Quadratic Models for Portfolio Credit Risk with Shot-Noise Effects

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Abstract

We propose a reduced form model for default that allows us to derive closed-form solutions to all the key ingredients in credit risk modeling: risk-free bond prices, defaultable bond prices (with and without stochastic recovery) and probabilities of survival. We show that all these quantities can be represented in general exponential quadratic forms, despite the fact that the intensity is allowed to jump producing shot-noise effects. In addition, we show how to price defaultable digital puts, CDSs and options on defaultable bonds.

Further on, we study a model for portfolio credit risk where we consider both firm specific and systematic risks. The model generalizes the attempt from Duffie and Gârleanu (2001). We find that the model produces realistic default correlation and clustering of defaults. Then, we show how to price first-to-default swaps, CDOs, and draw the link to currently proposed credit indices.

Contents

1 Introduction

Most of the quadratic term structure (QTS) literature has focused on analyzing risk-free bond prices and considering only Gaussian-quadratic models. Exceptions are Gaspar (2004) and Chen, Filipović, and Poor (2004). Gaspar (2004) introduces the so-called General Quadratic term structure (GQTS) models, which through an a-priori classification of factors include both the affine term structure (ATS) models and the Gaussian-QTS models as special cases. Instead, Chen, Filipović, and Poor (2004) study the traditional Gaussian-QTS, also in for both risk-free and defaultable bonds.

In this paper we use the concept of GQTS and augment it with a special type of jump-process, called shot-noise processes. By this, we do not only include the above mentioned models as a special case but also jump-diffusion models, like for example Duffie and Gârleanu (2001). While quadratic models naturally arise in intensity-based models, as the default intensity needs to be a positive process, the shot-noise component allows to obtain a suitable dynamic dependence structure for a market with a large number of defaultable entities. Needless to say, capturing dynamic dependencies is one of the most important points for modeling CDOs. Using the shot-noise process solves a basic problem in Duffie and Gârleanu (2001), namely that the mean-reversion speed of the diffusion part is the same as for the jump part. Besides this, shot-noise processes induce an interesting behavior to the process, which will result in clustering for defaults. It is this feature which seems very promising for capturing the complex dependencies which constitute the peculiarities embedded in a CDO.

As already mentioned, we consider an intensity-based approach to modeling default. This approach has always been very popular and has recently been justified by strong fundamental motivations. Indeed, Duffie and Lando (2001) show that the difference between the reduced-form approach and the economically more intuitive structural approach becomes irrelevant when one includes frictions in the structural model, such as imperfect information about the asset or the liability structure. Moreover, Collin-Dufrense, Goldstein, and Hugonnier (2004) proved that the price of a defaultable security is always the expectation of future discounted cash-flows, even when the so called "no-jump condition" is violated[1]. So, there are good reasons for expecting closed-form solutions for key ingredients of credit risk.

For a survey study on reduced form credit risk model we refer to Schmidt and Stute (2004).

The main goals of this paper are:

- To adapt the GQTS setup to model default risk using an intensity-based credit risk modeling approach *a la* Jarrow, Lando, and Turnbull (1997), Lando (1998) and Duffie and Singleton (1999) and to get closed-form solutions for all key ingredients in credit

[1] When the no-jump condition holds, the traditional risk-neutral measure can be used, basically adding to the discount rate some term which reflects the default risk. If the no-jump condition does not hold, using a new measure (the so-called survival measure) allows roughly the same to be done. The survival measure, also used in Schönbucher (2000) and Eberlein, Kluge, and Schönbucher (2005), is the measure that puts zero probability on those paths for which default occurs prior to maturity. As such this measure is only absolutely continuous with respect to the risk neutral probability and not equivalent to it.

risk modeling: risk-free bonds, defaultable bonds, probabilities of default, etc. This way we extend the Gaussian-QTS of defaultable bonds considered by Chen, Filipović, and Poor (2004).

- Use shot-noise processes to give a dynamic description of default dependence which is able to produce a high default correlation and contagion effects.
- Under some simplifying assumptions the model gives closed-form solution for the pricing of CDOs and other portfolio credit risk derivatives.

The paper is organized as follows. In Section 2 we review the basic setup of GQTS and present the main result on risk-free bond prices. In Section 3 we present the model for the defaultable bond market. Considering uncertainty effects, we find a motivation for shot-noise effects in the credit spreads. Then, the theoretical framework is given and we derive survival probabilities, defaultable bond prices, defaultable digital payoffs and show how to use these building blocks to price less trivial credit derivatives. We conclude the section by considering various recovery assumptions. In Section 4 we deal with portfolio credit risk issues. Special emphasis is put on default correlations and clustering effects implied by the considered framework. The subsection 4.3.2 is devoted to pricing CDOs, while the rest of the section deals with other portfolio credit derivatives, such as first-to-default swaps and options on credit indices. Section 5 illustrates the theoretical results by considering an easy three-factor model. Section 6 concludes the paper and discusses future research.

2 Risk-free Bond Market

For the risk-free bond market we use the general quadratic term structures setup studied in Gaspar (2004). Consider a finite set of time-dependent factors described by a $\mathbb{R}^m$-valued stochastic process $(Z_t)_{t\geq 0}$. The zero-coupon bond prices are assumed to depend on these factors by

$$p(t,T) = H(t,T,Z_t), \tag{1}$$

where H is a smooth function with the boundary condition $H(T,T,z) = 1$. In a general quadratic setting, H will turn out to have a quadratic form.

We propose the following dynamics for Z:

$$dZ_t = \alpha(t,Z_t)dt + \sigma(t,Z_t)dW_t, \tag{2}$$

where W is a m-dimensional Wiener Process, and it generates the filtration $\left(\mathcal{F}_t^W\right)_{t\geq 0}$.

The drift and volatility terms, α, σ, shall have the following form:

$$\alpha(t,z) = d(t) + E(t)z \tag{3}$$

$$\sigma(t,z)\sigma^\top(t,z) = k_0(t) + \sum_{i=1}^m k_i(t)z_i + \sum_{i,j=1}^m z_i\, g_{ij}(t)z_j. \tag{4}$$

Here, z_i is the i-th component of z. The deterministic and smooth functions d, k_0, k_i, g_{ij} for $i,j = 1,\cdots,m$ take values in $\mathbb{R}^m$ while E takes values in $\mathbb{R}^{m\times m}$ and $\cdot^\top$ denotes the transpose.

Also for the short rate we assume a quadratic form.

Assumption 2.1. *Assume that the risk-free short rate* $(r_t)_{t\geq 0}$ *is given by*

$$r(t, Z_t) = Z_t^\top Q(t) Z_t + g^\top(t) Z_t + f(t). \tag{5}$$

Here, Q, g *and* f *are deterministic and smooth functions with values in* $\mathbb{R}^{m\times m}$, $\mathbb{R}^m$ *and* $\mathbb{R}$, *respectively. Moreover,* $Q(t)$ *is assumed to be symmetric*[2] *for all* t.

Gaspar (2004) has shown how to identify these factors *a priori* from their impact on the drift α, volatility σ or the functional form of the short rate. We, thus, classify the components of Z in the following two groups.

Definition 2.2. (Classification of risk-free factors)

- Z_i is a *risk-free quadratic-factor* if it satisfies *at least one* of the following requirements:
  - (i) it has a quadratic impact on the short rate of interest $r(t)$, i.e., there exists t such that $Q_i(t) \neq 0$;
  - (ii) it has a quadratic impact on the functional form of the matrix $\sigma(t,z)\sigma^\top(t,z)$, i.e., there exist k and t such that $g_{ik}(t) \neq 0$;
  - (iii) it affects the drift term of the factors satisfying (i) or (ii), i.e., for Z_j satisfying (i) or (ii) we have $E_{ji}(t) \neq 0$, at least for some t.
- Z_i is a *risk-free linear-factor*, if it does not satisfy any of (i)-(iii).

We write $i \in Z^{(q)}$, if Z_i is a risk-free quadratic factor and $i \in Z^{(l)}$ if it is a risk-free linear factor.

The above classification immediately yields that Q, E and G have a certain form. To access this easily we introduce the following notation. We say a function $\mathcal{Q}$ has only *quadratic factors*, if its symbolic representation is of the form

$$\mathcal{Q}(t) = \begin{pmatrix} \mathcal{Q}^{(qq)}(t) & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{for all } t. \tag{6}$$

With this notation we have that Q and G have only quadratic factors, while for E

$$E(t) = \begin{pmatrix} E^{(qq)}(t) & 0 \\ E^{(lq)}(t) & E^{(ll)}(t) \end{pmatrix}.$$

From Gaspar (2004) it is known that, provided the factors have been reordered as $Z = \begin{bmatrix} Z^{(q)}, & Z^{(l)} \end{bmatrix}^\top$, the following conditions are sufficient for existence of a GQTS for risk-free bond prices. In this paper we assume that these conditions hold.

Assumption 2.3. *Assume that for* k_i *and* g_{ij} *in (4) the following holds:*

$$k_i = \begin{pmatrix} 0 & 0 \\ 0 & k_i^{(ll)} \end{pmatrix} \ \forall\ i \text{ and} \qquad g_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & g_{ij}^{(ll)} \end{pmatrix} \ \forall\ i,j \text{ s.t. } Z_i, Z_j \in Z^{(q)}.$$

[2]The symmetry assumption is not restrictive. Any non-symmetric quadratic form can be rewritten in an equivalent symmetric way with the advantage that the symmetric representation is unique.

The value of the bond-price can be determined by use of the Feynman-Kac formula in terms of certain ODEs. In our quadratic approach this will always lead to the following system of Riccati ODEs.

Definition 2.4 (Basic ODE System). Denote $\mathcal{T} := \{(t,T) \in \mathbb{R}^2 : 0 \leq t \leq T\}$ and consider functions A, B and C on $\mathcal{T}$ with values in $\mathbb{R}$, $\mathbb{R}^m$ and $\mathbb{R}^{m\times m}$, respectively. For functions ϕ_1 and ϕ_2, ϕ_3 on $\mathbb{R}^+$ with values in $\mathbb{R}$, $\mathbb{R}^m$ and $\mathbb{R}^{m\times m}$, respectively, we say that $(A, B, C, \phi_1, \phi_2, \phi_3)$ solves the *basic ODE system* if

$$\begin{aligned} \frac{\partial A}{\partial t} + d^\top(t)B + \frac{1}{2}B^\top k_0(t)B + \operatorname{tr}\{Ck_0(t)\} &= \phi_1(t) \\ \frac{\partial B}{\partial t} + E^\top(t)B + 2Cd(t) + \frac{1}{2}\tilde{B}^\top K(t)B + 2Ck_0(t)B &= \phi_2(t) \\ \frac{\partial C}{\partial t} + CE(t) + E^\top(t)C + 2Ck_0(t)C + \frac{1}{2}\tilde{B}^\top G(t)\tilde{B} &= \phi_3(t) \end{aligned}$$

subject to the boundary conditions $A(T,T) = 0, B(T,T) = 0, C(T,T) = 0$. A, B and C should always be evaluated at (t,T). E, d, k_0, are the functions from the above definitions (recall (3)-(4)) while

$$\tilde{B} := \begin{pmatrix} B & 0 & \cdots & 0 \\ 0 & B & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & B \end{pmatrix}, \quad K(t) = \begin{pmatrix} k_1(t) \\ \vdots \\ k_m(t) \end{pmatrix}, \quad G(t) = \begin{pmatrix} g_{11}(t) & \cdots & g_{1m}(t) \\ \vdots & \ddots & \vdots \\ g_{m1}(t) & \cdots & g_{mm}(t) \end{pmatrix}, \tag{7}$$

where we have $\tilde{B}, K \in \mathbb{R}^{m^2\times m}$ and $G \in \mathbb{R}^{m^2\times m^2}$.

We recall that the risk-free zero-coupon bond prices are given by

$$p(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)du} \middle| \mathcal{F}_t^W\right],$$

and that only in special cases can we obtain the bond prices in closed-form.

As proven in Gaspar (2004) the general quadratic case is one of those special cases and the zero-coupon bond prices can be easily obtained from solving the basic ODE system.

Result 2.5. *Suppose that Assumptions 2.1 holds. Furthermore assume Assumption 2.3 is verified when the factors Z are reordered as $Z = \begin{bmatrix} Z^{(q)}, & Z^{(l)} \end{bmatrix}^\top$. Then, the term structure of risk-free zero-coupon bond prices is given by*

$$p(t,T) = \exp\left[A(t,T) + B^\top(t,T)Z_t + Z_t^\top C(t,T)Z_t\right]$$

where (A, B, C, f, g, Q) solves the basic ODE from Definition 2.4. Recall that f, g and Q were given in Equation (5). Furthermore, C has only quadratic factors in the sense of (6).

3 Defaultable bond market

In this section we present the defaultable bond market. Before we present the actual model, we revise some general results needed for the valuation of defaultable bonds and related derivatives.

3.1 Doubly Stochastic Random Times

The results summarized in this section are all well-known, and may be found in any of the following books: McNeil, Frey, and Embrechts (2005), Lando (2004), Bielecki and Rutkowski (2002) or Schönbucher (2003).

We take the approach of explicitly constructing the doubly-stochastic random time τ, which will represent a single default, while the obtained results also hold in more general cases.

Definition 3.1 (Setup). Consider a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$. On this probability space there exists:

- a filtration $(\mathcal{F}_t)_{t\geq 0}$,
- a strictly nonnegative process $(\mu_t)_{t\geq 0}$ adapted to $(\mathcal{F}_t)_{t\geq 0}$,
- a random variable E_1 which is exponentially distributed with parameter 1 which is independent of $\mathcal{F}_\infty$.

Then, $\int_0^t \mu_u\, du$ is an increasing, continuous process. We define the *default time* τ as

$$\tau := \inf\{t \geq 0 : \int_0^t \mu_u\, du = E_1\}. \tag{8}$$

The *information on the default state* is denoted $\mathcal{H}_t := \sigma(\mathbf{1}_{\{\tau>s\}} : 0 \leq s \leq t)$ and the *total information* by $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$.

From the independence of μ and E_1 and under the assumption that E_1 is exponentially distributed, we directly obtain

Lemma 3.2. *For the random time τ, constructed in* (8)*, it holds that*

$$\mathbf{1}_{\{\tau>t\}}\mathbb{Q}\big(\tau > T|\mathcal{F}_T \vee \mathcal{H}_t\big) = \mathbf{1}_{\{\tau>t\}} \exp\Big(-\int_t^T \mu_u\, du\Big).$$

Proof. The essence of the proof is to use independency of E_1 and $\mathcal{F}_T$. First, observe that $\{\tau > T\} = \{\int_0^T \mu_u\, du < E_1\}$. So we have that on $\{\tau > t\}$

$$\mathbb{Q}(\tau > T|\mathcal{F}_T \vee \mathcal{H}_t) = \frac{\mathbb{Q}\Big(\int_0^T \mu_u\, du < E_1, \int_0^t \mu_u\, du < E_1|\mathcal{F}_T\Big)}{\mathbb{Q}\big(\int_0^t \mu_u\, du < E_1|\mathcal{F}_T\big)}.$$

As E_1 is exponentially distributed and independent from $\mathcal{F}_T$ we obtain

$$\mathbb{Q}\Big(E_1 > \int_0^t \mu_u\, du|\mathcal{F}_T\Big) = \exp\Big(-\int_0^t \mu_u\, du\Big),$$

and a similar result for the nominator. Observing, that the probability is zero on $\{\tau \leq t\}$, the conclusion follows. ■

The valuation of defaultable claims will base on the following two results

Theorem 3.3. *For a $\mathcal{F}$-adapted process $(X_t)_{t\geq 0}$ and the random time τ, constructed in* (8)*, it holds that*

$$(i) \qquad \mathbf{1}_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\Big(X_T\mathbf{1}_{\{\tau>T\}}\big|\mathcal{G}_t\Big) = \mathbf{1}_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\Big(X_T e^{-\int_t^T \mu_u\,du}\big|\mathcal{F}_t\Big),$$

$$(ii) \qquad \mathbf{1}_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\Big(X_\tau\mathbf{1}_{\{t<\tau\leq T\}}\big|\mathcal{G}_t\Big) = \mathbf{1}_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\Big(\int_t^T \Big[X_s\mu_s e^{-\int_t^s \mu_u\,du}\Big]\,ds\big|\mathcal{F}_t\Big).$$

Proof. We first prove (i). Using the definition of τ,

$$\mathbb{E}^{\mathbb{Q}}(X_T\mathbf{1}_{\{\tau>T\}}|\mathcal{G}_t) \quad = \quad \mathbf{1}_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\big[X_T\mathbb{E}^{\mathbb{Q}}\big(\mathbf{1}_{\{\tau>T\}}|\mathcal{F}_T\vee\mathcal{H}_t\big)|\mathcal{G}_t\big].$$

Now, Lemma 3.2 can be applied to obtain the inner probability. Finally, we use that $X_T e^{-\int_t^T \mu_u\,du}$ is $\mathcal{F}_T$-measurable and hence independent of E_1. In $\mathcal{G}_t = \mathcal{F}_t\vee\mathcal{H}_t$ the σ-algebra $\mathcal{H}_t$ contains additional information on E_1, but using independency this can be dropped, such that $\mathcal{G}_t$ can be replaced by $\mathcal{F}_t$.

For (ii), we give an intuitive argument[3] . Assume that $h:\mathbb{R}\to\mathbb{R}$ is a Borel-measurable function. Let $\tilde{\mathcal{F}}_T = \mathcal{F}_T\vee\mathcal{H}_t$, then,

$$\mathbf{1}_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\big(h(\tau)\mathbf{1}_{\{\tau\leq T\}}|\tilde{\mathcal{F}}_T\big) = \int_t^T h(s)f_{\tilde{\mathcal{F}}_T}(s)\,ds, \qquad (9)$$

where $f_{\tilde{\mathcal{F}}_t}(s)$ is the conditional density of τ given $\mathcal{G}_t$. f is derived with Lemma 3.2,

$$f_{\tilde{\mathcal{F}}_T}(s) = \lambda(s)\exp\big(-\int_t^s \mu_u\,du\big), \qquad \text{for } s\in(t,T].$$

Finally, on $\{\tau>t\}$,

$$\mathbb{E}^{\mathbb{Q}}\Big(X_\tau\mathbf{1}_{\{t<\tau\leq T\}}\big|\mathcal{G}_t\Big) = \mathbb{E}^{\mathbb{Q}}\Big[\mathbb{E}^{\mathbb{Q}}\Big(X_\tau\mathbf{1}_{\{t<\tau\leq T\}}\big|\tilde{\mathcal{F}}_T\Big)\big|\mathcal{G}_t\Big].$$

In the inner expectation X is measurable, such that we can apply (9) with X replacing h and obtain

$$\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T X_s f_{\tilde{\mathcal{F}}_T}(s)\,ds\big|\mathcal{G}_t\Big].$$

As previously, by independence we can replace $\mathcal{G}_t$ by $\mathcal{F}_t$. ■

3.2 Incomplete Information results in Shot-Noise Effects

Besides the already given motivation, in this section we look at a certain scenario which gives rise to shot-noise effects in intensities.

We consider the situation arising after the Enron accounting debacle[4] in 2001. As became clear to the investors that accounting manipulations hit the disastrous financial situation of Enron, a big trouble in credit markets arouse. Of course, investors were questioning how serious was the impact on other companies and if there had been other manipulations.

[3]A formal proof can be found in Bielecki and Rutkowski (2002, Prop. 8.2.1).

[4]See, e.g. http://en.wikipedia.org/wiki/Enron for a short starter.

Seen from a mathematical viewpoint, investors who want to estimate the default intensity of a company, say A, who might also be in difficulties face the following situation. Assume it is reasonable to consider two cases only, the more general case can be treated similarly: First, the case where the company A is also in difficulties, represented by a high default intensity μ_H and second, the case where it is not, represented by a much smaller μ_L. Denote the probability for the first case by p.

For a certain time, there is no new information to the investors except that the company did not default. If it defaults, there is no need anymore to worry about default intensity. Behaving rational, the investors would seek to determine the true default intensity by conditional expectation,

$$\mathbb{E}\big(\mu|\tau > t\big). \tag{10}$$

The default intensity μ is the random variable which takes the values μ_H, μ_L with probability p and $1-p$, respectively.

In a first step we compute

$$\begin{aligned}\mathbb{P}\big(\mu = \mu_H|\tau > t\big) &= \frac{\mathbb{P}(\mu = \mu_H, \tau > t)}{\mathbb{P}(\tau > t)} \\ &= \frac{pe^{-\mu_H t}}{pe^{-\mu_H t} + (1-p)e^{-\mu_L t}}.\end{aligned}$$

This yields, that the conditional expectation equals

$$(10) = \frac{\mu_H p e^{-\mu_H t} + \mu_L (1-p) e^{-\mu_L t}}{pe^{-\mu_H t} + (1-p)e^{-\mu_L t}}.$$

We plot the expectation in Figure 1. The result is quite intuitive.

First, it is clear that the expectation is between μ_H and μ_L and starts at $\bar{\mu} := p\mu_H + (1-p)\mu_L$, i.e. the average if $p = 0.5$. Second, if p is big enough the graph is not descending rapidly at the beginning, because of the high probability that the riskier case is true. Otherwise the function declines rapidly and converges to μ_L for larger t.

A word of caution is now due. The above considerations refer to $\mathbb{P}$ expectations, while the whole setup of this paper is under some equivalent martingale measure $\mathbb{Q}$. Following the argumentation in Elliott and Madan (1998) we argue, that it is reasonable to assume that shot-noise processes under $\mathbb{P}$ should also be shot-noise processes under $\mathbb{Q}$, just with different parameters. However, this is certainly not true for all martingale measures, but at least for some. Hence, the intensity under $\mathbb{Q}$ will also be a process with shot-noise effects. This assumption necessarily corresponds to an assumption on the market prices of risk. A thorough study of this would be far beyond the scope of this paper and will be treated elsewhere.

Remark 3.4. *In order to use the incomplete information argument as motivation for introducing shot-noise effects we implicitly assume that the market price of jump risk is such that the shot-noise behavior of the intensity holds both under the measures $\mathbb{P}$ and $\mathbb{Q}$.*

We now propose a quadratic model for the default events that includes shot-noise effects. We mainly seek for explicit expressions to all key elements. We start by presenting the setup for default. Then we compute explicitly all key building block as well as the price of some credit derivatives.

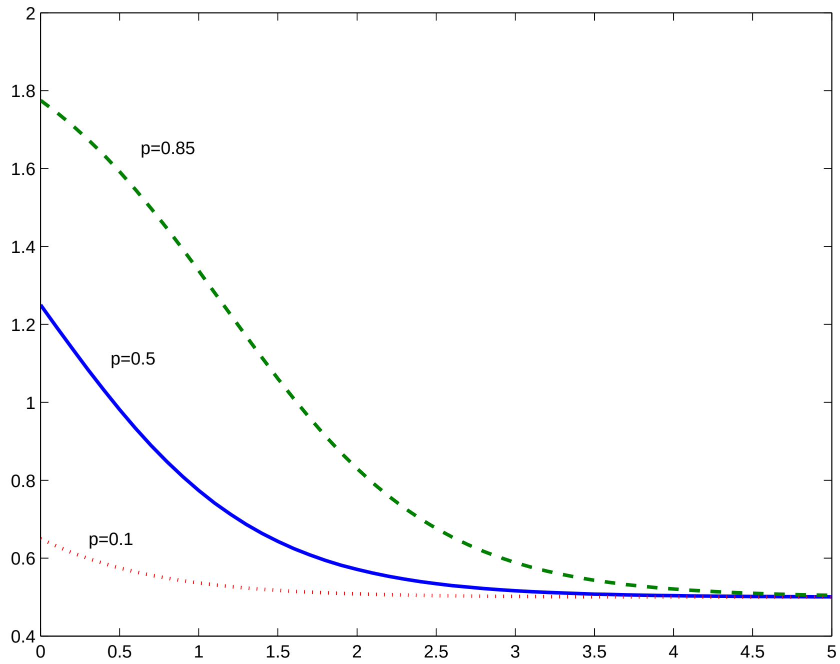


Figure 1: The Graph shows the conditional expectation (10) for several choices of p. On the x-axis we plot time in years. The values of μ_H and μ_L are 2.5 and 0.5, respectively.

