejo (2000). By Lemma 1 in Cochrane and Saá-Requejo (2000) $y = m - m^*$ can be orthogonally decomposed as follows

$$y = x^* + v\omega + \epsilon$$

if and only if

$$E\left(yx\right) = q \Longleftrightarrow E\left[\left(y + m^*\right)x\right] = p \Longleftrightarrow E\left(mx\right) = p$$

where v is an arbitrary constant, ϵ is a random variable that satisfies $E\left(\epsilon x\right) = E\left(\epsilon\omega\right) = 0$. Thus, we have that we can express the inequality constraint in (3.4) as

$$E\left(y^2\right) = E\left(x^{*2}\right) + v^2 E\left(\omega^2\right) + E\left(\epsilon^2\right) \leq \overline{d}^2$$

and the objective can be rewritten in the following way

$$\begin{aligned} E\left(mx^c\right) &= E\left[\left(y + m^*\right)x^c\right] = E\left(x^*x^c\right) + vE\left(\omega x^c\right) + E\left(\epsilon x^c\right) + E\left(m^*x^c\right) = \\ &= E\left(x^*x^c\right) + vE\left(\omega^2\right) + E\left(m^*x^c\right). \end{aligned}$$

Thus, solving problem (3.4) is equivalent to solving

$$\begin{aligned} &\min_v E\left(x^*x^c\right) + vE\left(\omega^2\right) + E\left(m^*x^c\right) \\ &\text{s.t. } E\left(x^{*2}\right) + v^2 E\left(\omega^2\right) + E\left(\epsilon^2\right) \leq \overline{d}^2 \end{aligned}$$

which in the optimum must satisfy $\epsilon = 0$ and

$$E\left(x^{*2}\right) + \underline{v}^2 E\left(\omega^2\right) + E\left(\epsilon^2\right) = \overline{d}^2$$

where $\underline{v}$ denotes the optimal value of v. Therefore,

$$\underline{v} = \sqrt{\frac{\overline{d}^2 - E\left(x^{*2}\right)}{E\left(\omega^2\right)}}$$

and since $m = y + m^*$ the optimal value of the stochastic discount factor is

$$\underline{m} = \underline{y} + m^* = x^* - \underline{v}\omega + m^*.$$

Replacing min with max in (3.4) and using identical arguments as the ones above we obtain the corresponding results for the upper bound. ■

Proof of Proposition 4.1. Our space of all possible payoffs is now

$$L_*^1 = \{x : E(m^*x) < \infty \text{ and } E(mx) < \infty \ \forall m \in M\}$$

and let $L_{*+}^1 = \{x \in L_*^1 : x \geq 0\}$ be the positive orthant of L_*^1.

As usual we will proof the equality in the proposition by showing that both sides are true. Let us start with $\leq$. For a given $m \in M$ we have that

$$-d(m, m^*) \leq \frac{m}{m^*} - 1 \leq d(m, m^*)$$

which gives

$$m^* - d(m, m^*)\, m^* \leq m \tag{A.4}$$

and

$$m \leq m^* + d(m, m^*)\, m^*. \tag{A.5}$$

For any $x \in L_*^1$ such that $\pi^*(x^+) + \pi^*(x^-) = 1$, multiply (A.4) and (A.5) by x^- and $-x^+$, respectively, and take expectations to obtain

$$-\pi_m\left(x^-\right) \leq -\pi^*\left(x^-\right) + d(m, m^*)\, \pi^*\left(x^-\right)$$

and

$$\pi_m\left(x^+\right) \leq \pi^*\left(x^+\right) + d(m, m^*)\, \pi^*\left(x^+\right).$$

By adding up these two inequalities we get

$$-d(m, m^*) \leq \pi^*(x) - \pi_m(x). \tag{A.6}$$

Now, multiply (A.4) and (A.5) by $-x^+$ and x^-, respectively. By identical arguments it follows that

$$d(m, m^*) \geq \pi^*(x) - \pi_m(x). \tag{A.7}$$

Finally, (A.6) and (A.7) give the desired inequality, that is

$$|\pi_m(x) - \pi(x)| \leq d(m, m^*).$$

Let us turn to the $\geq$ side of the equality and let $\overline{k}_m = \sup\left(\frac{m}{m^*}\right)$ and $\underline{k}_m = \inf\left(\frac{m}{m^*}\right)$. We have that

$$d(m, m^*) = \max\left\{\overline{k}_m - 1, 1 - \underline{k}_m\right\}.$$

Suppose that $d(m, m^*) = \overline{k}_m - 1$ and define

$$\lambda_m \equiv \max_{\substack{x \in L^1_* \\ \pi^*(x^+)+\pi^*(x^-)=1}} |\pi_m(x) - \pi^*(x)|.$$