3.3 Default events

In this section we will propose the model which will drive the default process. As already mentioned, we will combine a quadratic model with a shot-noise process. The shot-noise process will allow the default intensity to depend on past events, especially on its severity. Moreover, recent events will influence the intensity more than the distant past.

Assumption 3.5. *Consider as given a Wiener process W, a standard Poisson process $\tilde N$ with intensity l, both with respect to a common filtration*[5] *and an independent exponentially distributed variable with parameter 1, E_1. Denote the jumping times of $\tilde N$ by $\tilde\tau_i, i = 1, 2, \dots$.*

The state-variable Z is driven by W with quadratic dynamics as in (2)-(4).[6]

Define the strictly positive processes (η), (J) and (μ) as follows

$$\eta(t, Z_t) = Z_t^\top \mathsf{Q}(t) Z_t + \mathsf{g}^\top(t) Z_t + \mathsf{f}(t) \tag{11}$$

$$J_t = \sum_{\tilde\tau_i \le t} Y_i h(t - \tilde\tau_i) \tag{12}$$

$$\mu_t = \eta_t + J_t \tag{13}$$

where, Q, g and f are deterministic and smooth functions with values in $\mathbb{R}^{m\times m}$, $\mathbb{R}^m$ and $\mathbb{R}$, respectively. Moreover, $\mathsf{Q}(t)$ is assumed to be symmetric for all t. J is called a shot-noise

[5] If W and $\tilde N$ are a Wiener and a Poisson process w.r.t. a common filtration, they are independent. This is because $W + \tilde N$ then is a process with independent increments, hence a Lévy process. It is well known that continuous and jump part of a Lévy process are independent, compare, e.g. Sato (1999, Theorem...).

[6] Taking the same factors Z as for the risk-free process is no loss of generality.

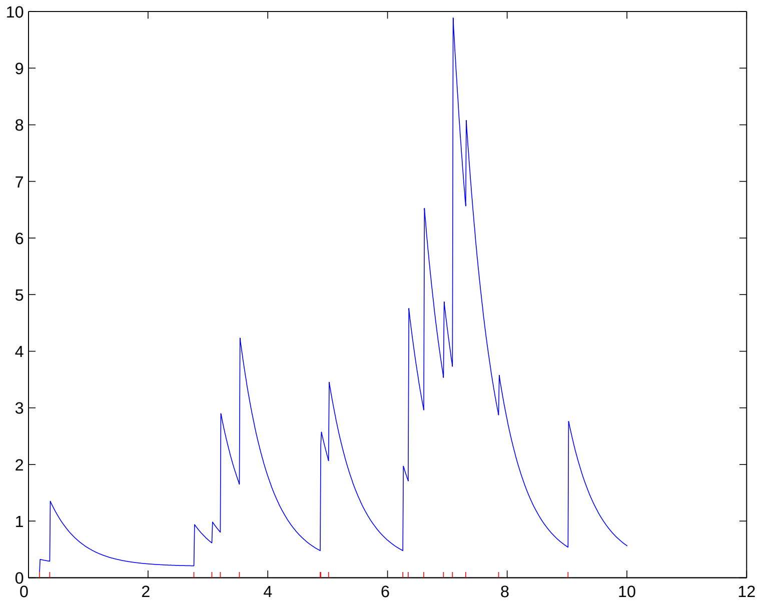


Figure 2: Possible realization of the process J with $h(x) = e^{-0.5x}$ and χ_2^2-distributed Y_i.

process, $Y_i, i = 1, 2, \dots$ are i.i.d. with distribution function F_Y and h is a differentiable function on $\mathbb{R}^+$.

Furthermore, we assume that the default time τ is given as in (8) with the intensity of the form (13).

The filtrations dealt with in Section 3.1 were rather general. In the following definition we specify precisely their meaning in the considered setup. [7]

Definition 3.6 (Filtrations). The filtration $(\mathcal{F})$ describes the accumulated information from market factors Z and J, defined by $\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_t^J = \sigma(Z_s, J_s : 0 \leq s \leq t)$. On the other side, $(\mathcal{H})$ represents the information on the default state, $\mathcal{H}_t := \sigma\left(\mathbf{1}_{\{\tau > s\}} : 0 \leq s \leq t\right)$. The total information to market participants is $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$.

Recall the incomplete information argument described in Section 3.2. The shot-noise effect in the argument is very well captured[8] by the process (J) proposed in Assumption 3.5.The function h describes the declining, more precisely the declining from $\bar{\mu} - \mu_L$ to 0. The jump height represents the market view on $\bar{\mu}$, while in our considerations the function h is very general.[9] Figure 2 show us a possible realization of the process (J).

The reason to include a quadratic component in the intensity has to do with the intuition that the intensity should be driven by a predictable component (the quadratic part) as well as by an unpredictable component (the jump part).

[7]The intensity μ is adapted to the filtration $\mathcal{F}$. Given the independence between W and $\tilde{N}$ we have $\mathcal{F}^W$ and $\mathcal{F}^J$ independent on one another. So, for any process independent of J, conditioning on $\mathcal{F}$ is the same as conditioning on $\mathcal{F}_t^W$. Likewise, for any process independent of r and η (and so of W), conditioning on $\mathcal{F}$ is the same as conditioning on $\mathcal{F}^J$.

[8]Up to a market price of jump risk consideration. See Remark 3.4.

[9]As will be shown, to impose Markovianity, h needs to be of the form ae^{-bt} (see Proposition 3.10 below).

The following lemma will be essential to guarantee non-negativity of the default intensity.

Lemma 3.7. *Consider an arbitrary vector* $Z \in \mathbb{R}^m$*, a symmetric, nonnegative definite matrix* $\mathsf{Q} \in \mathbb{R}^{m\times m}$ *and* $\mathsf{g} \in \mathbb{R}^m$ *such that* g *lies in the subspace spanned by the columns of* Q *and* $\mathsf{f} \in \mathbb{R}$*. Let* Z_* *be the solution of* $\mathsf{Q}Z = -\frac{1}{2}\mathsf{g}$*. Then the polynomial of degree two*

$$Z^\top \mathsf{Q} Z + \mathsf{g}^\top Z + \mathsf{f} \tag{14}$$

is nonnegative, if and only if $Z_* + \mathsf{f} \geq 0$*.*

Proof. According to Harville (1997, Section 19.1) and letting $\mathsf{f} = 0$, Z_* is the minimum of the polynomial in Equation (14). Then $Z_* + \mathsf{f} \geq 0$ implies nonnegativity of (14). ∎

If we have a linear factor, say Z^i, positivity follows if $Z^i \geq 0$ and $\mathsf{g}^i \geq 0$; or, alternatively from $Z^i \leq 0$ and $\mathsf{g}^i \leq 0$. After this, we can concentrate on the factors which have quadratic impact, which were denoted by $Z^{(q)}$. The respective part of Q and g were $\mathsf{Q}(qq)$ and $\mathsf{g}(q)$. With the aid of Lemma 3.7 we obtain the following.

Proposition 3.8. *Assume that* $\mathsf{Q}^{(qq)}(t)$ *is symmetric and nonnegative definite and* $\mathsf{g}^{(q)}(t)$ *lies in the subspace spanned by the columns of* $\mathsf{Q}^{(qq)}(t)$*, both for all* $t \geq 0$*. Denote by* $Z_*(t)$ *the solution of* $\mathsf{Q}^{(qq)}(t)Z = -\frac{1}{2}\mathsf{g}^{(q)}(t)$*. Then* $\eta(t, Z_t)$ *defined in* (11) *is positive, if*

1. *If* Z^i *is a linear factor then either* $Z^i \geq 0$ *and* $\mathsf{g}^i \geq 0$ *or* $Z^i \leq 0$ *and* $\mathsf{g}^i \leq 0$
2. *For all* $t \geq 0$ *it holds that* $Z_*(t) + \mathsf{f}(t) \geq 0$*.*

Using their impact on the drift α, on the volatility σ or on the funtional form of the intensity, we can provide an *intensity* classification of factors.

Definition 3.9. (Classification of intensity factors)

- Z_i is a *intensity quadratic-factor* if it satisfies *at least one* of the following requirements:
  - (i) it has a quadratic impact on (η), i.e. there exists some t such that $\mathsf{Q}_i(t) \neq 0$;
  - (ii) it has a quadratic impact on the functional form of the matrix $\sigma(t,z)\sigma^\top(t,z)$, i.e., there exist k and t such that $g_{ik}(t) \neq 0$;
  - (iii) it affects the drift term of the factors satisfying (i) or (ii), i.e., for Z_j satisfying (i) or (ii) we have $E_{ji}(t) \neq 0$ for some t.
- Z_i is a *intensity linear-factor* if it does not satisfy any of (i)-(iii).

As previously, we write in symbolic form $Z_i \in Z_\eta^{(q)}, Z_\eta^{(l)}$ for the quadratic intensity and linear intensity factors,respectively.

We use the symbolic notation $\bar{Z}^{(q)} = Z^{(q)} \cup Z_\eta^{(q)}$ and $\bar{Z}^{(l)} = Z^{(l)} \cap Z_\eta^{(l)}$, whenever the factors must be ordered according to both their impact on the risk-free short rate, r and on the quadratic part of the intensity, η.

In general, the considered shot-noise processes need not be Markovian. Anyway, from a computational point of view Markovianity is very important. There exists a clear classification, when the considered shot-noise process is Markovian or not.

Proposition 3.10. *Assume that for all $x \in [0, \infty)$ $h(x) \neq 0$. Then the process $(\mu_t)_{t \geq 0}$ is Markovian, if and only if h is of the form $h(t) = ae^{-bt}$.*

Proof. It is clear that for $b = 0$ the process is Markovian, so we need to consider the case where h is not constant.

Assume w.l.o.g. that $h(0) = 1$. As η is a Markovian process, we just have to look at J. To show that J is a Markov-process we calculate the conditional expectation. Consider $s < t$ and recall $\mathcal{F}_t^J := \sigma\{J_s : s \leq t\}$. Then

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[J_t | \mathcal{F}_s^J\right] &= \sum_{i=1}^{\tilde{N}_s} Y_i h(t - \tilde{\tau}_i) + \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=\tilde{N}_s+1}^{\tilde{N}_t} Y_i h(t - \tilde{\tau}_i) \,\Big|\, \mathcal{F}_s^J\right] \\
&= \sum_{i=1}^{\tilde{N}_s} Y_i h(t - \tilde{\tau}_i) + \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=\tilde{N}_s+1}^{\tilde{N}_t - \tilde{N}_s + \tilde{N}_s} Y_i h(t - \tilde{\tau}_i) \,\Big|\, \mathcal{F}_s^J\right].
\end{aligned} \tag{15}$$

In the last expectation, all terms are either measurable w.r.t. $\mathcal{F}_s^J$ or independent of $\mathcal{F}_s^J$. As the Y_i are identically distributed, we can shift the sum and obtain for the expectation

$$\mathbb{E}^{\mathbb{Q}}\left[\sum_{i=j+1}^{\tilde{N}_t - \tilde{N}_s + j} Y_i h\big(t - \tilde{\tau}_i\big) \,\Big|\, \tilde{N}_s = j\right] = \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{\tilde{N}_{t-s}} Y_i h\big(t - \tilde{\tau}_i\big)\right] =: f(s, t),$$

Hence Equation (15) equals

$$\sum_{i=1}^{\tilde{N}_s} Y_i h(t - \tilde{\tau}_i) + f(s, t). \tag{16}$$

As $f(s, t)$ is deterministic, necessary for Markovianity is that there exists a function $F(t, s, x)$, such that

$$\sum_{i=1}^{\tilde{N}_s} Y_i h(t - \tilde{\tau}_i) = F(t, s, J_s) = F\left(t, s, \sum_{i=1}^{\tilde{N}_s} Y_i h(s - \tilde{\tau}_i)\right), \tag{17}$$

so the first term in (16) can be represented as (measurable) function of J_s. We note that each Y_i is independent of all the other appearing terms. We will exploit this property to analyze the behavior of F.

Fix j and consider (17) on the set $\{\tilde{N}_t > j\}$. Taking the conditional expectation of (17) w.r.t. $Y_j = y$, we obtain

$$\mathbb{E}^{\mathbb{Q}}\Big(yh(t - \tilde{\tau}_j) + \sum_{i=1, i \neq j}^{\tilde{N}_s} Y_i h(t - \tilde{\tau}_i)\Big) = \mathbb{E}^{\mathbb{Q}}\Big(F\big(t, s, yh(s - \tilde{\tau}_j) + \sum_{i=1, i \neq j}^{\tilde{N}_s} Y_i h(s - \tilde{\tau}_i)\big)\Big).$$

Deriving w.r.t. y shows that

$$\mathbb{E}^{\mathbb{Q}}\big(h(t - \tilde{\tau}_j)\big) = \mathbb{E}^{\mathbb{Q}}\Big[F_x\Big(t, s, yh(s - \tilde{\tau}_j) + \sum_{i=1, i \neq j}^{\tilde{N}_s} Y_i h(s - \tilde{\tau}_i)\Big) h(s - \tilde{\tau}_j)\Big],$$

where we denoted the partial derivative of F w.r.t. x by F_x. As the l.h.s. does not depend on y, $F_x(t,s,x)$ must be constant in x, and we obtain that F must be of the form $\alpha(t,s)+\beta(t,s)x$.

Examining F on the set $\{\tilde{N}_t = 0\}$, we see that $\alpha(t,s)$ must necessarily be 0. In the next step we determine β. From Equation (17) we obtain for any i

$$h(t-\tilde{\tau}_i) = \beta(s,t)h(s-\tilde{\tau}_i).$$

Hence, $\beta(s,t) = h(t-y)/h(s-y)$ for any $y \geq 0$, and so $b(s,t) = h(t)/h(s)$. From this

$$\frac{h(t-y)}{h(s-y)} = \frac{h(t)}{h(s)}, \qquad \text{for all } t,s,y \geq 0.$$

By letting $s = y$ we obtain that $h(t-y) = h(0)h(t)/h(y)$ and so $h(t+y) = h(t)h(y)/h(0)$. We conclude $h'(y) = h'(0)h(y)/h(0)$. Therefore h is of the form ae^{-by}.

For the converse, note that for $h(y) = e^{-by}$

$$\sum_{i=1}^{\tilde{N}_t} Y_i h(t-\tilde{\tau}_i) = h(t)\sum_{i=1}^{\tilde{N}_t} Y_i h(-\tilde{\tau}_i),$$

and hence J is Markovian. ■

3.4 Building Blocks

In this section we give closed-form analytical expressions to what is known as building blocks in credit risk models. We make extensive use of Theorem 3.3, and thus we ask the reader to recall the various filtrations mentioned.

3.4.1 Implied Survival Probabilities

The survival probabilities under $\mathbb{Q}$ can explicitly be computed and are given in general quadratic form, which we will show in this section. First, observe that the survival probability will be denoted by $\mathbb{Q}_S$ and equals

$$\begin{aligned}
\mathbb{Q}_S(t,T) = \mathbb{Q}\left[\tau > T|\mathcal{G}_t\right] &= \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau>T\}}|\mathcal{G}_t\right] &(18)\\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \mu_u du}\Big|\mathcal{F}_t\right] &(19)\\
&= \mathbb{E}^{\mathbb{Q}}\Big[\exp(-\int_t^T \eta_u + J_u\, du)\Big|\mathcal{F}_t\Big]\\
&= \mathbb{E}^{\mathbb{Q}}\Big[\exp(-\int_t^T \eta_u\, du)\Big|\mathcal{F}_t^W\Big]\mathbb{E}^{\mathbb{Q}}\Big[\exp(-\int_t^T J_u\, du)\Big|\mathcal{F}_t^J\Big]. &(20)
\end{aligned}$$

The first term can be computed using Result 2.5. We note that in the result (r) has to be replaced by (η). Therefore we have to assume a different reordering of factors.

Lemma 3.11. *Suppose Assumption 3.5 hold. Furthermore, assume Assumption 2.3 is verified when the factors Z are reordered as $Z = \Big[Z_\eta^{(q)}, \quad Z_\eta^{(l)}\Big]^\top$. Then,*

$$\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^T \eta_u\, du\Big)\Big|\mathcal{F}_t^W\Big] = \exp\Big[\mathcal{A}(t,T) + \mathcal{B}^\top(t,T)Z_t + Z_t^\top \mathcal{C}(t,T)Z_t\Big]. \qquad (21)$$

where $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathsf{f}, \mathsf{g}, \mathsf{Q})$ *solve the basic ODE system in Definition 2.4. Recall that* f, g *and* Q *are given in Equation (11). Furthermore,* $\mathcal{C}$ *has only quadratic factors.*

Next, we consider the second term in (20):

$$\mathbb{E}^{\mathbb{Q}}\Big[\exp(-\int_t^T J_u\,du)\Big|\mathcal{F}_t^J\Big] \\ = \exp\Big(-\int_t^T \sum_{\tilde{\tau}_i \le t} Y_i h(u-\tilde{\tau}_i)\,du\Big)\mathbb{E}^{\mathbb{Q}}\Big[\exp\big(-\int_t^T \sum_{\tilde{\tau}_i \in (t,u]} Y_i h(u-\tilde{\tau}_i)\,du\big)\Big|\mathcal{F}_t^J\Big]\,. \tag{22}$$

The first term on the l.h.s. denotes the measurable part. It depends on the history of J and it equals

$$\exp\Big(-\int_t^T \sum_{\tilde{\tau}_i \le t} Y_i h(u-\tilde{\tau}_i)\,du\Big) = \exp\Big(-\sum_{\tilde{\tau}_i \le t} Y_i\big[H(T-\tilde{\tau}_i) - H(t-\tilde{\tau}_i)\big]\Big) \\ = \exp\Big\{\tilde{J}_t - \tilde{J}(t,T)\Big\}$$

where we have following notations:

$$H(x) = \int_0^x h(u)\,du\,. \tag{23}$$

and

$$\tilde{J}(t,T) = \sum_{\tilde{\tau}_i \le t} Y_i H(T-\tilde{\tau}_i) \qquad \tilde{J}(t,t) = \tilde{J}_t. \tag{24}$$

Remark 3.12. *Luckily, in the Markovian case the above term simplifies considerably. By Proposition 3.10 we necessarily have that* $h(x) = ae^{-bx}$ *and w.l.o.g. we can assume that* $a = 1$. *Then,*

$$H(x) = \int_0^x h(u)\,du = \frac{1}{b}\Big(1 - e^{-bx}\Big).$$

Therefore,

$$H(T-\tilde{\tau}_i) - H(t-\tilde{\tau}_i) = \frac{1}{b}\big[e^{-b(t-\tilde{\tau}_i)} - e^{-b(T-\tilde{\tau}_i)}\big] \\ = \frac{1}{b}\big[e^{-b(t-\tilde{\tau}_i)} - e^{-b(t-\tilde{\tau}_i)-b(T-t)}\big] \\ = h(t-\tilde{\tau}_i)\cdot H(T-t).$$

which implies

$$\tilde{J}_t - \tilde{J}(t,T) = -\sum_{\tilde{\tau}_i \le t} Y_i\big[H(T-\tilde{\tau}_i) - H(t-\tilde{\tau}_i)\big]\Big] \\ = -H(T-t)\sum_{\tilde{\tau}_i \le t} Y_i h(t-\tau_i) = -H(T-t)\,J_t. \tag{25}$$

Let us consider the remaining expectation (second term in (22)). First, recall that jumps occur with intensity l. We will use that a Poisson process has independent increments. Note that the number of jumps in $(t,u]$ is given by $\tilde{N}_u - \tilde{N}_t$, such that this term is independent of N_t. Then

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\Big[\exp\big(-\int_t^T \sum_{\tilde{\tau}_i \in (t,u]} Y_i h(u-\tilde{\tau}_i)\,du\big)\Big|\mathcal{F}_t^J\Big] \\
&\qquad = \sum_{k=0}^{\infty} \mathbb{E}^{\mathbb{Q}}\Big[\mathbf{1}_{\{\tilde{N}_T-\tilde{N}_t=k\}} \exp\Big(-\sum_{\tilde{\tau}_i \in (t,T]} Y_i \int_t^T \mathbf{1}_{\{\tilde{\tau}_i \le u\}} h(u-\tilde{\tau}_i)\,du\Big)\Big|\mathcal{F}_t^J\Big] \qquad (26)
\end{aligned}$$

It is well-known, that conditional on k jumps the jump times are distributed like the order statistics of uniform random variables over the interval, see for example Rolski, Schmidli, Schmidt, and Teugels (1999, p. 502). More precisely, denote by η_i, $i=1,\dots,k$ independent $U[0,1]$ random variables. Define and set $x = T-t$. Then, the expectation in (26) equals e^{-lx} for $k=0$ and for $k \ge 1$,

$$\begin{aligned}
e^{-lx}\frac{(lx)^k}{k!}\,&\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\sum_{i=1}^{k} Y_i \int_{\tilde{\tau}_i}^T h(u-\tilde{\tau}_i)\,du\right)\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\sum_{i=1}^{k} Y_i H(T-\,t-(T-t)\eta_{i:k})\right)\right].
\end{aligned}$$

As the Y_i are i.i.d. we can interchange the order of the sum. Denote by $\varphi_Y(\cdot)$ the Laplace transform of Y. Then

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\sum_{i=1}^{k} Y_i H\big(x(1-\eta_i)\big)\right)\right] = \left[\int_0^1 \varphi_Y\Big(H\big(xu\big)\Big)\,du\right]^k =: D(x)^k. \qquad (27)$$

The previous computations give the following lemma.

Lemma 3.13. *If $D(T-t)$ exists, then with $\tilde{J}$ as defined in (24) we have that*

$$\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^T J_u\,du\Big)\big|\mathcal{F}_t^J\Big] = \exp\Big[\tilde{J}_t - \tilde{J}(t,T) + (T-t)l\big(D(T-t)-1\big)\Big]. \qquad (28)$$

Summing up, we obtain the survival probabilities in the following form.

Proposition 3.14. *Denote by $x := T-t$ and consider $\mathcal{A},\mathcal{B},\mathcal{C}$ from Lemma 3.11 and $\tilde{J}$ as in (24). Then the survival probability on the interval (t,T), is given by*

$$\mathbb{Q}_S(t,T) = \exp\Big[\tilde{J}_t - \tilde{J}(t,T) + \mathcal{A}(t,T) + xl\big(D(x)-1\big) + \mathcal{B}^\top(t,T)Z_t + Z_t^\top \mathcal{C}(t,T)Z_t\Big].$$

Note that the exponent splits up into a deterministic part $\mathcal{A}(t,T) + x(lD(x)-l)$, a linear part $\mathcal{B}^\top(t,T)Z_t$ and a quadratic part $Z_t^\top\mathcal{C}(t,T)Z_t$ and the term $\tilde{J}_t - \tilde{J}(t,T)$, which is is affine in J in the Markovian case (recall (25)).

3.4.2 Defaultable bond prices with zero recovery

The price of a defaultable zero coupon bond under zero recovery, given by the risk-neutral expectation of its discounted payoff equals, on $\{\tau > t\}$,

$$\begin{aligned}
\bar{p}_0(t,T) &= \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^T r_u du\Big)\mathbf{1}_{\{\tau>T\}}\Big|\mathcal{G}_t\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^T r_u + \mu_u du\Big)\Big|\mathcal{F}_t\Big] = \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^T r_u + \eta_u + J_u du\Big)\Big|\mathcal{F}_t\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^T J_u\Big)|\mathcal{F}_t^J\Big]\cdot\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^T r_u + \eta_u du\Big)\Big|\mathcal{F}_t^W\Big]
\end{aligned}$$

The first expectation has been computed in (28). It remains to compute the second expectation. Once again we will use Result 2.5. To this, we need to replace r by $r + \eta$. Also, we have to consider the proper ordering.

Lemma 3.15. *Suppose Assumption 2.3 holds for Z, reordered as $Z = \big[\bar{Z}^{(q)}, \quad \bar{Z}^{(l)}\big]$. Under Assumption 3.5 and , we have that*

$$\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^T r_u + \eta_u du\Big)\Big|\mathcal{F}_t\Big] = \exp\big[\bar{A}(t,T) + \bar{B}^\top(t,T)Z_t + Z_t^\top\bar{C}(t,T)Z_t\big]. \tag{29}$$

Here $(\bar{A}, \bar{B}, \bar{C}, f + \mathsf{f}, g + \mathsf{g}, Q + \mathsf{Q})$ solve the basic ODE in Definition 2.4. Furthermore, $\bar{C}$ has only quadratic factors.

Using the above computations we obtain the following formula in general quadratic form.

Proposition 3.16. *Denote by $x := T - t$, consider $\bar{A}, \bar{B}, \bar{C}$ from Lemma 3.15, $\tilde{J}$ from (24) and D from (27). Then, the price of a defaultable zero-coupon bond under zero recovery is*

$$\bar{p}_0(t,T) = \exp\Big[\tilde{J}_t - \tilde{J}(t,T) + \bar{A}(t,T) + xl\big(D(x) - 1\big) + \bar{B}^\top(t,T)Z_t + Z_t^\top\bar{C}(t,T)Z_t\Big]. \tag{30}$$

In particular we note that

$$\bar{p}_0(t,T) \neq p(t,T)\mathbb{E}^{\mathbb{Q}}\Big[e^{-\int_t^T \mu_u du}\Big|\mathcal{F}_t\Big].$$

The reason why this equation does not hold is the dependence of r and η on the same state variable Z. It is not caused by J, because J is independent of X and, thus, of these two processes.

3.4.3 Default digital payoffs

It is well known, that evaluating a payment at default time, typically involves computing the following expectation

$$e(t,T) = \mathbb{E}^{\mathbb{Q}}\Big[\mu_T e^{-\int_t^T r_u + \mu_u du}|\mathcal{F}_t\Big]$$

which can be interpreted as the price of a security which pays 1 under the assumption that default happens at time T.[10]

[10] Formally, if we denote the price of security that pays 1 unit of currency if default happens between $[T, T+\delta]$ by $e^*(t,T,T+\delta)$. Then, $e(t,T) = \lim_{\delta\to 0}\frac{1}{\delta}e^*(t,T,T+\delta)$.

Luckily, we will be able to use already computed expectations for this. It may be recalled that by φ_Y we denote the Laplace transform of Y and the functions D, H were defined in (27) and (23), respectively.

Before we actually compute $e(t,T)$ we introduce the notion of *interlinked ODE system*.

Definition 3.17 (Interlinked ODE system). Denote $\mathcal{T} := \{(t,T) \in \mathbb{R}^2 : 0 \le t \le T\}$ and consider functions a, b, c, B and C on $\mathcal{T}$ with values in $\mathbb{R}$, $\mathbb{R}^m$,$\mathbb{R}^{m\times m}$, $\mathbb{R}^m$ and $\mathbb{R}^{m\times m}$, respectively. For functions ϕ_1 and ϕ_2, ϕ_3 with values in $\mathbb{R}$, $\mathbb{R}^m$ and $\mathbb{R}^{m\times m}$, respectively, we say that $(a,b,c,B,C,\phi_1,\phi_2,\phi_3)$ solves the *interlinked ODE system* if it solves

$$\frac{\partial a}{\partial t} + d^\top(t)b + B^\top k_0(t)b + \operatorname{tr}\{ck_0(t)\} = 0 \tag{31}$$

$$\frac{\partial b}{\partial t} + E^\top(t)b + 2cd(t) + \frac{1}{2}\tilde{B}^\top k_0(t)b + 2ck_0(t)B + 2Ck_0(t)b = 0 \tag{32}$$

$$\frac{\partial c}{\partial t} + cE(t) + E^\top(t)c + 4Ck_0(t)c + \frac{1}{2}\tilde{B}^\top G(t)\tilde{b} = 0 \tag{33}$$

subject to the boundary conditions $a(T,T) = \phi_1(T)$, $b(T,T) = \phi_2(T)$, $c(T,T) = \phi_3(T)$. a,b,c and B,C should always be evaluated at (t,T). E, d, k_0, are the functions from (4) while $\tilde{B}, K \in \mathbb{R}^{m^2\times m}$ and $G \in \mathbb{R}^{m^2\times m^2}$ are as in (7).

Proposition 3.18. *Let x:=T-t. The term $e(t,T)$ computes to*

$$\begin{aligned} e(t,T) = \bar{p}_o(t,T)\cdot \Big\{ & \bar{a}(t,T) + \bar{b}^\top(t,T)Z_t + Z_t^\top \bar{c}(t,T)Z_t \\ & + J(t,T) - l\cdot\Big[D(x)(1-x) - 1 + x\varphi_Y\big(H(x)\big)\Big]\Big\}, \end{aligned} \tag{34}$$

where [11]

$$J(t,T) := \sum_{\tilde{\tau}_i \le t} Y_i h(T-\tilde{\tau}_i), \tag{35}$$

and $(\bar{a}, \bar{b}, \bar{c}, \bar{B}, \bar{C}, \mathsf{f}, \mathsf{g}, \mathsf{Q})$ solve the interlinked ODE system of Definition 3.17 with $\bar{B}$,$\bar{C}$ are as in Lemma 3.15.

Proof. We start by noting that

$$\begin{aligned} e(t,T) &= \mathbb{E}^{\mathbb{Q}}\Big[\mu(T)e^{-\int_t^T r(u)+\mu(u)du}|\mathcal{G}_t\Big] \\ &= \mathbb{E}^{\mathbb{Q}}\Big[(\eta(T)+J(T))\,e^{-\int_t^T r(u)+\eta(u)+J(u)du}|\mathcal{G}_t\Big] \\ &= \mathbb{E}^{\mathbb{Q}}\Big[\eta(T)e^{-\int_t^T r(u)+\eta(u)du}|\mathcal{F}_t^W\Big]\underbrace{\mathbb{E}^{\mathbb{Q}}\Big[e^{-\int_t^T J(u)du}|\mathcal{F}_t^J\Big]}_{II} \\ &\quad + \mathbb{E}^{\mathbb{Q}}\Big[J(T)e^{-\int_t^T J(u)du}|\mathcal{F}_t^J\Big]\underbrace{\mathbb{E}^{\mathbb{Q}}\Big[e^{-\int_t^T r(u)+\eta(u)du}|\mathcal{F}_t^W\Big]}_{III} \end{aligned}$$

The expectations II and III have already been computed in Lemmas 3.13 and 3.15, respectively.

[11] This J notation is consistent with the use of J in (12), since we have $J(t,t) = J_t$.

It remains to compute the expectations

$$IV = \mathbb{E}^{\mathbb{Q}}\left[\eta(T)e^{-\int_t^T r(u)+\eta(u)du}|\mathcal{F}_t^W\right] \qquad \text{and} \qquad V = \mathbb{E}^{\mathbb{Q}}\left[J(T)e^{-\int_t^T J(u)du}|\mathcal{F}_t^J\right].$$

From Lemma A.2 in the appendix we know that

$$\begin{aligned} IV &= III \cdot \Big(\bar{a}(t,T) + \bar{b}(t,T)^\top(t)Z_t + Z_t^\top \bar{c}(t,T)(t)Z_t\Big) \\ V &= II \cdot \Big\{\sum_{\tilde{\tau}_i \le t} Y_i h(T-\tilde{\tau}_i) - l \cdot \Big[D(x)(1-x) - 1 + x\varphi_Y\big(H(x)\big)\Big]\Big\} \end{aligned}$$

To achieve the result, recall that $\bar{p}_o(t,T) = II \times II$ and observe that

$$\begin{aligned} e(t,T) &= IV \cdot II + III \cdot V \\ &= \bar{p}_o(t,T) \cdot \Big\{\bar{a}(t,T) + \bar{b}^\top(t,T)Z_t + Z_t^\top \bar{c}(t,T)Z_t \\ &+ \sum_{\tilde{\tau}_i \le t} Y_i h(T-\tilde{\tau}_i) - l \cdot \Big[D(x)(1-x) - 1 + x\varphi_Y\big(H(x)\big)\Big]\Big\}. \end{aligned}$$

■

Remark 3.19. *In the Markovian case, $h(x) = ae^{-bx}$ and the above formula may be simplified to*

$$\sum_{\tilde{\tau}_i \le t} Y_i h(T-\tilde{\tau}_i) = \frac{h(T)}{h(t)} \sum_{\tilde{\tau}_i \le t} Y_i h(t-\tilde{\tau}_i) = \frac{h(T)}{h(t)} J_t.$$

We note that

$$e(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\mu(T)e^{-\int_t^T r(u)+\mu(u)du}|\mathcal{G}_t\right] = \bar{p}_o(t,T)\bar{\mathbb{E}}^T\left[\mu(T)|\mathcal{G}_t\right]$$

thus using (34) we obtain the following.

Corollary 3.20. *By $\bar{\mathbb{E}}^T$ we denote the expectation under the T-survival measure. Then*

$$\begin{aligned} \bar{\mathbb{E}}^T\Big(\mu(T)|\mathcal{F}_t\Big) &= \bar{a}(t,T) + \bar{b}^\top(t,T)Z_t + Z_t^\top \bar{c}(t,T)Z_t \\ &+ J(t,T) - l \cdot \Big[D(x)(1-x) - 1 + x\varphi_Y\big(H(x)\big)\Big]. \end{aligned}$$

with $J(t,T)$ as in (35).

3.5 Incorporating positive Recovery

The expressions computed in the previous section mainly rely on the zero-recovery assumption. Of course, quantities like defaultable bonds typically have a positive recovery. In this section we will show how to extend the previous results to incorporate different recovery schemes.

We will consider two cases, recovery of treasury (RT) and recovery of market value (RMV). They differ in the interpretation of what is known as *loss quota q*. The exact meaning of q will be made clear in the descriptions below. Here we just point out that q is allowed to be some arbitrary random variable with values in $[0,1]$, as long as it is independent of everything else. [12]

Recall the definition of filtration $\mathcal{F}$ from Definition 3.6.

Definition 3.21. A *T-defaultable asset* is given by an $\mathcal{F}_T$-measurable random variable $\mathcal{X}$. At maturity T, the amount $\mathcal{X}$ is paid if no default happened until then. If a default happened before T, some recovery is paid.

Assumption 3.22. *The recovery of a T-defaultable asset $\mathcal{X}$ depends on the loss quota q, which is given by a random variable in the unit interval $[0,1]$ with distribution F_q. We assume that the loss quota q is independent of $\mathcal{G}_\infty$.*

Denote the expected value of q by $\bar{q} = \mathbb{E}^{\mathbb{Q}}[q]$.

3.5.1 Recovery of Treasury

In the recovery of treasury (RT) setup, the recovery of defaultable claims is expressed in terms of the market value of equivalent default-free assets. If a default happened before maturity, the final payoff is reduced to a proportion, $(1-q)$ times the promised payoff. q is revealed at default, and the reduced payment, the recovery, is paid at maturity. It is assumed to be no more subject to default risk.

Under RT it is straightforward to price any defaultable assets based on prices of equivalent risk-free and defaultable zero-recovery assets. The equivalent risk-free asset has the same payoff as the defaultable asset, but it is not subject to default risk. The next proposition states the general pricing rule under recovery of treasury.

Proposition 3.23. *Consider a T-defaultable asset. Let $\bar{\pi}_o(t)$ be the price of this defaultable asset under zero recovery and $\pi(t)$ be the price of the equivalent risk-free asset. Assume, that the recovery is of type RT and Assumption 3.22 holds. Then, on $\{t < \tau\}$, the price of the defaultable asset at time t is given by*

$$\bar{\pi}_{RT}(t) = \bar{q}\bar{\pi}_o(t) + (1-\bar{q})\pi(t).$$

Proof. We are working on $\{\tau > t\}$. Then, by definition, RT yields

$$\bar{\pi}_{RT}(t) \quad = \quad \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)du}\Big(\mathcal{X}\mathbf{1}_{\{\tau>T\}} + (1-q)\mathcal{X}\mathbf{1}_{\{\tau\leq T\}}\Big)\big|\mathcal{G}_t\right].$$

If we condition on q we have by independence

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)du}\Big((1-q)\mathcal{X} + q\mathcal{X}\mathbf{1}_{\{\tau>T\}}\Big)\big|\mathcal{G}_t \vee q\right]\\ &\qquad = (1-q)\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)du}\mathcal{X}\big|\mathcal{F}_t\right] + q\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(u)du}\mathcal{X}\mathbf{1}_{\{\tau>T\}}\big|\mathcal{G}_t\right]\\ &\qquad = (1-q)\pi(t) + q\bar{\pi}_o(t).\end{aligned}$$

[12] The assumption of independence between default events and recovery has been standard in the literature. In Gaspar and Slinko (2005) this assumption is relaxed to include realistic credit spreads features, at the cost of tractability. There it is shown that one must rely on numerical simulations to price any defaultable asset. In this paper we stick to the "traditional" assumption.

Besides q the above term is $\mathcal{G}_t$ measurable. As q is independent of $\mathcal{G}_\infty$, and $\pi(t)$, $\bar{\pi}_o(t)$ do not depend on q, we get

$$\bar{\pi}_{RT}(t) = \mathbb{E}^{\mathbb{Q}}[(1-q)]\pi(t) + \mathbb{E}^{\mathbb{Q}}[q]\bar{\pi}_o(t) \ ,$$

and the result follows. ∎

With this result we easily obtain zero-coupon defaultable bond prices under recovery of treasury.

Corollary 3.24. *Let $x := T - t$. Under RT, the price at time t of a zero-coupon bond maturing at T is*

$$\bar{p}_{RT}(t,T) = \bar{q}\ \exp\Big[\tilde{J}_t - \tilde{J}(t,T) + xl\Big(D(x) - 1\Big) + \bar{A}(t,T) + \bar{B}^\top(t,T)Z_t + Z_t^\top \bar{C}(t,T)Z_t\Big]$$
$$+ (1-\bar{q})\exp\Big[A(t,T) + B^\top(t,T)Z_t + Z_t^\top C(t,T)Z_t\Big].$$

where A, B, C are as in Result 2.5, $\bar{A}, \bar{B}, \bar{C}$ as in Lemma 3.15 and $\tilde{J}$ as defined in (24).

3.5.2 Recovery of market value

When we consider recovery of market value (RMV) we assume that if a default happens, then the recovery of the defaultable asset is $(1-q)$ times its pre-default value,[13]

$$(1-q)\bar{\pi}_{RMV}(\tau-). \tag{36}$$

The following result is a straightforward adaption to our setup of a well know result.[14]

Result 3.25. *Consider a T-defaultable asset $\mathcal{X}$ and assume that Assumption 3.22 is in force. Then the price of the defaultable asset under RMV equals*

$$\bar{\pi}_{RMV}(t) = \mathbf{1}_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s + \bar{q}\mu_s ds}\mathcal{X}|\mathcal{F}_t\right] + \mathbf{1}_{\{\tau\le t\}}e^{\int_\tau^t r_s\, ds}\,(1-q)\,\bar{\pi}_{RMV}(\tau-).$$

For a general payoff $\mathcal{X}$ there is not much more to say, but given a concrete situation more explicit formulas can be obtained. The next proposition gives the price of a defaultable zero-coupon bond under RMV in closed-form.

Proposition 3.26. *The price at time t of a zero-coupon bond maturing at T under RMV equals*

$$\bar{p}_{RMV}(t,T) = \mathbf{1}_{\{\tau\le t\}}e^{\int_\tau^t r_s\, ds}(1-q)\,\bar{p}_{RMV}(\tau-,T)$$
$$+ \mathbf{1}_{\{\tau>t\}}e^{\left\{\tilde{J}_t - \tilde{J}(t,T) + (T-t)l(D(\bar{q},T-t)-1) + \bar{A}(\bar{q},t,T) + \bar{B}^\top(\bar{q},t,T)Z_t + Z_t^\top \bar{C}(\bar{q},t,T)Z_t\right\}}$$

where $(\bar{A}, \bar{B}, \bar{C}, f + \bar{q}\,\mathsf{f}, g + \bar{q}\,\mathsf{g}, Q + \bar{q}\mathsf{Q})$ solves the basic ODE in 2.4 and we denote, with H from (23)*, and*

$$D(\bar{q},x) := \int_0^1 \varphi_Y(\bar{q}H(x(1-u)))du = \int_0^1 \varphi_Y(\bar{q}H(xu))du. \tag{37}$$

[13] The RMV model is inspired by the recovery rules of OTC derivatives.

[14] Note that it is a consequence of Definition 3.6 that the process $\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s + \bar{q}\mu_s ds}\mathcal{X}|\mathcal{F}_t\right]$ does not jump at τ. For the original results, compare Lando (2004) or Schönbucher (2003) to find an intuitive discretization of the result when q is assumed constant. The generalization to random q under Assumption 3.22 follows easily.

Proof. For a zero coupon bond price the payoff at maturity is $\mathcal{X} = 1$. We apply Result 3.25 and we need to compute

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s+\bar{q}\mu_s ds}|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s+\bar{q}\eta_s ds}|\mathcal{F}_t^W\right]\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T q\cdot J_s ds}|\mathcal{F}_t^J\right].$$

Note the similarities to the expectations computed in Lemmas 3.13 and 3.15. Following exactly the steps from the proofs while keeping track of the "$\bar{q}$" gives the result. Details may be found in the appendix. ■

3.6 Pricing Credit Derivatives

In this section we price credit derivatives using the prices and key ingredients previously derived. Among others, we show how prices for credit default swaps (CDS) can be obtained. The CDS is the most liquid credit risky product, so pricing formulas are necessary for calibration to real data.

3.6.1 Default Digital put

We start by pricing what is known as a default digital put (DDP) with maturity T. A DDP pays off 1 exactly at default if default happens before or at T. Its value at time t (given no previous default) is

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^\tau r_u\,du}\mathbf{1}_{\{\tau<T\}}\middle|\mathcal{G}_t\right]\\
&\quad= \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s r_u+\mu_u\,du}\mu_s\,ds\middle|\mathcal{F}_t\right]\\
&\quad= \int_t^T \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^s r_u+\mu_u\,du}\mu_s\middle|\mathcal{F}_t\right]ds = \int_t^T e(t,s)ds\\
&\quad= \int_t^T \bar{p}_o(t,s)\Big\{\bar{a}(t,s) + \bar{b}^\top(t,s)Z_t + Z_t^\top\bar{c}(t,s)Z_t\\
&\quad+ J(t,s) - l\cdot\Big[D(s-t)(1-s+t) - 1 + (s-t)\varphi_Y\big(H(s-t)\big)\Big]\Big\}\,ds,
\end{aligned}$$

where φ_Y is the Laplace transform of Y while a, b and c are solutions of (31)-(33), and $J(t,\cdot)$, D are defined in (35) and (27), respectively. The above integrals can easily be evaluated using the already obtained expressions of all ingredients.

3.6.2 Credit Default Swap

Definition 3.27. A *credit default swap* (CDS) consists of two legs, the fixed and the floating leg[15]. The fixed leg involves a regular *fee payment* and the floating leg offers a protection payment at default.

The CDS starts at some point T_0 and payments are done at the dates $T_1 < T_2 < \cdots < T_{N^*}$. At each T_n the following payments occur:

[15] The floating leg is also called the *default insurance*.

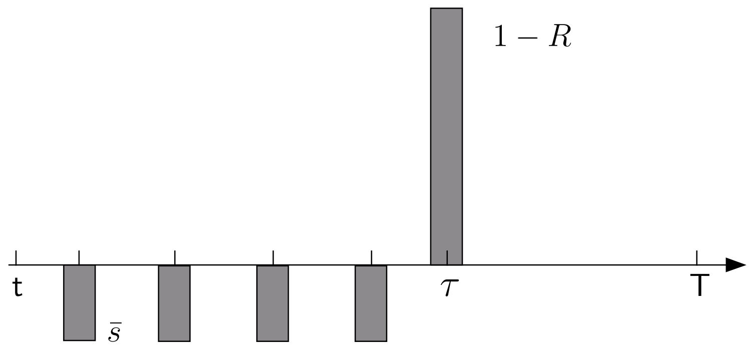


Figure 3: Cash flows for a CDS. Default occurs at τ before the option expires. The payoff is the difference of the par value of the bond to the bond's price at default, the recovery R. The default swap spread, $\bar{s}$, is paid regularly at times $T_1, \ldots$ until default.

- *Fixed leg:* pays a fixed amount called *the spread*, $\bar{s}$, times the length of the interval, $\bar{s} \cdot (T_n - T_{n-1})$ if there was no default in $(T_{n-1} - T_n]$
- *Floating leg:* pays the difference between the nominal value and the recovery value if default occured in $(T_{n-1}, T_n]$. Typically the nominal value is normalized to 1 u.c. and the payment is equal to the loss quota q. Of course the loss quota is related to the recovery R by $q = (1 - R)$,

Initially, the spread $\bar{s}$ of the CDS is determined in such a way that the initial value of the CDS is zero. The spread remains fixed such that as time passes by the value of the CDS can become quite different from zero.

Typically $t = T_0$. Otherwise the CDS is called a forward-start-CDS, and the spread can be computed using similar methods. The value at time t of the fixed leg is

$$\bar{s} \sum_{n=1}^{N^*} (T_n - T_{n-1}) \bar{p}_o(t, T_n) \ .$$

To compute the floating leg, we need the value of 1 unit of money payed at T_n if default happens in $(T_{n-1}, T_n]$. This value is denoted by $e^*(t, T_{n-1}, T_n)$. Observe that e^* was not computed in the previous section, but is closely related to e as

$$e(t, T_n) = \lim_{T_{n-1} \to T_n} \frac{1}{T_n - T_{n-1}} \, e^*(t, T_{n-1}, T_n).$$

A basic difference to a risk-free swap appears in the above formulation: not all terms needed to compute the credit spread are liquidly traded in the market: The $\bar{p}_o(t, T_n)$ in this case. Of course, under the assumption of fixed recovery one could compute these from ordinary bond prices, but nevertheless it is not a priori clear what the right recovery assumption is.

The following proposition gives an expression in closed form.