Note that

$$\begin{aligned}
\lambda_m &\geq \max_{\substack{x \in L^1_{*+} \\ \pi^*(x^+)+\pi^*(x^-)=1}} |\pi_m(x) - \pi^*(x)| = \\
&= \max_{\substack{x \in L^1_{*+} \\ E(m^*x)=1}} |\pi_m(x) - \pi^*(x)| = \\
&= \max_{\substack{x \in L^1_{*+} \\ E(m^*x)=1}} |E(mx) - 1| \geq \max_{\substack{x \in L^1_{*+} \\ E(m^*x)=1}} E(mx) - 1
\end{aligned} \tag{A.8}$$

Now, define the payoff

$$x_\epsilon = \frac{I\left[\frac{m}{m^*} \geq \overline{k}_m - \epsilon\right]}{m^* P\left[\frac{m}{m^*} \geq \overline{k}_m - \epsilon\right]} > 0$$

where I and P denote the indicator function and the real probability of the event in brackets, respectively. It is easy to see that $E(m^* x_\epsilon) = 1$; hence, it follows that

$$\max_{\substack{x \in L^1_{*+} \\ E(m^*x)=1}} E(mx) \geq E(mx_\epsilon) = E\left[\frac{m}{m^*} \middle| \frac{m}{m^*} \geq \overline{k}_m - \epsilon\right] \geq \overline{k}_m - \epsilon.$$

Since this holds from every $\epsilon > 0$, from (A.8) we have that

$$\lambda_m \geq \lim_{\epsilon \to 0} \overline{k}_m - \epsilon - 1 = \overline{k}_m - 1 = d(m, m^*).$$

Now, suppose $d(m, m^*) = 1 - \underline{k}_m$. By reasoning as in (A.8), it should be clear that

$$\lambda_m \geq \max_{\substack{x \in L^1_{*+} \\ E(m^*x)=1}} 1 - E(mx)$$

which by defining this time

$$x_\epsilon = \frac{I\left[\frac{m}{m^*} \leq \underline{k}_m + \epsilon\right]}{m^* P\left[\frac{m}{m^*} \leq \underline{k}_m + \epsilon\right]} > 0$$

and following symmetric arguments as the ones above gives

$$\lambda_m \geq d(m, m^*).$$

Proof of Proposition 4.2. We have that

$$\begin{aligned} d(m, m^*) &= \max_{\substack{x \in L \\ E(m^*|x|)=1}} |\pi_m(x) - \pi^*(x)| = \\ &= \max_{\substack{x \in L \\ E^*(|x|)=R^f}} \left|\pi_m(x) - \frac{E^*(x)}{R^f}\right| = \\ &= \max_{\substack{x \in L \\ E^*(|x|)=1}} \left|\pi_m(x) R^f - E^*(x)\right|. \end{aligned}$$

Hence, for any $x \in L$ except for the zero payoff, it must hold that

$$d(m, m^*) \geq \frac{\left|\pi_m(x) R^f - E^*(x)\right|}{E^*(|x|)} \qquad \text{(A.9)}$$

because the payoff $x/E^*(|x|)$ is an element of the feasible set in the maximization above. Obviously, (A.9) implies

$$d(m, m^*) \geq \frac{\left|E^*(x) - R^f\right|}{E^*(|x|)}$$

and

$$d(m, m^*) \geq \frac{|E^*(x)|}{E^*(|x|)}$$

for any $x \in R_m$ and any $x \in Z_m$, respectively. ■

References

Bernardo A.E. and O. Ledoit (2000), “Gain, Loss, and Asset Pricing”, *Journal of Political Economy*, vol. 108, no. 1, 144-172.

Chen, Z and P.J. Knez (1995), “Measurement of Market Integration and Arbitrage”, *The Review of Financial Studies*, 8, 287-325.

Cochrane, J. and J. Saá-Requejo (2000), “Beyond Arbitrage: Good-Deal Asset Price Bounds in Incomplete Markets”, *Journal of Political Economy,* vol. 108, no. 1, 79-119.

Hansen L.P. and R. Jagannathan (1997), “Assessing Specification Errors in Stochastic Discount Factor Models”, *Journal of Finance*, vol 52, no.2, 557-590.

Luenberger D.G. (1969), *Optimization by Vector Space Methods,* John Wiley & Sons, Inc., New York.