Proposition 3.28. *We have the following*

$$e^*(t, T_{n-1}, T_n) = \bar{p}_o(t, T_{n-1}) e^{\alpha(t, T_{n-1}, T_n) + \beta^\top(t, T_{n-1}, T_n) Z_t + Z_t^\top \gamma(t, T_{n-1}, T_n) Z_t} - \bar{p}_o(t, T_n), \quad (38)$$

where α, β and γ are deterministic functions and solve the follwoing system of ODE

$$\left\{\begin{array}{rcl} \frac{\partial \alpha}{\partial t}+d^{\top}(t)\beta+\frac{1}{2}\beta^{\top}k_0(t)\beta+\operatorname{tr}\gamma k_0(t)+\beta^{\top}k_0(t)\bar{B} &=& 0 \\ \alpha(T_{n-1},T_{n-1},T_n) &=& A(T_{n-1},T_n) \end{array}\right. \tag{39}$$

$$\left\{\begin{array}{rcl} \frac{\partial \beta}{\partial t}+E^{\top}(t)\beta+2\gamma d(t)+\frac{1}{2}\tilde{\beta}^{\top}K(t)\beta+2\gamma k_0(t)\beta & & \\ +2\bar{C}k_0(t)\beta+2\gamma k_0(t)\bar{B}+\tilde{\beta}^{\top}K(t)\bar{B} &=& 0 \\ \beta(T_{n-1},T_{n-1},T_n) &=& B(T_{n-1},T_n) \end{array}\right. \tag{40}$$

$$\left\{\begin{array}{rcl} \frac{\partial \gamma}{\partial t}+\gamma E(t)+E^{\top}(t)\gamma+2\gamma k_0(t)\gamma+\frac{1}{2}\tilde{\beta}^{\top}G(t)\tilde{\beta} & & \\ +4\bar{C}k_0(t)\gamma+\tilde{\bar{B}}^{\top}G(t)\tilde{\beta} &=& 0 \\ \gamma(T_{n-1},T_{n-1},T_n) &=& C(T_{n-1},T_n) \end{array}\right. \tag{41}$$

A, B and C are from Result 2.5, while $\bar{B}$ and $\bar{C}$ from Proposition 3.16. α,β,γ should be evaluated at (t,T_{n-1},T_n) and $\bar{B},\bar{C}$ at $(t,T-n-1)$.

Proof. We first note that, the expected discounted value of 1 payed at T_n if default happens in $(T_{n-1},T_n]$ is given by

$$\begin{aligned} e^*(t,T_{n-1},T_n) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_n} r_s ds}\Big(\mathbf{1}_{\{\tau>T_{n-1}\}}-\mathbf{1}_{\{\tau>T_n\}}\Big)\Big|\mathcal{G}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_n} r_s ds}\left(e^{-\int_t^{T_{n-1}}\mu_s ds}-e^{-\int_t^{T_n}\mu_s ds}\right)\Big|\mathcal{F}_t\right], \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_n} r_s ds}e^{-\int_t^{T_{n-1}}\mu_s ds}\Big|\mathcal{F}_t\right]-\bar{p}_o(t,T_n). \end{aligned}$$

It remains to compute the expectation. Note that

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_n} r_s ds}e^{-\int_t^{T_{n-1}}\mu_s ds}\Big|\mathcal{F}_t\right] \\ &\quad= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_{n-1}} r_s+\mu_s ds}p(T_{n-1},T_n)\big|\mathcal{F}_t\right] \\ &\quad= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_{n-1}} r_s+\eta_s ds}p(T_{n-1},T_n)\big|\mathcal{F}_t^W\right]\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_{n-1}} J_s ds}\big|\mathcal{F}_t^J\right]. \end{aligned}$$

The last step is due to the independence between (J) and the other terms. The second expectation was computed in Lemma 3.13. Observe that $p(T_{n-1},T_n)$ has the well-known form given in Result 2.5. Then, Lemma A.1 allows us to derive the above expectation. We give the full details in the appendix, which show (38). ■

For $T_{n-1}\to T_n$ we recover many well-known functions out of α,β and γ as shown in the following lemma.

Lemma 3.29. *For the triple $(\alpha,\bar{A},a)$ we have the following relation:*

$$\lim_{\delta\to 0}\frac{\partial\alpha}{\partial\delta}(t,T,T+\delta)-\frac{\partial\bar{A}}{\partial\delta}(t,T+\delta)=a(t,T). \tag{42}$$

Here, α is as in Proposition 3.28, $\bar{A}$ as in Lemma 3.15 and a as in Result 2.5. The result also holds for $(\beta,\bar{B},b)$ and $(\gamma,\bar{C},c)$.

Proof. First, note that

$$
\begin{aligned}
e(t,T) &= \lim_{\delta\to 0}\frac{1}{\delta}e^*(t,T,T+\delta)\\
&= \lim_{\delta\to 0}\bar{p}_o(t,T)\left(\frac{\partial\alpha}{\partial\delta}(t,T,T+\delta)+\frac{\partial\beta}{\partial\delta}^{\top}(t,T,T+\delta)Z_t+Z_t^{\top}\frac{\partial\gamma}{\partial\delta}(t,T,T+\delta)Z_t\right)\\
&- \lim_{\delta\to 0}\frac{\partial\bar{p}_o}{\partial\delta}(t,T+\delta) \qquad (43)
\end{aligned}
$$

We use the representation of $\bar{p}_o(t,T+\delta)$ derived in Proposition 3.16 and obtain

$$
\begin{aligned}
\frac{\partial}{\partial\delta}\bar{p}_o(t,T,T+\delta) \quad = \quad & \Big\{\frac{\partial}{\partial\delta}\bar{A}(t,T+\delta)+lD(T+\delta-t)+l(T+\delta-t)\frac{\partial D}{\partial\delta}(T+\delta-t)-l\\
&+\frac{\partial}{\partial\delta}\bar{B}(t,T+\delta)Z_t-J_t+Z_t^{\top}\frac{\partial}{\partial\delta}\bar{C}(t,T+\delta)Z_t\Big\}\bar{p}(t,T+\delta)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
(43) = \bar{p}_o(t,T)\Big\{&\left(\lim_{\delta\to 0}\frac{\partial\alpha}{\partial\delta}(t,T,T+\delta)\right)+\left(\lim_{\delta\to 0}\frac{\partial\beta}{\partial\delta}(t,T,T+\delta)\right)^{\top}Z_t\\
&+Z_t^{\top}\left(\lim_{\delta\to 0}\frac{\partial\gamma}{\partial\delta}(t,T,T+\delta)\right)Z_t-\left(\lim_{\delta\to 0}\frac{\partial\bar{A}}{\partial\delta}(t,T,T+\delta)\right)-lD(T-t)\\
&-l(T-t)\frac{\partial D}{\partial T}(T-t)+l-\left(\lim_{\delta\to 0}\frac{\partial\bar{B}}{\partial\delta}(t,T+\delta)\right)^{\top}Z_t+J_t-Z_t^{\top}\left(\lim_{\delta\to 0}\frac{\partial\bar{C}}{\partial\delta}(t,T+\delta)\right)Z_t\Big\}.
\end{aligned}
$$

and the result follows from $\frac{\partial D}{\partial T}(T-t)=[1-\varphi_Y(H(T-t))]$ comparing the above expression with (43). ■

With the above results the value of the floating leg can be obtained in closed form:

$$
q\sum_{n=1}^{N^*}e^*(t,T_{n-1},T_n).
$$

Finally, the spread $\bar{s}$ that leads to equal value of both legs at time t is

$$
\bar{s}=q\frac{\sum_{i=1}^{N^*}e^*(t,T_{n-1},T_n)}{\sum_{i=1}^{N^*}(T_n-T_{n-1})\bar{p}_o(t,T_n)}. \qquad (44)
$$

It is straightforward to generalize to random recovery, which is independent of all the other factors which . Then R simply has to be replaced by $\bar{R}=\mathbb{E}^{\mathbb{Q}}(R)$ in the above formulas.

3.6.3 Options on defaultable bonds

In this section we consider a put option on a zero-recovery defaultable bond. The payoff at maturity of a put option with maturity T written on a bond with maturity $T^*>T$

and with strike X is given by $\max(X - \bar{p}_o(T,T^*), 0)$. Here $\bar{p}_o(T,T^*)$ denotes the price of a zero-recovery bond, compare Proposition 3.16.

In the Markovian case we are able to deduce a quite concrete formula for European option prices. We define

$$\Delta_Z(Z_T,T,T^*) = \bar{A}(T,T^*) + (T^*-T)l[D(T^*-T)-l] + \bar{B}^\top(T,T^*)Z_T + Z_T^\top \bar{C}(T,T^*)Z_T$$

For motivation, take a put on a zero-recovery defaultable bond. The price of the put equals

$$\begin{aligned}\text{put}(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}[X - \bar{p}_0(T,T^*)]\mathbf{1}_{\{\bar{p}_0(T,T^*)<X\}}\Big|\mathcal{G}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}X\mathbf{1}_{\{\bar{p}_0(T,T^*)<X\}}\Big|\mathcal{G}_t\right] - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\bar{p}_0(T,T^*)\mathbf{1}_{\{\bar{p}_0(T,T^*)<X\}}\Big|\mathcal{G}_t\right].\end{aligned}$$

We look carefully at the second expectation above. To this, we use the explicit form of $\bar{p}_0$ from Proposition 3.16,

$$\bar{p}_o(T,T^*) = \exp\Big(\tilde{J}_T - \tilde{J}(T,T^*)\Delta_Z(Z_T,T,T^*)\Big),$$

where Δ_Z is defined above and we note that it is of quadratic form. It is clear that if J is not Markovian, one has to look more closely at $\tilde{J}$ terms. For the Markovian case, however, we recall (25) and obtain that

$$\bar{p}_o(T,T^*) = \exp\Big(-H(T^*-T)\,J_T + \Delta_Z(Z_T,T,T^*)\Big).$$

In Appendix B we show how to determine the conditional distribution of J_T given $\mathcal{F}_t^J$, if not explicitly then by inverting the Laplace-transform. At this point of generality one can not get much further, but in concrete examples (i.e. for specific distributions of Y) it is possible to derive more detailed formulas.

For now, we denote the conditional density of J_T given $\mathcal{F}_t^J$ by $F_{J_T|J_t}$ and for every European claim with the payoff $X(Z_T,J_T)$ at T we can use independence of Z and J. Define

$$\tilde{X}(z,J_t) := \int X(z,j)F_{J_T|J_t}(dj).$$

The first step in evaluating a derivative is to compute $\tilde{X}$ on basis of J_t. With this, one can use in a second step the structure of the quadratic setup to derive the price of the derivative:

$$\mathbb{E}\Big(e^{-\int_t^T r_s\,ds}X(Z_T,J_T)\Big|\mathcal{F}_t\Big) = \mathbb{E}\Big(e^{-\int_t^T r_s\,ds}\tilde{X}(Z_T,J_t)\Big|Z_t,J_t\Big).$$

The remaining expectations can be computed numerically or using inverse Fourier/Laplace transform.[16]

We now go on with the analysis and consider several firms issuing default securities. This will allow us to address issues of portfolio credit risk.

[16] This technique was originally proposed by Duffie, Pan, and Singleton (2000), generalized by Heston (1993) and Leippold and Wu (2002). A clever step used in Eberlein and Raible (1999) improves the computational speed.

4 Portfolio Credit Risk

4.1 Setup

To study portfolio credit risk we need to consider defaultable securities, from several firms $k = 1, \cdots, \bar{K}$ also called *names.* We denote the notional associated with each firm by M^k.

Each firm may default only once and its default time is denoted by T^k. The counting process counting *all* defaults is denoted by $N_t := \sum \mathbf{1}_{\{T^k \leq t\}}$. If a default of name k happens, we denote the loss quota by q^k.

We order the default times $T^1, \ldots, T^{\bar{K}}$ and denote the outcome by $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_{\bar{K}}$.

Furthermore, we need to know which company defaulted at τ_k, and we therefore define the *identity of the j-th default* by

$$a_j = k \qquad \text{if} \qquad \tau_j = T^k.$$

At time t we therefore know $a_1, \cdots, a_{N_t}$.

For modeling individual defaults we take a setup similar to the previous section, except that each firm's default intensity is now driven by firm specific as well as systematic risks, which are common to all firms. Assumption 4.1 formally states the new intensity form.

Assumption 4.1. *Set* $\mathbf{k} = \left\{1, \cdots, \bar{K}\right\}$*. Consider independent processes* μ^i *of the form quadratic* [17] *plus jump and identical in distribution, for* $i \in \mathbf{k} \cup \{c\}$*, i.e.,*

$$\mu_t^i = \eta_t^i + J_t^i \quad \text{and} \quad J_t^i = \sum_{\tilde{\tau}_j^i \leq t} Y_j^i h^i(t - \tilde{\tau}_j^i), \qquad \eta_t^i = Z_t^\top \mathsf{Q}^i(t) Z_t + \mathsf{g}^i(t)^\top Z_t + \mathsf{f}^i(t)$$

The default intensity of each defaultable firm $k \in \mathbf{k}$ *is modeled as*[18]

$$\lambda_t^k = \mu_t^k + \epsilon^k \mu_t^c. \tag{45}$$

Furthermore, we assume that the risk-free short rate r *is independent of the firm specific intensity* μ^k *but not necessarily of the common intensity* μ^c*.*

The higher ϵ_i the bigger is the dependence of the common default risk driven by μ^c.

For intuition take $\epsilon_i \equiv \epsilon$. Then, if μ^c jumps then suddenly the default risk of all the assets increase a lot and we will see numerous defaults. This can also be caused by a rise in the quadratic part to a high level, but then it is more or less predictable. The first effect causes some clustering similar to contagion effects, which means if one company defaults and others are closely related to this company, they are very likely to default also. The latter effect is more like a business cycle effect, so on bad days more companies default than on good days.

The formulas listed in the following remark are fundamental building blocks for the portfolio setup. They are more ore less straightforward generalizations of the results given in the

[17] We note that to get indepence of μ^i we also need, in particular, independence of η^i. Given that we are dealing with the same Z state variables independence is achieved imposing, for a given i, that if we have $(Q^i)_j \neq 0$ or $(g^i)_j \neq 0$, then $(Q^i)_j = 0$, $(g^k)_j = 0$ for all $k \neq i$. In words, any element in Z can only appear in *one* η^i.

[18] When dealing with only one firm, as in Section 3, the distinction between firm specific and systematic risks becomes irrelevant. This distinction only makes sense in a portfolio context.

previous sections. We give full details in Lemma A.2 in the appendix. We also introduce a concise short hand notation for the different expressions which will be helpful in the computations to come.

Remark 4.2. *Set $x = T - t$ and $\mathbf{k} = \{1, \cdots, \bar{K}\}$. We take $\theta \in \mathbb{R}$ and $i = \mathbf{k} \cup \{c\}$. Furthermore, we introduce the short hand notation on the l.h.s:*

$$\begin{aligned} S^i_\eta(\theta,t,T) = \quad & \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \theta\eta^i_s ds}|\mathcal{F}^W_t\right] & = e^{\mathcal{A}^i(\theta,t,T)+\mathcal{B}^{i\top}(\theta,t,T)Z_t+Z_t^\top\mathcal{C}^i(\theta,t,T)Z_t} \\ S^i_J(\theta,t,T) = \quad & \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \theta J^i_s ds}|\mathcal{F}^J_t\right] & = e^{\theta(\tilde{J}_t-\tilde{J}(t,T))+l^i x[D^i(\theta,x)-1]} \\ \bar{S}^c_\eta(\theta,t,T) = \quad & \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s+\theta\eta^c_s ds}|\mathcal{F}^W_t\right] & = e^{\bar{A}^c(\theta,t,T)+\bar{B}^{c\top}(\theta,t,T)Z_t+Z^\top\bar{C}^c(\theta,t,T)Z_t} \end{aligned}$$

where $(\mathcal{A}^i,\mathcal{B}^i,\mathcal{C}^i,\theta\mathsf{Q}^i,\theta\mathsf{g}^i,\theta\mathsf{f}^i)$, $(\bar{A}^c,\bar{B}^c,\bar{C}^c,Q+\theta\mathsf{Q}^c,g+\theta\mathsf{g}^c,f+\theta\mathsf{f}^c)$ solve the basic ODE system of Definition 2.4, $D^i(\theta,x) = \int_0^1 \varphi[\theta H^i(x(1-u))]du$, $H^i(x) = \int_0^x h^i(u)du$, and $\tilde{J}^i$ is defined similarly to (24) (using H^i and Y^i).

$$\begin{aligned} \Gamma^i_\eta(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\theta\eta^i_T e^{-\int_t^T \theta\eta^i_s ds}|\mathcal{F}^W_t\right] \\ &= S^i_\eta(\theta,t,T)\cdot\Big(a^i(\theta,t,T)+b^{i\top}(\theta,t,T)Z_t+Z_t^\top c^i(\theta,t,T)Z_t\Big) \\ \Gamma^i_J(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\theta J^i_T e^{-\int_t^T \theta J^i_s ds}|\mathcal{F}^J_t\right] \\ &= S^i_J(\theta,t,T)\cdot\left[\theta J^i(t,T) - l^i\cdot\Big(D^i(\theta,x)(1-x)-1+x\varphi^i\big(\theta H^i(x)\big)\Big)\right] \\ \bar{\Gamma}^c(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\theta\eta^c_T e^{-\int_t^T r_s+\theta\eta^c_s ds}|\mathcal{F}^W_t\right] \\ &= \bar{S}^c_\eta(\theta,t,T)\cdot\Big(\bar{a}^c(\theta,t,T)+\bar{b}^{c\top}(\theta,t,T)Z_t+Z_t^\top\bar{c}^c(\theta,t,T)Z_t\Big) \end{aligned}$$

where $(a^i,\ b^i,\ c^i,\ \mathcal{B}^i,\ \mathcal{C}^i,\ \theta\mathsf{f}^i,\ \theta\mathsf{g}^i,\ \theta\mathsf{Q}^i)$, $(\bar{a}^c,\ \bar{b}^c,\bar{c}^c,\ \bar{B}^c,\ \bar{C}^c,\ \theta\mathsf{f}^c,\ \theta\mathsf{g}^c,\ \theta\mathsf{Q}^c)$ solve the interlinked system of Definition 3.17 and $J^i(t,T)$ is defined similarly to (35) (using h^i and Y^i).

Furthermore, we have

$$\begin{aligned} S^i(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \theta\mu^i_s ds}|\mathcal{F}^W_t\right] = S^i_\eta(\theta,t,T)\cdot S^i_J(\theta,t,T) \\ \bar{S}^c(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s+\theta\mu^c_s ds}|\mathcal{F}^W_t\right] = \bar{S}^c_\eta(\theta,t,T)\cdot S^c_J(\theta,t,T) \\ \Gamma^i(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\theta\mu^i e^{-\int_t^T \theta\mu^i_s ds}|\mathcal{F}^W_t\right] = \Gamma^i_\eta(\theta,t,T)S^i_J(\theta,t,T)+\Gamma^i_J(\theta,t,T)S^i_\eta(\theta,t,T) \\ \bar{\Gamma}^c(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\theta\mu^c e^{-\int_t^T r_s+\theta\mu^c_s ds}|\mathcal{F}^W_t\right] = \bar{\Gamma}^c_\eta(\theta,t,T)S^c_J(\theta,t,T)+\Gamma^c_J(\theta,t,T)\bar{S}^c_\eta(\theta,t,T). \end{aligned}$$

Finally, for $\theta = 1$ we use (t,T) instead of $(1,t,T)$ on the l.h.s. notation.

We keep all the notation from the previous section but we have to add a superscript "$\cdot^k$" to be able to distinguish across firms. This way, $\mathbb{Q}^k_S$, denotes the survival probability of firm k, $\bar{p}^k(t,T)$ is the price of a T-defaultable bond issued by firm k, $e^k(t,T)$ can be interpreted as the price of a payoff of 1 u.c if the firm k defaults at T, while $e^{*k}(e,T_{n-1},T_n)$ is the price if you get 1 u.c. paid if the firm k defaults in $(T_{n-1},T_n]$.

In the next Lemma we derive the key building blocks using the new intensity (45).

Lemma 4.3. *Given Assumption 4.1 we have the following closed form solutions:*

$$\begin{aligned}
\mathbb{Q}_S^k(t,T) &= S^k(t,T)\cdot S^c(\epsilon^k,t,T)\\
\bar{p}_o^k(t,T) &= S^k(t,T)\cdot \bar{S}^c(\epsilon^k,t,T)\\
e^k(t,T) &= \Gamma^k(t,T)\cdot \bar{S}^c(\epsilon^k,t,T)+\bar{\Gamma}^c(\epsilon^k,t,T)\cdot S^k(t,T)\\
e^{*k}(t,T_{n-1},T_n) &= e^{\alpha^k(t,T_{n-1},T_n)+\beta^{k\top}(t,T_{n-1},T_n)Z_t+Z_t^\top\gamma^k(t,T_{n-1},T_n)Z_t}\cdot \bar{p}_o^k(t,T_{n-1})-\bar{p}_o^k(t,T_n)
\end{aligned}$$

where all the $S^\cdot$ *and* $\Gamma^\cdot$ *are as in Remark 4.2 and* α,β,γ *are as in Proposition 3.28.*

Proof. All results follow from the independence of μ^k and μ^c. Concretely, for $\mathbb{Q}_S^k(t,T)$ and $e^k(t,T)$ we have,

$$\begin{aligned}
\mathbb{Q}_S^k(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T\lambda_s^k ds}\middle|\mathcal{F}_t\right]\\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T\mu_s^k ds}\cdot e^{-\int_t^T\epsilon^k\mu_s^c ds}\middle|\mathcal{F}_t\right].
\end{aligned}$$

As μ^k and μ^c are independent we immediately obtain $\mathbb{Q}_S^k(t,T)=S^k(t,T)\cdot S^c(\epsilon^k,t,T)$. Similarly,

$$\begin{aligned}
e^k(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\lambda_T^k e^{-\int_t^T r_s+\lambda_s^k ds}\middle|\mathcal{F}_t\right]\\
&= \mathbb{E}^{\mathbb{Q}}\left[\mu_T^k e^{-\int_t^T\mu_s^k ds}\cdot e^{-\int_t^T(r_s+\epsilon^k\mu_s^c)ds}\middle|\mathcal{F}_t\right]+\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T\mu_s^k ds}\cdot\epsilon^k\mu_T^c e^{-\int_t^T(r_s+\epsilon^k\mu_s^c)ds}\middle|\mathcal{F}_t\right]\\
&= \Gamma^k(t,T)\cdot\bar{S}^c(\epsilon^k,t,T)+S^k(t,T)\cdot\bar{\Gamma}^c(\epsilon^k,t,T).
\end{aligned}$$

The same type of argument can be used to compute $p_o^k(t,T)$ and $e^{*k}(t,T_{n-1},T_n)$. ■

4.2 Default correlation and Clustering

It is often argued that in the framework used here, where the default times are conditionally independent, the resulting default correlation is not high enough. However, already Duffie and Gârleanu (2001) showed that this is not the case. Especially through jumps or, more precisely, high peaks in the intensity a high default correlation is induced.

A problem showing up in the jump-diffusion setting of Duffie and Gârleanu (2001) is the right choice of the mean reversion speed which affects both the diffusion and the jump part.[19] In their model, big jumps are necessary to induce high default correlation. To avoid that the intensity stays on a very high level for a long time, the mean reversion speed must be quite high. On the other hand, such a high mean reversion speed gives unrealistic behavior for the diffusive part. In the framework presented here, this problem is solved, as the mean reversion speeds can be different.

The so-called default correlation is basically the correlation between the default indicators of two companies. Denote by $\mathbb{Q}_D^i$ the probability of company i defaulting in $(t,T]$ and by

[19] In their work the intensity is of the form

$$d\mu_t=\kappa\big(\theta-\mu_t\big)\,dt+\sigma\sqrt{\mu_t}dW_t+dJ_t,$$

where (W) is a Brownian motion and (J) is a pure jump process and thus a special case of a shot-noise process. With this formulation the authors obtain bond prices in an affine form. A problem of this approach is to adjust κ in the right way. This is because κ controls the mean reversion speed of the diffusive as well as of the jump part.

$\mathbb{Q}_D^{i,j}(t,T)$ the probability that companies i and j default in $(t,T]$. The default correlation is defined as

$$\rho^{i,j}(t,T) = \frac{\mathbb{Q}_D^{i,j}(t,T) - \mathbb{Q}_D^i(t,T)\mathbb{Q}_D^j(t,T)}{\sqrt{\mathbb{Q}_D^i(t,T)[1-\mathbb{Q}_D^i(t,T)]\mathbb{Q}_D^j(t,T)[1-\mathbb{Q}_D^j(t,T)]}}.$$

The default probabilities relate to the survival probabilities by $\mathbb{Q}_D^k(t,T) = 1 - \mathbb{Q}_S^k(t,T)$ where $\mathbb{Q}_S$ is given in Lemma 4.3.

Proposition 4.4. *Suppose Assumption 4.1 holds. Then, the default correlation of two different companies i and j is given by*

$$\rho^{i,j}(t,T) = \frac{S^i(t,T)S^j(t,T)\left[S^c(\epsilon^i+\epsilon^j,t,T) - S^c(\epsilon^i,t,T)S^c(\epsilon^j,t,T)\right]}{\sqrt{\mathbb{Q}_D^i(t,T)[1-\mathbb{Q}_D^i(t,T)]\mathbb{Q}_D^j(t,T)[1-\mathbb{Q}_D^j(t,T)]}}.$$

where $S^\cdot$ are as in Remark 4.2 and we recall $\mathbb{Q}_D^\cdot = 1 - \mathbb{Q}_S^\cdot$.

Proof. The probability of joint default of the firms i,j until time T given that none has defaulted until t is given by

$$\begin{aligned}
\mathbb{Q}_D^{i,j}(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{T^i<T\}}\mathbf{1}_{\{T^j<T\}}\middle|\mathcal{G}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\left(1-e^{-\int_t^T\lambda_s^i ds}\right)\left(1-e^{-\int_t^T\lambda_s^j ds}\right)\middle|\mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\left(1-e^{-\int_t^T\mu_s^i+\epsilon^i\mu_s^c ds}\right)\left(1-e^{-\int_t^T\mu_s^j+\epsilon^j\mu_s^c ds}\right)\middle|\mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[1-e^{-\int_t^T\mu_s^i+\epsilon^i\mu_s^c ds} - e^{-\int_t^T\mu_s^j+\epsilon^j\mu_s^c ds} + e^{-\int_t^T\mu_s^i+\mu_s^j+(\epsilon^i+\epsilon^j)\mu_s^c ds}\middle|\mathcal{F}_t\right] \\
&= 1-\mathbb{Q}_S^i(t,T)-\mathbb{Q}_S^j(t,T)+\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T\mu_s^i+\mu_s^j+(\epsilon^i+\epsilon^j)\mu_s^c ds}\middle|\mathcal{F}_t\right].
\end{aligned}$$

Again using independence of μ^i, μ^j and μ^c we obtain

$$\mathbb{Q}_D^{i,j}(t,T) = 1-\mathbb{Q}_S^i(t,T)-\mathbb{Q}_S^j(t,T)+S^i(t,T)S^j(t,T)S^c(\epsilon^i+\epsilon^j,t,T).$$

By the definition of $\rho^{i,j}$ the result follows from

$$\begin{aligned}
\mathbb{Q}_D^i(t,T)\mathbb{Q}_D^j(t,T) &= (1-\mathbb{Q}_S^i(t,T))(1-\mathbb{Q}_S^j(t,T)) \\
&= 1-\mathbb{Q}_S^i(t,T)-\mathbb{Q}_S^j(t,T)+\mathbb{Q}_S^i(t,T)\mathbb{Q}_S^j(t,T) \\
&= 1-\mathbb{Q}_S^i(t,T)-\mathbb{Q}_S^j(t,T)+S^i(t,T)S^c(\epsilon^i,t,T)S^j(t,T)S^c(\epsilon^j,t,T). \quad \blacksquare
\end{aligned}$$

4.3 Portfolio Credit Derivatives

To obtain concrete formulas, we will make a simplifying assumption about homogeneity of the portfolio. Assumptions like this are quite usual in the literature on CDOs. However, it should be pointed out that the following calculations go through in a similar fashion without these assumptions, but the expressions will get more involved. Nonetheless, typically, portfolio credit derivatives base on a portfolio of homogeneous credits, therefore the following assumption is quite plausible for practical purposes.

Assumption 4.5. *We consider a portfolio of homogeneous credits, i.e. the notionals are equal, $M^i = M$, the recoveries are assumed to be time-independent and i.i.d. $q^{a_j}(\tau_j) = q_j$ and the correlation factors are the same, $\epsilon^i = \epsilon$. Moreover, we assume that Assumption 4.1 holds, and that the processes J^k for all $k = 1 \cdots \bar{K}$ therein are equivalent, i.e. they are based on the same set of parameters F_Y, h, l. On the other hand, J^c has the parameters F_Y^c, h^c, l^c.*

4.3.1 First-to-Default Swaps

A *first-to-default swap* (FtDS) is a contract which offers protection on the first default of a portfolio only. The two counterparties which exchange payments are named protection seller and protection buyer. Payments are due at fixed payment dates, say $t_1, \cdots, t_{N^*}$. Moreover, there is an initiation date $t_0 < t_1$. If t_0 is in the future, the FtDS is called *forward-start FtDS*. The FtDS is characterized by the so-called *first-to-default spread* s^{FtD} which is fixed at initiation of the contract.

- The *protection seller* pays at t_n, if the first default occurred in $(t_{n-1}, t_n]$ the *default payment*. Assume that name k is the one which defaulted first. Then the default payment equals $M^k \cdot q^k$. If no default happens until t_{N^*} the protection seller pays nothing.
- The *protection buyer* pays the spread s^{FtD}, until the maturity of the FtDS, t_{N^*}, or until the first default (whichever comes first).

As the protection buyer has only fixed payments, the payments due to him are also called *fixed leg*, while the payments of the protection seller are called *floating leg*.

The spread $\bar{s}^{\text{FtD}}$ is chosen in such a way that at initiation of the FtDS its value at t_0 equals zero. Note that there are no payments until t_1. If a default happens before t_0, the contract is worthless. To emphasize the dependence of the spread on the current time write $\bar{s}^{\text{FtD}}(t)$.

The following results rely on the distribution of the first default time, which is the minimum of all default times. The main result is Theorem 4.7

We will make use of Assumption 4.5 to ease exposition. Denote the probability that the first default, τ_1, occurs in $(t, T]$ by $\mathbb{Q}_S^{\text{FtD}}(t, T)$. The next lemma deals with properties of the first default time. Recall from Assumption 4.5 that ϵ the sensitivity of each intensity λ^i to the common part μ^c.

Lemma 4.6. *Suppose that Assumption 4.5 is in force. Consider a portfolio of $\bar{K}$ names and assume no default has occurred up to time t. Then, the survival probability of the first default is given by*

$$\mathbb{Q}_S^{FtD}(t, T) = \mathbf{1}_{\{\tau_1 > t\}}\, S^c(\epsilon \bar{K}, t, T) \cdot \prod_{k=1}^{\bar{K}} S^k(t, T).$$

Furthermore, the value of one unit of currency paid at T only if $\tau_1 > T$ is given by

$$\bar{p}^{FtD}(t, T) = \mathbf{1}_{\{\tau_1 > t\}}\, \bar{S}^c(\epsilon \bar{K}, t, T) \cdot \prod_{k=1}^{\bar{K}} S^k(t, T). \tag{46}$$

Proof. The result is trivial on $\{\tau_1 \leq t\}$, so we consider $\{\tau_1 > t\}$ from now on. Then, by definition $\mathbb{Q}_S^{\text{FtD}}(t,T) = \mathbb{Q}(\tau_1 > T|\mathcal{G}_t)$. We start by conditioning on $\mu^c_{[t,T]} \vee \mathcal{G}_t$. Recall that the default time of name k is denoted by T_k. Then,

$$\mathbb{Q}\left(\tau_1 > T\big|\mu^c_{[t,T]} \vee \mathcal{G}_t\right) = \mathbb{Q}\left(\min(T_1, T_2, \cdots, T_{\bar{K}}) > T\big|\mu^c_{[t,T]} \vee \mathcal{G}_t\right)$$
$$= \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{T_1 > T, T_2 > T, \cdots, T_{\bar{K}} > T\}}\big|\mu^c_{[t,T]} \vee \mathcal{G}_t)\right]. \tag{47}$$

As $T_1, \ldots, T_{\bar{K}}$ are independent, conditionally on $\mu^c_{[t,T]}$, we obtain

$$(47) = \mathbb{E}^{\mathbb{Q}}\bigg(\exp\Big[-\sum_{k=1}^{\bar{K}}\int_t^T \lambda_s^k ds\Big]\,\Big|\mu^c_{[t,T]} \vee \mathcal{F}_t\bigg)$$
$$= e^{-\bar{K}\epsilon\int_t^T \mu_s^c ds} \cdot \mathbb{E}^{\mathbb{Q}}\bigg(\exp\Big[-\sum_{k=1}^{\bar{K}}\int_t^T (J_s^k + \eta_s^k) ds\Big]\Big|\mu^c_{[t,T]} \vee \mathcal{F}_t\bigg)$$

As $\eta^1, \ldots, \eta^{\bar{K}}, J^1, \ldots, J^{\bar{K}}$ are mutually independent we obtain using the expressions given in Remark 4.2

$$(47) = e^{-\epsilon\bar{K}\int_t^T \mu_s^c ds} \cdot \prod_{k=1}^{\bar{K}} S^k(t,T).$$

It may be recalled that $S^k = S_\eta^k S_J^k$. Thus,

$$\mathbb{Q}(\tau_1 > T|\mathcal{G}_t) = \mathbb{E}^{\mathbb{Q}}\bigg(e^{-\epsilon\bar{K}\int_t^T \mu_s^c ds} \cdot \prod_{k=1}^{\bar{K}} S^k(t,T)\Big|\mathcal{F}_t\bigg)$$
$$= S^c(\epsilon\bar{K}, t, T) \cdot \prod_{k=1}^{\bar{K}} S^k(t,T).$$

Using the same methodology with the fact that r is independent of μ^k for all $k \in \mathbf{k}$ but not of μ^c determines $\bar{p}^{\text{FtD}}(t,T)$. ∎

The spread of the FtDS is given by the following result. Recall that $\bar{p}^{\text{FtD}}(t,T)$ was computed in (46).

Theorem 4.7. *Suppose Assumption 4.5 is in force. Consider a portfolio of $\bar{K}$ names and assume no default has occurred up to time t. Then, the spread of the FtDS is given by*

$$\bar{s}^{FtD}(t) = \bar{q}\frac{\sum_{n=1}^{N^*} e^{FtD*}(t, t_{n-1}, t_n)}{\sum_{n=1}^{N^*}(t_n - t_{n-1})\bar{p}^{FtD}(t, t_n)}$$

where

$$e^{FtD*}(t, t_{n-1}, t_n) = e^{\alpha^c(t,t_{n-1},t_n) + \beta^{c\top}(t,t_{n-1},t_n)Z_t + Z_t^\top\gamma^c(t,t_{n-1},t_n)Z_t} \cdot \bar{p}^{FtD}(t, t_{n-1}) - \bar{p}^{FtD}(t, t_n). \tag{48}$$

Here α, β, γ solve the system in (39)-(41). Furthermore, there α, β, γ must be evaluated at (t, T_{n-1}, T_n) while $\bar{B}, \bar{C}$ must be evaluated at $(\epsilon\bar{K}, t, t_{n-1})$.

Proof. For ease of notation we write s^{FtD} instead of $s^{\mathrm{FtD}}(t)$. The value at time t of the fixed leg of the FtDS follows from the results in the previous lemma:

$$\mathbb{E}^{\mathbb{Q}}\Big[\sum_{n=1}^{N^*} e^{-\int_t^{t_n} r_s\,ds}\, s^{\mathrm{FtD}}\, \mathbf{1}_{\{\tau>t_n\}}\big|\mathcal{G}_t\Big] = s^{\mathrm{FtD}} \sum_{n=1}^{N^*}(t_n - t_{n-1})\cdot \bar{p}^{FtD}(t,T_n).$$

For the pricing of the floating leg we need to compute

$$\begin{aligned} e^{\mathrm{FtD}*}(t,T_{n-1},T_n) \quad &:= \quad \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t_n} r_s ds}\mathbf{1}_{\{\tau_1\in(t_n,t_{n-1}]\}}\Big|\,\mathcal{G}_t\right] \\ &= \quad \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t_n} r_s ds}\mathbf{1}_{\{\tau_1>t_{n-1}\}}\Big|\,\mathcal{G}_t\right] - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t_n} r_s ds}\mathbf{1}_{\{\tau_1>t_n\}}\Big|\,\mathcal{G}_t\right], \end{aligned}$$

where the second expectation equals $\bar{p}^{\mathrm{FtD}}(t,t_n)$. Furthermore,

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t_n} r_s ds}\mathbf{1}_{\{\tau_1>t_{n-1}\}}\Big|\,\mathcal{G}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[p(t_{n-1},t_n)\cdot e^{-\int_t^{t_{n-1}} r_s ds}\mathbf{1}_{\{\tau_1>t_{n-1}\}}\Big|\,\mathcal{F}_t\right]. \tag{49}$$

Following the steps from the previous lemma we can deduce the following. Alternatively, in the conditionally independent approach the default intensity of the minimum of the default times is simply the sum over all intensities. However, we get the following

$$(49) = \mathbb{E}^{\mathbb{Q}}\left[p(t_{n-1},t_n)\cdot e^{-\int_t^{t_{n-1}}\left[r_s+\epsilon\bar{K}\mu_s^c+\sum_{k=1}^{\bar{K}}\mu_s^k\right]ds}\Big|\mathcal{F}_t\right]$$

We write short $\tilde{\mathcal{F}}_{t,t_{N^*}}$ for $\mathcal{F}_t\vee\sigma(\mu_s^c, r_s : t\le s\le t_{N^*})$. Conditioning on $\tilde{\mathcal{F}}$ we obtain

$$(49) = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left(e^{-\sum_{k=1}^{\bar{K}}\int_t^{t_{n-1}}\mu_s^k\,ds}\big|\tilde{\mathcal{F}}_{t,t_{N^*}}\right)\cdot p(t_{n-1},t_n)e^{-\int_t^{t_{n-1}}\left(r_s+\epsilon\bar{K}\mu_s^c\right)ds}\Big|\mathcal{F}_t\right].$$

Let us consider the inner expectation more closely. By Assumption 3.22 we have that $\eta^1,\dots,\eta^{\bar{K}}$ are independent of μ^c and r, so that

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\left(e^{-\sum_{k=1}^{\bar{K}}\int_t^{t_{n-1}}\mu_s^k\,ds}\big|\tilde{\mathcal{F}}_{t,t_{N^*}}\right) &= \mathbb{E}^{\mathbb{Q}}\left(e^{-\sum_{k=1}^{\bar{K}}\int_t^{t_{n-1}}\eta_s^k\,ds}\big|\mathcal{F}_t\right)\cdot\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^{t_{n-1}}\sum_{k=1}^{\bar{K}}J_s^k\,ds}\big|\mathcal{F}_t\right) \\ &= \prod_{k=1}^{\bar{K}} S^k(t,t_{n-1}). \end{aligned}$$

It may be recalled that $S^k = S^k_\eta\cdot S^k_J$. To evaluate (49) we can proceed exactly as in Proposition 3.28. In analogy to Proposition 3.28 we obtain that

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\left[p(t_{n-1},t_n)e^{-\int_t^{t_{n-1}}\left(r_s+\epsilon\bar{K}\eta_s^c\right)ds}\Big|\mathcal{F}_t\right] &= \bar{S}^c_\eta(\epsilon\bar{K},t,t_{n-1})\cdot \\ &\cdot\exp\left(\alpha^c(t,t_{n-1},t_n)+\beta^{c\top}(t,t_{n-1},t_n)Z_t+Z_t^\top\gamma^c(t,t_{n-1},t_n)Z_t\right). \end{aligned}$$

The remaining part with J^c is given by S^c_J such that by $\bar{S}^c = \bar{S}^c_\eta\cdot S^c_J$ expression (49) equals

$$e^{\alpha^c(t,t_n,t_{n-1})+\beta^{c\top}(t,t_n,t_{n-1})Z_t+Z_t^\top\gamma^c(t,t_n,t_{n-1})Z_t}\cdot\underbrace{\bar{S}^c(\epsilon\bar{K},t,t_{n-1})\cdot\prod_{k=1}^{\bar{K}}S^k(t,t_{n-1})}_{\bar{p}^{\mathrm{FtD}}(t,t_{n-1})},$$

where α,β,γ are as stated in the theorem. ■

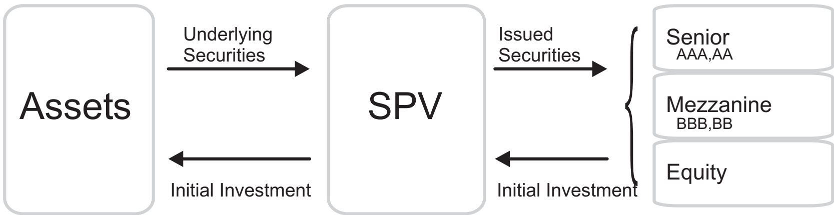
4.3.2 CDOs

We introduce the concept of Collateralized Debt Obligations (CDOs) as in Duffie and Gârleanu (2001). The aim of this section is to price the so-called *synthetic CDOs*.

A synthetic CDO is an asset-backed security whose underlying collateral is a portfolio of CDSs. A CDO allocates interest income and principal repayments from a collateral pool of CDSs to a prioritized collection of CDO securities, called *tranches*.

While there are many variations, a standard prioritization scheme is simple subordination: *senior* CDO notes are paid before *mezzanine* and *equity pice* is paid with the any residual cash-flow.

The following picture clarifies the structure of a CDO. In addition to the general portfolio



setup introduced in Section 4.1 we need to introduce some additional notation to describe the cash-flow of CDOs.

We consider a CDO with several tranches $i = 1, \cdots, \bar{I}$. In the case were we have senior, mezzanine and equity tranches only we would simply take $\bar{I} = 3$. The tranches are separated according to fixed barriers b_i. That is, b_1 separates tranche 1 from tranche 2, b_2 separated the tranche 2 from tranche 3, and so on - compare figure 4.

The loss at each default time τ_j is generally given by $M^{a_j} q^{a_j}(\tau_j)$.[20] However, under the homogeneity Assumption 4.5, it simplifies to $\xi_j := Mq_j$.

The *loss process* of the CDO is given by

$$L(t) := \sum_{j=1}^{N_t} \xi_j.$$

It describes the reduction in face value of the whole underlying portfolio due to according defaults. The *loss of tranche i* is given by

$$L^i(t) = \begin{cases} 0 & \text{if } \ L(t) < b^{i-1} \\ L(t) - b^{i-1} & \text{if } \ b^{i-1} \leq L(t) < b^i \\ b^i - b^{i-1} & \text{if } \ L(t) \geq b^i \end{cases} \tag{50}$$

Figure 4 illustrates the CDO setup with a possible loss path affecting various tranches.

We start by computing the *distribution of portfolios losses* under both the martingale measure and the T- forward measure. This will serve as building block for the pricing of CDOs.

[20] We note that the loss at each default time would depend on the notional amount of the defaulted firm and on the recovery process of that firm evaluated at the default time.

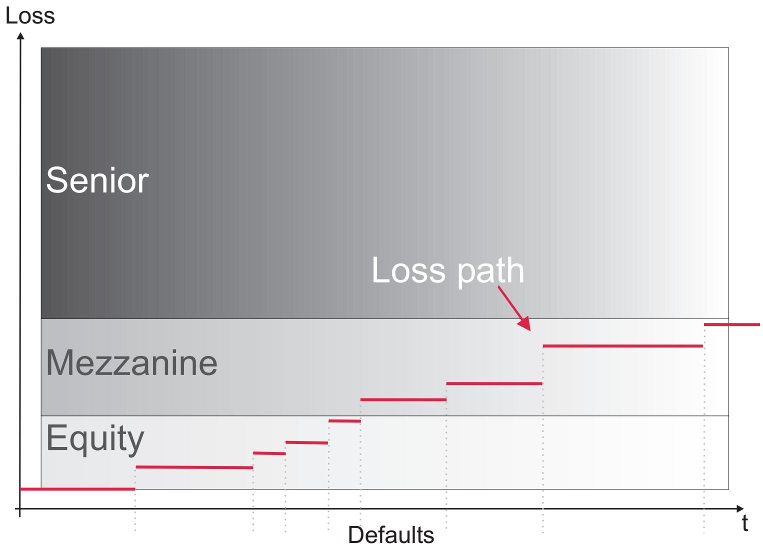


Figure 4: Tranches' losses in CDOs.

In the following we will need the distribution of sums of losses, for which we define

$$\bar{F}_{q,k}(y) := \mathbb{Q}\Big(\sum_{j=1}^{k} q_j > y\Big). \tag{51}$$

Depending on the distribution of the losses, this could be in more ore less closed form. At this point we stick with this abbreviation.

Remark 4.8. *It is somehow a natural choice to model q with a beta-distribution. The beta-distribution is a flexible class of distributions which have support $[0,1]$. Unfortunately, the convolution is not given in closed form. However, $F_{q,k}$ can be obtained via inverting the Fourier-transform. There exists numerical algorithms to do this quite efficiently. Note also, that this has to be computed once and therefore does not effect speed of the valuation algorithm.*

Alternatively, one could use the uniform distribution, and obtain a closed form solution.

Given our setup we can always conclude for the unconditional distribution of the loss function L. However, for pricing and risk management it is necessary to consider L after some time passed by, and we therefore will be interested in the *conditional* distribution of the loss function. To this it will be convenient to require the processes $(\lambda_t^k)_{t\geq 0}$, $k = 1, \dots, \bar{K}$ to be Markovian. We recall that this is equivalent to $h(t) = a\exp(-bt)$ by Proposition 3.10.

Can use Markovianity to conclude for the conditional distribution of L? Lemma 4.9 gives the answer. Before, however, to handle defaulted and non-defaulted companies in a concise way, we need to introduce some more notation.

Denote by $\mathcal{S}_t$ the set which contains the indices of assets not defaulted until t, the "survivors":

$$\mathcal{S}_t := \{1 \leq k \leq \bar{K} : T^k > t\}.$$

In the following Lemma we will fix the number of defaults in the interval $(t, T]$ and then sum over all possible combinations defaults.

We write $\sum_{\mathbf{k}_n \in \mathcal{S}_t}$ for the sum over all sets $\mathbf{k}_n = \{k_1, \dots, k_n\}$ of size n with pairwise different elements and $k_1, \dots, k_n \in \mathcal{S}_t$. $\mathbf{k}_n$ represents the n companies which default in $(t, T]$.

Given $\mathbf{k}_n$, the companies *not* defaulting are denoted by

$$\mathcal{S}_t \backslash \mathbf{k}_n := \{1 \le l \le n : l \in \mathcal{S}_t, l \notin \mathbf{k}_n\}.$$

Furthermore, we write short $\{T^{\mathbf{k}_n} \in (t, T]\}$ for $\{T^{k_1} \in (t, T], \dots, T^{k_n} \in (t, T]\}$.

Lemma 4.9. *Suppose the function $h(x)$ and $h^c(x)$ in Assumption 4.5 are of the form ae^{-bx}. Then the conditional distribution of the portfolio losses, L, is given by*

$$\mathbb{Q}(L_T \le x | \mathcal{G}_t) = \mathbf{1}_{\{T^{\mathcal{S}_t} > t\}} \sum_{n=0}^{\bar{K}-N_t} \cdot F_{q,n}\Big(\frac{x - L_t}{M}\Big)$$
$$\cdot \sum_{\mathbf{k}_n \in \mathcal{S}_t} \Big\{ S^c(\epsilon(\bar{K} - N_t - n), t, T) \Big(\prod_{k \in \mathcal{S}_t \backslash \mathbf{k}_n} S^k(t, T) \Big) - S^c(\epsilon(\bar{K} - N_t), t, T) \Big(\prod_{k \in \mathcal{S}_t} S^k(t, T) \Big) \Big\}$$

where $F_{q,n}(\cdot)$ is defined in (51) and S^k and S^c are of exponential quadratic form as defined in Remark 4.2.

Furthermore, if $t = 0$, the above expression gives the unconditional expectation and the functions $h(x)$, $h^c(x)$ need not have any special form.

Proof. The conditional distribution of L is given by

$$\begin{aligned}\mathbb{Q}(L_T \le x | \mathcal{G}_t) &= \mathbb{Q}\big(L_T - L_t \le x - L_t | \mathcal{G}_t\big) = \mathbb{Q}\Big(\sum_{j=1}^{N_T - N_t} \xi_j \le x - L_t \big| \mathcal{G}_t\Big) \\ &= \mathbb{Q}\Big(\sum_{j=1}^{N_T - N_t} q^j \le \frac{x - L_t}{M} \big| \mathcal{G}_t\Big) = F_{q, N_T - N_t}\Big(\frac{x - L_t}{M}\Big).\end{aligned}$$

Recall that (N) is the counting process of *all* defaults. For the following, we first condition on μ^c. Then all individual defaults τ^i are independent and stem from independent Cox-processes with (also independent) intensities $(\lambda^k(t))_{t \ge 0}$, $k = 1, \dots, \bar{K}$. Observe that $N_T - N_t$ is not independent from N_t[21]. But, it is not difficult to compute the conditional distribution. However, in contrast to the unconditional distribution, we need to distinguish which company defaults.

Using the Markovianity of the processes μ^k we need to determine

$$\mathbb{Q}\big(N_T - N_t = k | \mathcal{S}_t, N_t, \mu^c_{[t,T]}, \mathcal{F}_t\big). \tag{52}$$

We write $\tilde{\mathcal{F}}_t := \sigma(\mathcal{S}_t, N_t, \mu^c_{[t,T]}, \mathcal{F}_t)$. In the above probability we will have k companies defaulting in $(t, T]$. Summing over all possible indices was denoted by $\sum_{\mathbf{k}_n \in \mathcal{S}_t}$. Then,

$$(52) = \sum_{\mathbf{k}_n \in \mathcal{S}_t} \mathbb{Q}\big(T^{\mathbf{k}_n} \in (t, T] | \tilde{\mathcal{F}}_t\big)\, \mathbb{Q}\big(T^{\mathcal{S}_t \backslash \mathbf{k}_n} > T | \tilde{\mathcal{F}}_t\big).$$

[21] For example, if all companies default before t, hence $N_t = \bar{K}$ it follows that $N_T - N_t = 0$.

Note that the survival probability of asset k is given by

$$\mathbb{Q}\big(T^k > T|\tilde{\mathcal{F}}_t\big) = \mathbb{Q}\big(T^k > T|\mathbf{1}_{\{T^k>t\}}, \mu^c_{[t,T]}, \mathcal{F}_t\big)$$
$$= \mathbf{1}_{\{T^k>t\}} \exp\Big(- \epsilon \int_t^T \mu^c_s \, ds\Big) \underbrace{\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(- \int_t^T \mu^k_s \, ds\Big)\big|\mathcal{F}_t\Big]}_{=S^k(t,T)}.$$

The expectation on the r.h.s. is of the exponential quadratic from as given by Remark 4.2. In the Markovian case, (25) can be used to simplify this even further. Furthermore, since, conditionally on μ^c the defaults occur independently, we have

$$\mathbb{Q}\big(T^{\mathbf{k}_n} > T|\tilde{\mathcal{F}}_t\big) = \mathbf{1}_{\{T^{\mathbf{k}_n}>t\}} \exp\Big(- n\epsilon \int_t^T \mu^c_s \, ds\Big) \prod_{k\in\mathbf{k}_n} S^k(t,T).$$

Note that S^k takes the form given in Remark 4.2 and can moreover be simplified according to Equation (25).

On $\{T^k > t\}$ we also have that $\mathbb{Q}\big(T^k \in (t,T]|\tilde{\mathcal{F}}_t\big) = 1 - \mathbb{Q}\big(T^k > T|\tilde{\mathcal{F}}_t\big)$. Hence,

$$\begin{aligned}
&\mathbb{Q}\big(N_T - N_t = n|\tilde{\mathcal{F}}_t\big) = \\
&\quad = \sum_{\mathbf{k}_n\in\mathcal{S}_t} \Big\{1 - e^{-n\epsilon \int_t^T \mu^c_s \, ds} \prod_{k\in\mathbf{k}_n} S^k(t,T)\Big\} \cdot e^{-(\bar{K}-N_t-n)\epsilon \int_t^T \mu^c_s \, ds} \prod_{k\in\mathcal{S}_t\setminus\mathbf{k}_n} S^k(t,T) \\
&\quad = \sum_{\mathbf{k}_n\in\mathcal{S}_t} \Big[e^{-(\bar{K}-N_t-n)\int_t^T \mu^c_s \, ds} \prod_{k\in\mathcal{S}_t\setminus\mathbf{k}_n} S^k(t,T) - e^{-\epsilon(\bar{K}-N_t)\int_t^T \mu^c_s \, ds} \prod_{k\in\mathcal{S}_t} S^k(t,T)\Big]
\end{aligned} \tag{53}$$

After we have done all calculation conditioned on μ^c we finally have to consider the unconditional expectation. This is, on $\{T^{\mathcal{S}_t} > t\}$,

$$\begin{aligned}
&\mathbb{Q}\big(N_T - N_t = n|\mathcal{S}_t, N_t, \mathcal{F}_t\big) \\
&\quad = \sum_{\mathbf{k}_n\in\mathcal{S}_t} \Big[S^c(\epsilon(\bar{K} - N_t - n), t, T) \prod_{k\in\mathcal{S}_t\setminus\mathbf{k}_n} S^k(t,T) - S^c(\epsilon(\bar{K} - N_t), t, T) \prod_{k\in\mathcal{S}_t} S^k(t,T)\Big].
\end{aligned}$$

■

Proposition 4.10. *Denote by $\mathbb{Q}^T$ the T-forward measure. With the above notation we have*

$$\mathbb{Q}^T\big(L_T \leq x|\mathcal{G}_t\big) = \mathbf{1}_{\{T^{\mathcal{S}_t}>t\}} \frac{1}{p(t,T)} \sum_{n=0}^{\bar{K}-N_t} F_{q,n}\Big(\frac{x - L_t}{M}\Big)$$
$$\cdot \sum_{\mathbf{k}_n\in\mathcal{S}_t} \Big\{\bar{S}^c(\epsilon(\bar{K} - N_t - n), t, T)\Big(\prod_{k\in\mathcal{S}_t\setminus\mathbf{k}_n} S^k(t,T)\Big) - \bar{S}^c(\epsilon(\bar{K} - N_t), t, T)\Big(\prod_{k\in\mathcal{S}_t} S^k(t,T)\Big)\Big\}$$

$\bar{S}^c$ and S^k are of exponential quadratic form as in Remark 4.2 and A, B, C are given in Result 2.5.

Note that $p(t,T) = \exp\big(A(t,T) + B^\top(t,T)Z_t + Z_t^\top C^(t,T)Z_t\big)$ by Result 2.5.

Proof. First, observe that

$$p(t,T)\mathbb{Q}^T\left(L_T \le x \middle| \mathcal{G}_t\right) = \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}\mathbf{1}_{\{L_T\le x\}}\middle|\mathcal{G}_t\right).$$

We therefore just need to compute $\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}\mathbf{1}_{\{L_T\le x\}}\middle|\mathcal{G}_t\right)$.

To this, let $\tilde{\mathcal{G}}_t := \sigma(\mathcal{S}_t, N_t, \mu^c_{[t,T]}, \mathcal{G}_t)$ and recall r has common factors, i.e., conditional on μ^c it is known. We thus have

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\mathbf{1}_{\{L_T\le x\}}\middle|\mathcal{G}_t\right] &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{L_T\le x\}}\middle|\tilde{\mathcal{G}}_t\right]\middle|\mathcal{G}_t\right]\\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\mathbb{Q}\left(L_T\le x|\tilde{\mathcal{G}}_t\right)\middle|\mathcal{G}_t\right]\end{aligned}$$

For the inner expectation we may use Equation (53) to obtain that the above equals

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}}\left\{e^{-\int_t^T r_s ds}\sum_{n=0}^{\bar{K}-N_t}F_{q,n}\left(\frac{x-L_t}{M}\right)\right.\\ &\left.\cdot\sum_{\mathbf{k}_n\in\mathcal{S}_t}\left[e^{-\epsilon(\bar{K}-N_t-n)\int_t^T\mu^c_s ds}\left(\prod_{k\in\mathcal{S}_t\setminus\mathbf{k}_n}S^k(t,T)\right)-e^{-\epsilon(\bar{K}-N_t)\int_t^T\mu^c_s ds}\left(\prod_{k\in\mathcal{S}_t}S^k(t,T)\right)\right]\middle|\mathcal{G}_t\right\}.\end{aligned}$$

Recalling the notation defined in Remark 4.2 we have that

$$\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}e^{-\epsilon(\bar{K}-N_t-n)\int_t^T\mu^c_s ds}\middle|\mathcal{G}_t\right) = \bar{S}^c(\epsilon(\bar{K}-N_t-n),t,T)$$

such that we obtain the given expression immediately. ■

We can now focus on the *pricing of tranches of synthetic CDOs*. We make the following normalizations:

- The CDO offers notes on each tranche with par value 1.
- Interest is paid at times $t_1,\cdots,t_{N^*}$
- The value of the entire tranche at the time the CDO is issued (time zero) is

$$V^i(0) = b^i - b^{i-1}$$

- At each intermediate time $t_j < t_{N^*}$ we receive the coupon c^i (the coupon is, obviously, tranche dependent). The payment is on the remaining principal in the trance, so that the payments due at t_j are

$$\left(1-\frac{L^i(t_j)}{b^i-b^{i-1}}\right)c^i(t_j-t_{j-1})$$

- At maturity t_{N^*} the coupon plus the remain value of the tranche is paid:

$$\left(1-\frac{L^i(t_{N^*})}{b^i-b^{i-1}}\right)\left(c^i(t_{N^*}-t_{N^*-1})+1\right).$$

The critical point here is the reinvestment of the recovery payment. Note that in reality, the default of an entity from the underlying pool leads to a non-payment of the future coupons. The recovery has to be re-invested at the current market level and possibly gets a lower coupon. In this section, we assume that these missing future coupons are included in the recovery. This means, that the recovery after default is the actual recovery minus financing cost of the future coupons (which also could be a gain if the market offers better conditions at default).

Define $c_j^i := c^i(t_j - t_{j-1})$ for $i < N^*$ and $c_{N^*}^i := c^i(t_{N^*} - t_{N^*-1}) + 1$ and denote the value of the tranche i at time t by $V^i(t)$. Then $V^i(t)$ is given by

$$\begin{aligned} V^i(t) &= \mathbb{E}^{\mathbb{Q}}\left[\sum_{j=1}^{N^*} e^{-\int_t^{t_j} r(u)du}\left(1-\frac{L^i(t_j)}{b^i-b^{i-1}}\right)c_j^i \middle| \mathcal{G}_t\right] \\ &= \sum_{j=1}^{N^*} p(t,t_j)\left(1-\frac{\mathbb{E}^{t_j}\left[L^i(t_j)\middle|\mathcal{G}_t\right]}{b^i-b^{i-1}}\right)c_j^i, \end{aligned}$$

where $\mathbb{E}^{t_j}[\cdot|\mathcal{G}_t]$ denotes conditional expectation under the t_j-forward measure.

The price of risk-free bonds have been computed in Result 2.5 and the only difficulty is in computing $\mathbb{E}^{t_j}\left[L^i(t_j)\middle|\mathcal{G}_t\right]$. In literature on CDOs it is quite common to assume independence of interest rates and all processes related to the loss process. As we saw when computing the distribution of the portfolio losses, in our framework dealing with the T-forward measure requires only little additional effort, and so here we relax this assumption.

The next theorem gives $\mathbb{E}^T\left[L^i(T)\middle|\mathcal{G}_t\right]$ for all $T > t$ in closed form and concludes the CDO analysis. It uses the notation introduced on page 35. We denote the density of the sum of k losses, similar to the distribution function defined in (51), by $f_{q,k}(\cdot)$.

Theorem 4.11. *Consider $t < T$. The conditional distribution function of L_{t_j} is defined by $F_{\mathcal{G}_t}^{L_T}(x) := \mathbb{Q}^T(L_{t_j} \le x|\mathcal{G}_t)$. We have that its density is given by*

$$\begin{aligned} f_{\mathcal{G}_t}^{L_T}(x) &= \mathbf{1}_{\{T^{S_t}>t\}}\frac{1}{p(t,T)}\sum_{n=0}^{\bar{K}-N_t}\cdot f_{q,n}\Big(\frac{x-L_t}{M}\Big) \\ &\cdot\sum_{\mathbf{k}_n\in\mathcal{S}_t}\left\{\bar{S}^c(\epsilon(\bar{K}-N_t-n),t,T)\left(\prod_{k\in\mathcal{S}_t\setminus\mathbf{k}_n}S^k(t,T)\right)-\bar{S}^c(\epsilon(\bar{K}-N_t),t,T)\left(\prod_{k\in\mathcal{S}_t}S^k(t,T)\right)\right\} \end{aligned}$$

and the conditional expected value of tranche i equals

$$\mathbb{E}^T(L_T^i|\mathcal{G}_t) = (b^i-b^{i-1})\Big[1-F_{\mathcal{G}_t}^{L_T}(b^i)\Big] - b^{i-1}\Big[F_{\mathcal{G}_t}^{L_T}(b^i)-F_{\mathcal{G}_t}^{L_T}(b^{i-1})\Big] + \int_{b^{i-1}}^{b^i} x f_{\mathcal{G}_t}^{L_T}(x)\,dx.$$

Proof. By the definition of the tranche loss in (50) we have that

$$\begin{aligned} \mathbb{E}^T(L_T^i|\mathcal{G}_t) &= \mathbb{E}^T\big(L_T\mathbf{1}_{\{L_T\in(b^{i-1},b^i]\}}|\mathcal{G}_t\big) - b^{i-1}\mathbb{Q}^T\big(L_T\in(b^{i-1},b^i]|\mathcal{G}_t\big) \\ &\quad + (b^i-b^{i-1})\mathbb{Q}^T\big(L_T>b^i|\mathcal{G}_t\big). \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{Q}^T\big(L_T\in(b^{i-1},b^i]|\mathcal{G}_t\big) &= \mathbb{Q}^T\big(L_T\le b^i|\mathcal{G}_t\big) - \mathbb{Q}^T\big(L_T\le b^{i-1}|\mathcal{G}_t\big) \\ \mathbb{E}^T\big(L_T\mathbf{1}_{\{L_T\in(b^{i-1},b^i]\}}|\mathcal{G}_t\big) &= \int_{b^{i-1}}^{b^i} x f_{\mathcal{G}_t}^T(x)\,dx. \end{aligned}$$

The result follows from the closed form for $\mathbb{Q}^T(L_{t_j} \le x|\mathcal{G}_t)$ computed in Proposition 4.10. ■

4.3.3 Link to Credit Indices

In this section we draw the link to currently traded credit indices and discuss the pricing of options written on those indices.

Quite recently, there evolved a liquid market for credit indeces, the so-called CDX or i-Traxx. There is not much literature available, but in Pedersen (2003) and Felsenheimer, Gisdakis, and Zaiser (2004) some informations may be found. The Dow Jones iTraxx emerged from two other indices, the iBoxx and the iTraxx, on 21st of June 2004.

The iTraxx is effectively a portfolio of 125 single CDS. To guarantee liquidity, the portfolio is reorganized on quarterly time points by a certain voting scheme and defaulted entities are removed. The aim of this procedure is to guarantee that the underlying portfolio stays in a certain class of credit worthiness (or rating, respectively).

Especially for the fast growing and very liquid market of credit indices, there is an increasing demand on options raising naturally the question on how to price them.

The mathematical setting for an credit index is as follows. W.l.o.g. we assume that the notional is 1. The credit index is on $\bar{K}$ names, each represented by a CDS with spread $s^i(t)$. Each names are in the same credit class, so that the homogeneous pool Assumption 4.1 will hold. Especially, the single names have equal weight $\frac{1}{\bar{K}}$.

The payment stream of the credit index is as follows. Recall that $\bar{K} - N_t$ is the number of CDS alive at time t

- *Fixed leg:* The spread is paid on the remaining notional, i.e. at each time t_n of the tenor $t_1, \dots, t_{N^*}$ the payoff is

$$\bar{S}(t_n - t_{n-1})\frac{\bar{K} - N_{t_n}}{\bar{K}}.$$

- *Defaulting leg:* We assume the payments of default protection occur at the end of the defaulting period, i.e. the payments of the floating or protection leg in the interval $(t_{n-1}, t_n]$ due at t_n are

$$\sum_{T^k \in (t_{n-1}, t_n]} (1 - R^k(t_n)).$$

  Here, $R^k(t_n)$ is the value of the recovery[22] of the underlying k at time t_n.

Before any default happens and if the recovery is paid as in the underlying CDS, it is clear that the payment streams of the index are equivalent to the payment streams of the portfolio of the equally weighted underlying CDS (with spread denoted by $\bar{s}_i$) and so the spread of the index is simply

$$\bar{S}_t = \frac{1}{\bar{K}}\sum_{k=1}^{\bar{K}} \bar{s}^k(t).$$

[22] For completeness it should be mentioned, that a recovery payment R^k due directly at default time can be incorporated in this setting by setting the recovery to $R^k(t_n) = R^k \exp\big(\int_{T_k}^{t_n} r_u\,du\big)$. This is equivalent to the payment $R^k(T^k)$ at T^k directly.

Now, if a default happens, the situation gets more complicated. One entity is removed and the index still pays the spread $\bar{S}$. However, the spread of the portfolio with equally weighted CDS, where now the defaulting entity is removed has a possibly different spread:

$$\frac{1}{\bar{K}} \sum_{k=1}^{\bar{K}} \bar{s}^k(t) \mathbf{1}_{\{T^k > t\}}.$$

For example, if $\bar{K} = 2$ and s^1 equals 100 and s^2 equals 200, both constant, we obtain for the index spread 150, but after default of name 1 the portfolio with equal weights pays the spread 100 while the index pays the spread 75. This may show that for pricing some more effort has to be done.

We start by determining the value of the index spread at a certain time t. The spread offered by the index is chosen, such that fixed and defaulting leg equal in value. We denote this spread by $\bar{S}_t$.

Using the above formulation, the value of the fixed leg at time t is

$$\bar{S}_t \sum_{t_n \ge t} (t_n - t_{n-1}) \frac{1}{\bar{K}} \sum_{k=1}^{\bar{K}} \mathbb{E}^{\mathbb{Q}} \Big(e^{-\int_t^{t_n} r_u \, du} \mathbf{1}_{\{T^k > t_n\}} \big| \mathcal{G}_t \Big),$$

where $\sum_{t_n \ge t}$ is, more precisely, the sum over all $t_n \in \{t_1, \dots, t_{N^*}\}$ with $t_n \ge t$. The last expectation is equal to $\bar{p}_0^k(t, t_n)$, the appropriate zero-recovery bond for the kth underlying. The value of the floating leg equals

$$\sum_{t_n \ge t}^{N^*} \mathbb{E}^{\mathbb{Q}} \Big(\sum_{T^j \in (t_{n-1}, t_n]} e^{-\int_t^{t_n} r_u \, du} (1 - R_k(t^n)) \big| \mathcal{G}_t \Big).$$

Under the assumption of homogeneity of the recovery as well as time-independent recovery, which is also independent of the other factors, we can replace $1 - R^k(t_n) = q^k$ simply by $\bar{q}$, where $\bar{q}$ is the expectation of the loss quota, q^k. The remaining term has already been calculated in Lemma 4.3. Form there it may be recalled that the value of one unit of currency, paid at t_n, when name k defaults in $(t_{n-1}, t_n]$ was named $e^{*k}(t, t_{n-1}, t_n)$ and can be calculated closed form. With this, the value of the floating leg is

$$\begin{aligned}
\sum_{t_n \ge t} \mathbb{E}^{\mathbb{Q}} \Big(& \sum_{T^j \in (t_{n-1}, t_n]} e^{-\int_t^{t_n} r_u \, du} (1 - R_k(t^n)) \big| \mathcal{G}_t \Big) \\
&= \bar{q} \sum_{t_n \ge t} \sum_{k=1}^{\bar{K}} \mathbb{E}^{\mathbb{Q}} \Big(e^{-\int_t^{t_n} r_u \, du} \mathbf{1}_{\{T^k \in (t_{n-1}, t_n]\}} \big| \mathcal{G}_t \Big) \\
&= \bar{q} \sum_{t_n \ge t} \sum_{k=1}^{\bar{K}} e^{*k}(t, t_{n-1}, t_n).
\end{aligned}$$

We obtain the following formula for the spread of the credit index:

$$\bar{S}_t = \bar{q} \bar{K} \frac{\sum_{t_n \ge t} \sum_{k=1}^{\bar{K}} e^{*k}(t, t_{n-1}, t_n)}{\sum_{t_n \ge t} (t_n - t_{n-1}) \sum_{k=1}^{\bar{K}} \bar{p}_o^k(t, t_n)}.$$

Typically, it is the case that $t_n - t_{n-1} = \Delta$ and the above formula simplifies a bit more.

5 Illustration

In this section we illustrate the results derived in the previous sections with a concrete three-factor model.

5.1 The model

We take

$$Z = \begin{pmatrix} Z^1 \\ Z^2 \\ r \end{pmatrix}$$

as the state variable, and assume its $\mathbb{Q}$-dynamics given by

$$dZ_t^1 = \left[\beta_1(t) - \alpha_1 Z_t^1\right] dt + \sigma_1 dW_t^1 \tag{54}$$

$$dZ_t^2 = \left[\beta_2(t) - \alpha_2 Z_t^2\right] dt + \sigma_2 dW_t^2 \tag{55}$$

$$dr_t = \alpha_r \left[\beta_r - r_t\right] dt + \sigma_r \sqrt{r_t} dW_t^r \tag{56}$$

where α_i, σ_i, for $i = 1, 2, r$ and β_r are constants, while $\beta_1(\cdot)$, $\beta_2(\cdot)$ are deterministic functions of t and W^1, W^2 and W^r are independent $\mathbb{Q}$-Wiener processes.

We will analyze two firms, denoted 1 and 2. Each firm's intensity is driven by firms specific as well as common factors in accordance with Assumption 4.1.

For each firm the intensity is given by

$$\lambda_t^k = \mu_t^k + \epsilon^k \mu_t^c, \quad \mu_t^k = \eta_t^k = \left(Z_t^k\right)^2, \quad k = 1, 2 \ \epsilon^1, \epsilon^2 \in \mathbb{R} \tag{57}$$

$$\mu^c = J^c + \delta r, \quad J_t^c = \sum_{\tilde{\tau}_i < t} Y_i h^c(t - \tilde{\tau}_i), \quad Y_i \sim \chi^2(2) \quad h^c(x) = e^{-bx}, \ b \in \mathbb{R}_+ \ . \tag{58}$$

and the $\tilde{\tau}_i$ are the jumps of a Poisson process with intensity l^c.

We note that the firms specific terms do not have jumps and are purely quadratic terms, while the common factors depend linearly on the short rate and allow for jumps. The common jumps follow the shot-noise formulation in Assumption 4.1, the $\tilde{\tau}_i$'s stem from a standard Poisson distribution with constant parameter l^c and the Y_i's have a χ^2 distribution with two degrees of freedom.

Figure 5 shows simulated default times for different choices of ϵ_i. The left plot has $\epsilon_i = 0.1$ while the right plot has $\epsilon_i = 0.5$. Especially the plot on the right hand side shows a strong dependence of the two default times.

This kind of strong dependence illustrates why the shot-noise feature enables us to reproduce contagion effects. Typically, contagion refers to the following effect: default of company A leads to serious problems in related companies, such that at least some of them default close in time. The shot-noise model is not able to directly produce such an effect, because in the presented model the default of a pre-specified company does not have any effect on the default intensity of other companies. However, as the simulations show, the jumps in intensities induces quite a big number of defaults close in time, which mimics the contagion effect. Ongoing research incorporates a self-exciting feature, which in turn will enable the model to directly induce contagion.

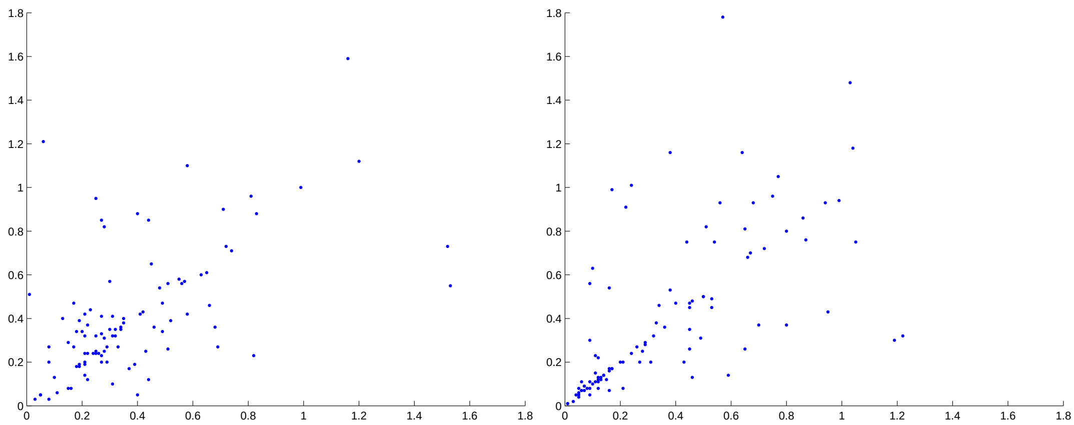


Figure 5: Simulated defaults of two companies according to the model in Section 5.1. Parameters are for $i = 1, 2$: $\beta_i = 1, \alpha_i = 0.5, \sigma_i = 0.2, l^c = 2, b = 0.5$. The jumps are χ_2^2-distributed. The left picture has $\epsilon_i = 0.1$, the right $\epsilon_i = 0.5$.

Using the notation previously described we identify all the needed matrices:

$$\text{drift as in (3):} \qquad d(t) = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \alpha_r \beta_r \end{bmatrix} \qquad E(t) = \begin{bmatrix} -\alpha_1 & 0 & 0 \\ 0 & -\alpha_2 & 0 \\ 0 & 0 & -\alpha_r \end{bmatrix}$$

$$\text{variance as in (4):} \qquad k_0(t) = \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad k_r(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_r^2 \end{bmatrix}$$

$$k_i(t) = g_{uj}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad i = 1, 2; \quad uj = 1, 2, r \ .$$

Furthermore we also have

$$\eta^k = (Z^k)^2 \quad \Rightarrow \quad (\mathsf{Q}^k)_{ij}(t) = \begin{cases} 1 & ij = kk \\ 0 & \text{otherwise} \end{cases}, \quad \mathsf{g}^k(t) = \mathbf{0}, \quad \mathsf{f}^k(t) = 0; \quad k = 1, 2$$

$$r = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad Q(t) = \mathbf{0}, \quad g(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad f(t) = 0 \ .$$

5.2 Risk-free term structure

From (56) we recognize the CIR model for the short rate, and thus the risk-free bond prices have an ATS.

Result 2.5 yields

$$p(t, T) = e^{A(t,T) + B^\top(t,T) Z_t + Z_t^\top C(t,T) Z_t}$$

and solving the basic system of ODEs we get with notation $x = T - t$, $\gamma_r = \sqrt{\alpha_r^2 + 2\sigma_r^2}$

$$A(t,T) = \frac{2\alpha_r\beta_r}{\sigma_r^2} \ln\left(\frac{2\gamma_r e^{(\alpha_r+\gamma_r)\frac{x}{2}}}{(\gamma_r+\alpha_r)(e^{\gamma_r x}-1)+2\gamma_r}\right), \quad B(t,T) = \begin{bmatrix} 0 \\ 0 \\ \dfrac{2(e^{\gamma_r x}-1)}{(\gamma_r+\alpha_r)(e^{\gamma_r x}-1)+2\gamma_r} \end{bmatrix},$$

and $C(t,T) = \mathbf{0}$.

Result 5.1. *The closed-formula solution for the risk-free bond prices is*

$$p(t,t+x) = \left(\frac{2\gamma_r e^{(\alpha_r+\gamma_r)\frac{x}{2}}}{(\gamma_r+\alpha_r)(e^{\gamma_r x}-1)+2\gamma_r}\right)^{\dfrac{2\alpha_r\beta_r}{\sigma_r^2}} \times \exp\left\{\left(\frac{2(1-e^{\gamma x})}{(\gamma+\alpha_r)(e^{\gamma x}-1)+2\gamma}\right) r_t\right\},$$

where $\gamma_r = \sqrt{\alpha_r^2 + 2\sigma_r^2}$.

5.3 Key building blocks for credit risk

We will now compute the basic quantities from Remark 4.2. In the following we always set $x := T - t$.

- $S^k(\theta,t,T)$:

We start by noting that the firm specific components have no jumps, so we have

$$S^k(\theta,t,T) = S_\eta^k(\theta,t,T) = \exp\left(\mathcal{A}^k(\theta,t,T) + \mathcal{B}^{k\top}(\theta,t,T)Z_t + Z_t^\top \mathcal{C}^k(\theta,t,T)Z_t\right)$$

where $(\mathcal{A}^k, \mathcal{B}^k, \mathcal{C}^k, \theta\mathsf{Q}^k, 0, 0)$ solve the basic ODE system of Definition 2.4.

We also note that, due to independence of the three factors in Z, we immediately obtain

$$\left(\mathcal{B}^k\right)_{ij}(\theta,t,T) = \begin{cases} \mathcal{B}^k(\theta,t,T) & i = k \\ 0 & \text{otherwise} \end{cases}, \quad \left(\mathcal{C}^k\right)_{ij}(\theta,t,T) = \begin{cases} \mathcal{C}^k(\theta,t,T) & i = j = k \\ 0 & \text{otherwise} \end{cases}. \tag{59}$$

$\mathcal{B}^k$ and $\mathcal{C}^k$ on the r.h.s. are now *scalar* functions [23] that solve the scalar ODE system

$$\begin{cases} \dfrac{\partial \mathcal{B}^k}{\partial t} - \alpha_k\mathcal{B}^k + 2\mathcal{C}^k\beta_k + 2\sigma_k^2\mathcal{C}^k\mathcal{B}^k = 0 \\ \mathcal{B}^k(\theta,T,T) = 0 \end{cases}$$
$$\begin{cases} \dfrac{\partial \mathcal{C}^k}{\partial t} - 2\alpha_k\mathcal{C}^k + 2\sigma_k^2\left(\mathcal{C}^k\right)^2 = \theta \\ \mathcal{C}^k(\theta,T,T) = 0 \end{cases}$$

[23] Even though we acknowledge that using the same notation on both the vector/matrix on the l.h.s and the scalar functions on the r.h.s of (59) may be misleading, we believe it is better than introducing more notation. Moreover, given the independence between our three factors, it should be clear at each point which entrance in a vector/matrix entrance is not zero and we will always be refering to that one.

whose solution is, with $\gamma_k = \sqrt{\alpha_k^2 + 2\sigma_k^2\theta}$, given by:

$$\mathcal{C}^k(\theta,t,T) = \frac{\theta\left[1-e^{2\gamma_k x}\right]}{(\gamma_k+\alpha_k)\left[e^{2\gamma_k x}-1\right]+2\gamma_k}, \tag{60}$$

$$\begin{aligned} \mathcal{B}^k(t,T) &= \int_t^T \beta_k(s) e^{\int_t^s \alpha_k - 2\mathcal{C}(\theta,u,T)du} 2\mathcal{C}(\theta,s,T)ds \\ &= \left[2(\alpha_k+\gamma_k)(e^{2\gamma_k x}-1)+4\gamma_k\right]^{-1/\sigma_k^2} \times \qquad (61) \\ &\quad \times \int_t^T \left(\beta_k(s) \frac{4\theta e^{\left(\alpha_k + \frac{4\theta}{\gamma_k-\alpha_k}\right)(s-t)}(1-e^{2\gamma_k(T-s)})}{\left[2(\alpha_k+\gamma_k)(e^{2\gamma_k(T-s)}-1)+4\gamma_k\right]^{1-\frac{1}{\sigma_k^2}}} \right) ds. \end{aligned}$$

Finally, the ODE for $\mathcal{A}^k$ reduces to

$$\begin{cases} \dfrac{\partial \mathcal{A}^k}{\partial t} + \beta^k(t)\mathcal{B}^k + \dfrac{1}{2}\sigma_k^2\left(\mathcal{B}^k\right)^2 + \sigma_k^2\mathcal{C}^k = 0 \\ \mathcal{A}(\theta,T,T) = 0 \end{cases}$$

and integrating we get

$$\begin{aligned} \mathcal{A}^k(\theta,t,T) &= -\int_t^T \beta_k(s)\mathcal{B}^k(\theta,s,T) + \frac{1}{2}\sigma_k^2\left(\mathcal{B}^k(\theta,s,T)\right)^2 + \sigma_k^2\mathcal{C}^k(\theta,s,T)ds \\ &= \frac{1}{2}\ln\left(\frac{2\gamma_k e^{(\gamma_k+\alpha_k)x}}{(\gamma_k+\alpha_k)(e^{2\gamma_k x}-1)+2\gamma_k}\right) - \int_t^T \beta_k(s)\mathcal{B}^k(\theta,s,T) + \frac{1}{2}\sigma_k^2\left(\mathcal{B}^k(\theta,s,T)\right)^2 ds. \end{aligned}$$

for $\mathcal{B}^k$ as in (61).

Result 5.2. *So, for k=1,2, we have $S^k(\theta,t,T) = S_\eta^k(\theta,t,T)$ and*

$$\begin{aligned} S^k(\theta,t,T) &= \sqrt{\frac{2\gamma_k e^{(\gamma_k+\alpha_k)x}}{(\gamma_k+\alpha_k)(e^{2\gamma_k x}-1)+2\gamma_k}} \times \exp\left\{ \frac{\theta\left[1-e^{2\gamma_k x}\right]}{(\gamma_k+\alpha_k)\left[e^{2\gamma_k x}-1\right]+2\gamma_k}\left(Z_t^k\right)^2 \right\} \times \\ &\quad \times \exp\left\{ -\int_t^T \left(\beta_k(s)\mathcal{B}^k(\theta,s,T) + \frac{1}{2}\sigma_k^2\left(\mathcal{B}^k(\theta,s,T)\right)^2\right) ds + \mathcal{B}^k(\theta,t,T)Z_t^k \right\}, \end{aligned} \tag{62}$$

where $\mathcal{B}^k$ is as in (61).

For arbitrary β_k the above expressions involving $\mathcal{B}^k$ must be evaluated numerically.

- $S_\eta^c(\theta,t,T)$, $\bar{S}_\eta^c(\theta,t,T)$:

We note that $\eta_t^c = \delta r_t$. So,

$$\begin{aligned} S_\eta^c(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \theta\eta_s^c ds} \middle| \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \theta\delta r_s ds} \middle| \mathcal{F}_t \right] \\ &= \exp\left\{ \mathcal{A}^c(\theta,t,T) + \mathcal{B}^{c\top}(\theta,t,T)Z_t + Z_t^\top \mathcal{C}^c(\theta,t,T)Z_t \right\} \\ \bar{S}_\eta^c(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s+\theta\eta_s ds} \middle| \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T (1+\theta\delta) r_s ds} \middle| \mathcal{F}_t \right] \\ &= \exp\left\{ \bar{A}^c(\theta,t,T) + \bar{B}^{c\top}(\theta,t,T)Z_t + Z_t^\top \bar{C}^c(\theta,t,T)Z_t \right\}. \end{aligned}$$

Note that the above expectations are very similar to the expectation needed to compute for the risk-free bond prices.

Indeed, it is easy to show that both quantities can be obtained from $\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \Delta r_s ds}\Big|\,\mathcal{F}_t\right]$ and we have

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \Delta r_s ds}\Big|\,\mathcal{F}_t\right] = \exp\left\{A(\Delta,t,T) + B^\top(\Delta,t,T)Z_t + Z_t^\top C(\Delta,t,T)Z_t\right\} \tag{63}$$

where, setting $\hat{\gamma}_r = \sqrt{\alpha_r^2 + 2\sigma_r^2\Delta}$,

$$A(\Delta,t,T) = \frac{2\alpha_r\beta_r}{\sigma_r^2}\ln\left[\frac{2\hat{\gamma}_r e^{(\hat{\gamma}_r+\alpha_r)\frac{T-t}{2}}}{(\alpha_r+\hat{\gamma}_r)[e^{\hat{\gamma}_r(T-t)}-1]+2\hat{\gamma}_r}\right] \tag{64}$$

$$B(\Delta,t,T) = \begin{bmatrix} 0 \\ 0 \\ \dfrac{2\Delta\left[e^{\hat{\gamma}_r(T-t)}-1\right]}{(\hat{\gamma}_r+\alpha_r)\left[e^{\hat{\gamma}_r(T-t)}-1\right]+2\hat{\gamma}_r} \end{bmatrix}, \tag{65}$$

$$C(\Delta,t,T) = \mathbf{0}. \tag{66}$$

We finally obtain $\bar{A}^c$, $\bar{B}^c$, $\bar{C}^c$ and $\mathcal{A}^c$, $\mathcal{B}^c$, $\mathcal{C}^c$ as follows:[24]

$$\bar{A}^c(\theta,t,T) = A((1+\theta\delta),t,T) \qquad \bar{B}^c(\theta,t,T) = B((1+\theta\delta),t,T) \qquad \bar{C}^c(\theta,t,T) = \mathbf{0}$$
$$\mathcal{A}^c(\theta,t,T) = A(\theta\delta,t,T) \qquad \mathcal{B}^c(\theta,t,T) = B(\theta\delta,t,T) \qquad \mathcal{C}^c(\theta,t,T) = \mathbf{0}$$

where A, B, C are as in (64)-(66).

Summarizing we obtain:

Result 5.3. *With $\gamma_\eta^c = \sqrt{\alpha_r^2 + 2\sigma_r^2\theta\delta}$ and $\bar{\gamma}_\eta^c = \sqrt{\alpha_r^2 + 2\sigma_r^2(1+\theta\delta)}$ we have the following expressions:*

$$S_\eta^c(\theta,t,T) = \left(\frac{2\gamma_\eta^c e^{(\gamma_\eta^c+\alpha_r)\frac{x}{2}}}{(\alpha_r+\gamma_\eta^c)[e^{\gamma_\eta^c x}-1]+2\gamma_\eta^c}\right)^{\frac{2\alpha_r\beta_r}{\sigma_r^2}} \times \exp\left\{\left(\frac{2\theta\delta\left[1-e^{\gamma_\eta^c x}\right]}{(\gamma_\eta^c+\alpha_r)\left[e^{\gamma_\eta^c x}-1\right]+2\gamma_\eta^c}\right)r_t\right\} \tag{67}$$

$$\bar{S}_\eta^c(\theta,t,T) = \left(\frac{2\bar{\gamma}_\eta^c e^{(\bar{\gamma}_\eta^c+\alpha_r)\frac{x}{2}}}{(\alpha_r+\bar{\gamma}_\eta^c)[e^{\bar{\gamma}_\eta^c x}-1]+2\bar{\gamma}_\eta^c}\right)^{\frac{2\alpha_r\beta_r}{\sigma_r^2}} \times \exp\left\{\left(\frac{2(1+\theta\delta)\left[1-e^{\bar{\gamma}_\eta^c x)}\right]}{(\bar{\gamma}_\eta^c+\alpha_r)\left[e^{\bar{\gamma}_\eta^c x}-1\right]+2\bar{\gamma}_\eta^c}\right)r_t\right\} \tag{68}$$

•$S_J^c(\theta,t,T)$:

[24] The alternative to these computations is to solve the basic ODE system of Definition 2.4 for $(\bar{A}^c, \bar{B}^c, \bar{C}^c, 0, (1+\delta)g, 0)$ and $(\mathcal{A}^c, \mathcal{B}^c, \mathcal{C}^c, 0, \delta\theta, 0)$. Given the specificities of our model the above is faster.

Recalling, that we consider the Markovian case of our model, we get from Remark 4.2 that

$$S_J^c(\theta,t,T) = e^{-H^c(T-t)J_t + l^c x[D^c(\theta,x)-1]}.$$

The functions H^c and D^c can be obtained directly from $h^c(x) = e^{-bx}$ and the fact that Y_i^cs are $\chi^2(2)$ distributed (recall equations (23) and (27)).

We start by computing

$$H^c(x) = \int_0^x e^{-bs} = -\frac{1}{b}\left[e^{-bx} - 1\right] = \frac{1}{b}\left[1 - e^{-bx}\right] \ . \tag{69}$$

To compute D^c we will make use of the Laplace transform of the $\chi^2(\nu)$ distribution.[25] For $\nu = 2$,

$$\varphi^c(u) = \frac{1}{1+2u} \ , \tag{70}$$

and D^c will have a simple formula. However, for any choice of ν there is an explicit expression, just the formulas will get lengthier.

With $\nu = 2$ we find

$$\begin{aligned} D(\theta,s) &= \int_0^1 \varphi^c(\theta H(s-su))\,du \\ &= \int_0^1 \Big(1 + 2\theta H\big(s(1-u)\big)\Big)^{-1} du \\ &= \int_0^1 \Big[1 + \frac{2\theta}{b}\Big(1 - e^{-bs(1-u)}\Big)\Big]^{-1} du. \end{aligned}$$

Integrating gives[27]

$$D^c(\theta,s) = \frac{1}{b+2\theta}\left[b + \frac{1}{s}\ln\Big(1 + \frac{2\theta}{b}(1-e^{-bs})\Big)\right]. \tag{71}$$

Putting all the information together, we conclude the following.

Result 5.4.

$$\begin{aligned} S_J^c(\theta,t,T) &= \Big[1 + \frac{2\theta}{b}\big(1 - e^{-b(T-t)}\big)\Big]^{\frac{l^c}{b+2\theta}} \\ &\quad \times \exp\left\{\frac{1}{b}\left[e^{-bx} - 1\right] J_t + l^c(T-t)\Big[\frac{b}{b+2\theta} - 1\Big]\right\}. \end{aligned} \tag{72}$$

[25] Recall that for $u \geq 0$ the Laplace transform of random variable which has χ^2 distribution with ν degrees of freedom, equals [26],

$$\varphi_{\chi^2_\nu}(u) = \mathbb{E}(e^{-u\chi^2_\nu}) = (1+2u)^{-\nu/2}.$$

[27] Note that the primitive of $(a + be^{cu})^{-1}$ is

$$\frac{u}{a} - \frac{1}{ac}\ln\Big(a + be^{cu}\Big).$$

Furthermore, from Remark 4.2, we know

$$S^c(\theta,t,T) = S^c_\eta(\theta,t,T)S^c_J(\theta,t,T) \tag{73}$$

$$\bar{S}^c(\theta,t,T) = \bar{S}^c_\eta(\theta,t,T)S^c_J(\theta,t,T) \tag{74}$$

with S^c_η as in (67), $\bar{S}^c_\eta$ as in (68) and S^c_J as in (72).

- $\Gamma^k(\theta,t,T)$:

$$\Gamma^k(\theta,t,T) = \Gamma^k_\eta(\theta,t,T) = S^k_\eta(\theta,t,T)\exp\left(a^k(\theta,t,T) + b^{k\top}(\theta,t,T)Z_t + Z_t^\top c^k(\theta,t,T)Z_t\right)$$

where a^k, b^k and c^k solve the interlinked system of Definition 3.17. In our case, and for each fixed k, the system can be simplified since

$$(b^k)_i(\theta,t,T) = \begin{cases} b^k(\theta,t,T) & i=k \\ 0 & \text{otherwise} \end{cases} \quad (c^k)_{ij}(\theta,t,T) = \begin{cases} c^k(\theta,t,T) & i=j=k \\ 0 & \text{otherwise} \end{cases}. \tag{75}$$

For a^k and b^k, c^k (on the l.h.s above) we get the scalar system of ODE

$$\begin{cases} \dfrac{\partial a^k}{\partial t} + \beta_k b^k + \sigma_k^2 \mathcal{B}^k b^k + \sigma_k^2 c = 0 \\ a^k(\theta,T,T) = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial b^k}{\partial t} - \alpha_k b^k + 2\beta_k c + 2\sigma_k^2 \mathcal{C}^k b^k + 2\sigma_k^2 \mathcal{B}^k c^k = 0 \\ b^k(\theta,T,T) = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial c^k}{\partial t} - 2\alpha_k c^k + 4\sigma_k^2 \mathcal{C}^k c^k = 0 \\ c^k(\theta,T,T) = \theta \end{cases}$$

where $\mathcal{B}^k$ and $\mathcal{C}^k$ are as in (60)-(61).

Solving first for c^k we get

$$c^k(\theta,t,T) = \theta\exp\left\{-2\int_t^T \alpha_k - 2\sigma_k^2\mathcal{C}^k(s,T)ds\right\}$$

$$= \frac{\theta\left[(\gamma_k+\alpha_k)(e^{\gamma_k x}-1)+2\gamma_k\right]}{\gamma_k e^{(\gamma_k+3\alpha_k)x}} \tag{76}$$

where $\gamma_k = \sqrt{\alpha_k^2 + 2\sigma_k^2\theta}$.

Then b^k equals

$$b^k(\theta,t,T) = -2\int_t^T e^{\int_t^s \alpha_k - 2\sigma_k^2\mathcal{C}^k(u,T)du}\left(\beta_k(s) - \sigma^2\mathcal{B}^k(s,T)\right)c^k(s,T)ds \tag{77}$$

where $\mathcal{B}^k$ and $\mathcal{C}^k$ are as in (60)-(61) and c^k as in (76). Finally,

$$a^k(\theta,t,T) = -\int_t^T \beta_k(s) + \sigma_k^2\mathcal{B}^k(s,T)b + \sigma_k^2 c^k(s,T)ds. \tag{78}$$

with $\mathcal{B}$ as in (61) and c^k as in (76). Summarizing, we get the following result.

Result 5.5. *With c^k as in (76) and b^k and a^k numerically evaluated using (77)-(78), we have that*

$$\Gamma^k(\theta,t,T) = \Gamma^k_\eta(\theta,t,T) = S^k_\eta(\theta,t,T)\exp\left(a^k(\theta,t,T) + b^k(\theta,t,T)Z^k_t + c^k(\theta,t,T)\left(Z^k_t\right)^2\right) \tag{79}$$

- $\Gamma^c_\eta(\theta,t,T)$, $\bar{\Gamma}^c_\eta(\theta,t,T)$:

Once again, we note that in our special case, to obtain $\Gamma^c_\eta(\theta,t,T)$ and $\bar{\Gamma}^c_\eta(\theta,t,T)$ we need to solve expressions of the type $\mathbb{E}^{\mathbb{Q}}\left[\theta r_T e^{-\int_t^T \Delta r_s ds}\middle|\mathcal{F}_t\right]$, which can be easily proven to be of the form

$$\mathbb{E}^{\mathbb{Q}}\left[\theta r_T e^{-\int_t^T \Delta r_s ds}\middle|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \Delta r_s ds}\middle|\mathcal{F}_t\right]e^{a(\Delta,t,T)+b(\Delta,t,T)r_t}.$$

The expectation on the r.h.s. has been previously computed (compare with equation (63)) and a, b solve a simplified scalar version of the interlinked ODE system in Definition 3.17

$$\begin{cases} \dfrac{\partial a}{\partial t} + \alpha_r\beta_r b &= 0 \\ a(\Delta,T,T) &= 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial b}{\partial t} - \alpha_r b + \dfrac{1}{2}\sigma_k^2 Bb &= 0 \\ b(\Delta,T,T) &= \theta \end{cases}$$

where B can be obtained from (65).

The solution to the above system is given by

$$a(\Delta,t,T) = \int_t^T \alpha_r\beta_r b(\Delta,s,T)ds \tag{80}$$

$$\begin{aligned} b(\Delta,t,T) &= \theta e^{-\int_t^T \alpha_r - \frac{1}{2}\sigma_r^2 B(s,T)ds} \\ &= \frac{\theta\left[(\alpha_r+\hat{\gamma}_r)(e^{\hat{\gamma}_r x}-1)+2\hat{\gamma}_r\right]}{2\hat{\gamma}_r e^{(3\alpha_r+\hat{\gamma}_r)\frac{x}{2}}} \end{aligned} \tag{81}$$

where a must be evaluated numerically using b in (81) and $\hat{\gamma}_r = \sqrt{\alpha_r + 2\sigma_r^2\Delta}$

Using the above derived equations (80)-(81) we can finally obtain and

$$\begin{aligned} \Gamma^c_\eta(\theta,t,T) &= S^c_\eta(\theta,t,T)\exp\left\{a(\theta\delta,t,T) + b(\theta\delta,t,T)r_t\right\} \\ \bar{\Gamma}^c_\eta(\theta,t,T) &= \bar{S}^c_\eta(\theta,t,T)\exp\left\{a((1+\theta\delta),t,T) + b((1+\theta\delta),t,T)r_t\right\} \end{aligned}$$

Result 5.6. *With $\gamma^c_\eta = \sqrt{\alpha_r^2 + 2\sigma_r^2\theta\delta}$ and $\gamma^c_\eta = \sqrt{\alpha_r^2 + 2\sigma_r^2(1+\theta\delta)}$,*

$$\Gamma^c_\eta(\theta,t,T) = S^c_\eta(\theta,t,T)\exp\left\{a(\theta\delta,t,T) + \left(\frac{\theta\left[(\alpha_r+\gamma^c_\eta)(e^{\gamma^c_\eta x}-1)+2\gamma^c_\eta\right]}{2\gamma^c_\eta e^{(3\alpha_r+\gamma^c_\eta)\frac{x}{2}}}\right)r_t\right\} \tag{82}$$

$$\bar{\Gamma}^c_\eta(\theta,t,T) = \bar{S}^c_\eta(\theta,t,T)\exp\left\{a((1+\theta\delta),t,T) + \left(\frac{\theta\left[(\alpha_r+\bar{\gamma}^c_\eta)(e^{\bar{\gamma}^c_\eta x}-1)+2\bar{\gamma}^c_\eta\right]}{2\bar{\gamma}^c_\eta e^{(3\alpha_r+\bar{\gamma}^c_\eta)\frac{x}{2}}}\right)r_t\right\}. \tag{83}$$

$S_\eta^c(\theta,t,T)$ and $\bar{S}_\eta^c(\theta,t,T)$ can be obtained from (67), (68) and a must be evaluated numerically using (80)-(81).

- $\Gamma_J^c(\theta,t,T)$:

$$\Gamma_J^c(\theta,t,T) \quad = \quad S_J^c(\theta,t,T)\left[\theta J^k(t,T) - l^c[D^c(\theta,x)(1-x)-1] + x\varphi^c(\theta H^c(x))\right]$$

$H^c(x)$ and $D^c(\theta,x)$ are as in (69), (71). While from the Laplace transform of the $\chi^2(2)$ in (70) we immediately obtain

$$\varphi^c(\theta H^c(x)) = \frac{1}{1+\frac{2\theta}{b}(1-e^{-bx})}$$

Result 5.7. *With $J^c(t,T) = \sum_{\tilde{\tau}_i \le t} Y_i e^{-b(T-\tilde{\tau}_i)}$, thus known at time t, and $S_J^c(\theta,t,T)$ as in (72) we have that*

$$\Gamma_J^c(\theta,t,T) = S_J^c(\theta,t,T)\Bigg\{\theta J^c(t,T) - l^c\left[\frac{1}{2+b\theta}\left(b+\frac{1}{x}\ln\left(1+\frac{2\theta}{c}(1-e^{-bx})\right)\right)(1-x)-1\right] + x\frac{1}{1+\frac{2\theta}{b}(1-e^{-bx})}\Bigg\} \quad (84)$$

Furthermore, using Remark 4.2,

$$\Gamma^c(\theta,t,T) \quad = \quad \Gamma_\eta^c(\theta,t,T)S_J^c(\theta,t,T) + \Gamma_J^c(\theta,t,T)S_\eta^c(\theta,t,T) \quad (85)$$
$$\bar{\Gamma}^c(\theta,t,T) \quad = \quad \bar{\Gamma}_\eta^c(\theta,t,T)S_J^c(\theta,t,T) + \bar{\Gamma}_J^c(\theta,t,T)\bar{S}_\eta^c(\theta,t,T) \quad (86)$$

In Results 5.1 to 5.7, we have computed in closed-form (up to the numerical integration of some integrals), all the needed ingredients to derive explicit expressions of, e.g. , credit derivatives, CDOs and FtDs.

For instance,

- Firm k's survival probability is given by

$$\mathbb{Q}_S^k(t,T) = S^k(t,T) \times S^c(\epsilon^k,t,T) \quad (87)$$

  with $S^k(t,T) = S^k(1,t,T)$ from (62), S^c from (73).

- Firm k's zero-recovery defaultable bond has the price

$$\bar{p}_o^k(t,T) = S^k(t,T) \times \bar{S}^c(\epsilon^k,t,T) \quad (88)$$

  with $S^k(t,T) = S^k(1,t,T)$ from (62), $\bar{S}^c$ from (74).

- The price of 1 unit of currency if firm k defaults in $(t,T]$ is given by

$$e^k(t,T) \quad = \quad \Gamma^k(t,T) \times \bar{S}^c(\epsilon^k,t,T) + \bar{\Gamma}^c(\epsilon^k,t,T) \times S^k(t,T)$$

  with $S^k(t,T) = S^k(1,t,T)$ from (62), $\Gamma^k(t,T) = \Gamma^k(1,t,T)$ from (79), $\bar{S}^c$ from (74) and $\bar{\Gamma}^c$ from (86).

- The default correlation between firm 1 and 2 is given by

$$\rho^{1,2}(t,T) = \frac{S^1(t,T)S^2(t,T)\left[S^c(\epsilon^1+\epsilon^2,t,T) - S^c(\epsilon^1,t,T)S^c(\epsilon^2,t,T)\right]}{\sqrt{\mathbb{Q}_D^1(t,T)[1-\mathbb{Q}_D^1(t,T)]\mathbb{Q}_D^2(t,T)[1-\mathbb{Q}_D^2(t,T)]}}$$

where for $k = 1, 2$ we have $S^k(t,T) = S^k(1,t,T)$ from (62) and $\mathbb{Q}_D^k(t,T) = 1 - \mathbb{Q}_S^k(t,T)$. $\mathbb{Q}_S^k$ is given in (87) and S^c in (73).

6 Conclusion

We have presented a class of reduced-form models for credit risk for which it is possible to compute, in closed form, all relevant quantities in credit risk modeling both at firm level and at portfolio level. In addition we computed explicit formulas to price several credit derivatives like CDSs, CDOs and FtDS.

In the presented class of models intensities were used which have both a predictable component and an unpredictable one. The predictable component is of the general quadratic type as introduced in Gaspar (2004), while the unpredictable component is modeled as a shot-noise process. Quadratic models are particularly useful for modeling intensities since they naturally lead to strictly positive processes. Furthermore, it is well-known that it is the largest polynomial order one can deal with without introducing arbitrage (see Filipović (2002) or the discussion in Gaspar (2004)). The shot-noise component is essential in producing realistic default correlation levels across firms.

In Section 5 we presented an easy example illustrating and clarifying the use of the derived results.

In our opinion, the class of models proposed in this paper is particularly suited to fit real data and handle portfolio issues in closed-form. A natural step of future research would be the calibration of a concrete model to market data. Ongoing research extends the shot-noise component of the model to self-exciting processes, which will directly lead to contagious effects. In terms of prices, it would be interesting to formalize and handle cash-flow CDOs which are much less studied in the literature than synthetic CDOs and possess interesting embedded options.

A Appendix: Technical details and Proofs

Some of the proofs make use of the following Lemmas.

Lemma A.1. *For any deterministic function G of the state variable Z, and any function F, quadratic in the state variable Z, s.t.*

$$F(t,z) = \phi_1(t) + \phi_2^\top(t)z + z^\top\phi_3(t)z$$

the following property holds

$$\mathbb{E}^{\mathbb{Q}}\left[G(Z_T,T)e^{-\int_t^T F_s ds}|\mathcal{F}_t\right] = g(t,Z_t,T)e^{A(t,T)+B^\top(t,T)Z_t+Z_t^\top C(t,T)Z_t} \tag{89}$$

where $(A, B, C, \phi_1, \phi_2, \phi_3)$ *solve the basic ODE system of Definition 2.4 and* g *solves the following PDE*

$$\begin{cases} \frac{\partial g}{\partial t} + \sum_i \frac{\partial g}{\partial z_i}\alpha_i + \frac{1}{2}\sum_{ij}\left(\frac{\partial^2 g}{\partial z_i \partial z_j} + \frac{\partial g}{\partial z_i}\frac{\partial h}{\partial z_j} + \frac{\partial g}{\partial z_j}\frac{\partial h}{\partial z_i}\right)\sigma_i\sigma_j &= 0 \\ g(T, z, T) &= G(z, T) \end{cases}$$

Proof. Let $y(t, Z_t, T) = \mathbb{E}^{\mathbb{Q}}\left[G(T, Z_T, T)e^{-\int_t^T F_s ds}|\mathcal{F}_t\right]$. Then it must solve the PDE

$$\begin{cases} \frac{\partial y}{\partial t} + \sum_i \frac{\partial y}{\partial z}\alpha_i + \frac{1}{2}\sum_{ij}\frac{\partial^2 y}{\partial z_i \partial z_j}\sigma_i\sigma_j &= Fy \\ y(T, z, T) &= G(z, T) \end{cases} \tag{90}$$

where all partial derivatives should be evaluated at (t, T) and α and σ are the drift and diffusion of the state variable Z as defined in (4).

We start by doing some computations that turn out to be useful later on. If above expectation would be the form

$$y(t, z, T) = g(t, z, T)e^{A(t,T)+B^\top(t,T)z+z^\top C(t,T)z} = g(t, z, T)e^{h(t,z,T)}$$

where z is allowed to be multi-dimensional, we have the following partial derivatives

$$\begin{aligned} \frac{\partial y}{\partial t} &= \frac{\partial g}{\partial t}\cdot e^h + \frac{\partial h}{\partial t}\cdot g\cdot e^h = \frac{\partial g}{\partial t}\cdot e^h + \frac{\partial h}{\partial t}\cdot y \\ \frac{\partial y}{\partial z_i} &= \frac{\partial g}{\partial z_i}e^h + g\cdot\frac{\partial h}{\partial z_i}\cdot e^h = \frac{\partial g}{\partial z_i}e^h + \frac{\partial h}{\partial z_i}\cdot y \\ \frac{\partial^2 y}{\partial z_i \partial z_j} &= \left[\frac{\partial^2 g}{\partial z_i \partial z_j}\cdot e^h + \frac{\partial g}{\partial z_i}\frac{\partial h}{\partial z_j}\cdot e^h + \frac{\partial g}{\partial z_j}\frac{\partial h}{\partial z_i}e^h + g\left(\frac{\partial^2 h}{\partial z_i \partial z_j}\cdot e^h + \frac{\partial h}{\partial z_i}\frac{\partial h}{\partial z_j}\cdot e^h\right)\right] \\ &= \frac{\partial^2 g}{\partial z_i \partial z_j}\cdot e^h + \frac{\partial g}{\partial z_i}\frac{\partial h}{\partial z_j}\cdot e^h + \frac{\partial g}{\partial z_j}\frac{\partial h}{\partial z_i}\cdot e^h + \frac{\partial^2 h}{\partial z_i \partial z_j}\cdot y + \frac{\partial h}{\partial z_i}\frac{\partial h}{\partial z_j}\cdot y \end{aligned}$$

And if that is so the PDE (90) becomes

$$\begin{cases} \frac{\partial g}{\partial t}\cdot e^h + \frac{\partial h}{\partial t}\cdot y + \sum_i\left(\frac{\partial g}{\partial z_i}e^h + \frac{\partial h}{\partial z_i}\cdot y\right)\alpha_i + \\ \qquad + \frac{1}{2}\sum_{ij}\left(\frac{\partial^2 g}{\partial z_i \partial z_j}\cdot e^h + \frac{\partial g}{\partial z_i}\frac{\partial h}{\partial z_j}\cdot e^h + \frac{\partial g}{\partial z_j}\frac{\partial h}{\partial z_i}\cdot e^h\right)\sigma_i\sigma_j \\ \qquad + \frac{1}{2}\sum_{ij}\left(\frac{\partial^2 h}{\partial z_i \partial z_j}\cdot y + \frac{\partial h}{\partial z_i}\frac{\partial h}{\partial z_j}\cdot y\right)\sigma_i\sigma_j = Fy \\ y(T, z, T) = G(z, T) \end{cases}$$

which using separation of variables in y and e^h can be splitted into two PDEs, one for g and one for h.

$$\begin{cases} \dfrac{\partial h}{\partial t} + \sum_i \dfrac{\partial h}{\partial z_i}\alpha_i + \dfrac{1}{2}\sum_{ij}\left(\dfrac{\partial^2 h}{\partial z_i \partial z_j} + \dfrac{\partial h}{\partial z_i}\dfrac{\partial h}{\partial z_j}\right)\sigma_i\sigma_j &= F \\ h(T,z,T) &= 0 \end{cases} \tag{91}$$

$$\begin{cases} \dfrac{\partial g}{\partial t} + \sum_i \dfrac{\partial g}{\partial z_i}\alpha_i + \dfrac{1}{2}\sum_{ij}\left(\dfrac{\partial^2 g}{\partial z_i \partial z_j} + \dfrac{\partial g}{\partial z_i}\dfrac{\partial h}{\partial z_j} + \dfrac{\partial g}{\partial z_j}\dfrac{\partial h}{\partial z_i}\right)\sigma_i\sigma_j &= 0 \\ g(T,z,T) &= G(z,T) \end{cases} \tag{92}$$

To prove the result it remains to show that $h(t,z,T) = A(t,T) + B^\top(t,T)z + z^\top(t,T)z$ with A, B and C from the basic ODE system of Definition 2.4 solves the PDE (91). The result follows from

$$\frac{\partial h}{\partial t} = \frac{\partial \bar{A}}{\partial t} + \frac{\partial \bar{B}}{\partial t}^\top z + z^\top \frac{\partial \bar{C}}{\partial t} z \qquad \frac{\partial h}{\partial z_i} = \left(\bar{B}_i + 2\bar{C}_i z\right) \qquad \frac{\partial^2 h}{\partial z_i \partial z_j} = 2\bar{C}_{ij}$$

and the fact that the PDE (91) becomes a separable equation equivalent to the basic ODE system. ■

With the notation $\tilde{J}$ from Equation (24), $J(t,T)$ in (35) and $D(\theta,\cdot)$ from Remark 4.2 we have the following lemma.

Lemma A.2. *Let $x = T - t$ and consider r as in (5), J as in (12) and η as in (11) and some constant $\theta \in \mathbb{R}$. For (ii) we require existence of $D(\theta, x)$ and for (v) also in some surrounding of x. Then,*

$$\begin{aligned}
(i) \qquad S_\eta(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \theta\eta_s ds}|\mathcal{F}_t^W\right] \\
&= \exp\Big(\mathcal{A}(\theta,t,T) + \mathcal{B}^\top(\theta,t,T)Z_t + Z_t^\top \mathcal{C}(\theta,t,T)Z_t\Big) \\
(ii) \qquad S_J(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \theta J_s ds}|\mathcal{F}_t^J\right] \\
&= \exp\Big(\theta(\tilde{J}_t - \tilde{J}(t,T)) + lx[D(\theta,x) - 1]\Big) \\
(iii) \qquad \bar{S}(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s + \theta\eta_s ds}|\mathcal{F}_t^W\right] \\
&= \exp\Big(\bar{A}(\theta,t,T) + \bar{B}^\top(\theta,t,T)Z_t + Z^\top(t)\bar{C}(\theta,t,T)Z_t\Big)
\end{aligned}$$

$$\begin{aligned}
(iv) \qquad \Gamma_\eta(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\theta\eta_T e^{-\int_t^T \theta\eta_s ds}|\mathcal{F}_t^W\right] \\
&= \big(a(\theta,t,T) + b^\top(\theta,t,T)Z_t + Z_t^\top c(\theta,t,T)Z_t\big)\cdot S_\eta(\theta,t,T)
\end{aligned}$$

$$\begin{aligned}
(v) \qquad \Gamma_J(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\theta J_T e^{-\int_t^T \theta J_s ds}|\mathcal{F}_t^J\right] \\
&= S_J(\theta,t,T)\cdot\left\{\theta J(t,T) - l\cdot\Big[D(\theta,x)(1-x) - 1 + x\varphi_Y\big(\theta H(x)\big)\Big]\right\}
\end{aligned}$$

$$\begin{aligned}
(vi) \qquad \bar{\Gamma}(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\theta\eta_T e^{-\int_t^T r_s + \theta\eta_s ds}|\mathcal{F}_t^W\right] \\
&= \big(\bar{a}(\theta,t,T) + \bar{b}^\top(\theta,t,T)Z_t + Z_t^\top \bar{c}(\theta,t,T)Z_t\big)\cdot \bar{S}_\eta(\theta,t,T)
\end{aligned}$$

where $(\mathcal{A},\ \mathcal{B},\ \mathcal{C},\ \theta\mathsf{f},\ \theta\mathsf{g},\ \theta\mathsf{Q})$ *and* $(\bar{A},\ \bar{B},\ \bar{C},\ f+\theta\mathsf{f},\ g+\theta\mathsf{g},\ Q+\theta\mathsf{Q})$ *solve the basic ODE system of Definition 2.4, while* $(a,\ b,\ c,\ \mathcal{B},\ \mathcal{C},\ \theta\mathsf{f},\ \theta\mathsf{g},\ \theta\mathsf{Q})$ *and* $(\bar{a},\ \bar{b},\ \bar{c},\ \bar{B},\ \bar{C},\ \theta\mathsf{f},\ \theta\mathsf{g},\ \theta\mathsf{Q})$ *solve the interlinked system of Definition (3.17).*
Furthermore,

$$\begin{aligned}
(vii)\quad S(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \theta\mu_s ds}|\mathcal{F}_t\right] = S_\eta(\theta,t,T)\cdot S_J(\theta,t,T)\\
(viii)\quad \bar{S}(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s+\theta\mu_s ds}|\mathcal{F}_t\right] = \bar{S}_\eta(\theta,t,T)\cdot S_J(\theta,t,T)\\
(ix)\quad \Gamma(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\theta\mu_T e^{-\int_t^T \theta\mu_s ds}|\mathcal{F}_t\right]\\
&= \Gamma_\eta(\theta,t,T)\cdot S_J(\theta,t,T) + \Gamma_J(\theta,t,T)\cdot S_\eta(\theta,t,T)\\
(x)\quad \bar{\Gamma}(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\theta\mu_T e^{-\int_t^T r_s+\theta\mu_s ds}|\mathcal{F}_t\right]\\
&= \bar{\Gamma}_\eta(\theta,t,T)\cdot S_J(\theta,t,T) + \Gamma_J(\theta,t,T)\cdot \bar{S}_\eta(\theta,t,T)
\end{aligned}$$

Proof. Properties $(vii)-(ix)$ follow from those in $(i)-(vi)$ by independence between $\mathcal{F}^W$ and $\mathcal{F}^J$ and the fact that $\mu=\eta+J$. Note also that (i) follows from (iii) as well as (iv) from (vi) when we take $f(t)=0, g(t)=0, Q(t)=0 \Rightarrow r_t=0, \forall t$. Thus, it remains to prove (ii), (iii), (v), (vi). These four expectations, however, are quite similar to some expectation computed in the main text. They can be computed using the more ore less the same methodology as already laid out, being cautious with the constant "θ".

Proof of (ii). We basically mimic the proof of Lemma 3.13. Recall the notation of $\tilde{J}$ from (24) and set $x=T-t$. Then,

$$\begin{aligned}
S_J(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \theta J_u\,du}|\mathcal{F}_t^J\right]\\
&= \exp\left\{\theta(\tilde{J}_t - \tilde{J}(t,T))\right\}\mathbb{E}^{\mathbb{Q}}\left[\exp\Big(-\sum_{\tilde{\tau}_i\in(t,T]}\theta Y_i\int_t^T \mathbf{1}_{\{\tilde{\tau}_i\le u\}}h(u-\tilde{\tau}_i)\,du\Big)|\mathcal{F}_t^J\right].
\end{aligned}$$

The expectation equals

$$e^{-lx}+\sum_{k=1}^{\infty}e^{-lx}\frac{(lx)^k}{k!}\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\sum_{i=1}^{k}Y_i\theta H\big(x(1-\eta_i)\big)\right)\right] = e^{lx(D(\theta,x)-1)}, \tag{93}$$

as

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-Y_1\theta H\big(x(1-\eta_1)\big)\right)\right] = \int_0^1 \varphi_Y\Big(\theta H(xu)\Big)\,du = D(\theta,x).$$

Proof of (iii)

$$\begin{aligned}
\bar{S}(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s+\theta\eta_s ds}|\mathcal{F}_t^W\right]\\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T\left(Z_s^\top(Q+\theta\mathsf{Q}^c(s))Z_s+(g+\theta\mathsf{g}^c(s))^\top Z_s+(f+\theta\mathsf{f}^c(s))\,ds\right)}|\mathcal{F}_t^W\right]\\
&= \exp\left\{\bar{A}(\theta,t,T)+\bar{B}^\top(\theta,t,T)Z_t+Z_t^\top\bar{C}(\theta,t,T)Z_t\right\}.
\end{aligned}$$

Comparing with (29) and as $(\bar{A},\bar{B},\bar{C},f+\theta\mathsf{f},g+\theta\mathsf{g},Q+\theta\mathsf{Q})$ solve the basic system of ODEs from Definition 2.4 the result follows.

Proof of (*v*) Recall the notations for $\tilde{J}(t,T)$ and $J(t,T)$ introduced in (24) and (35), respectively. Proceeding similar to the proof of (ii) we split in a measurable and future part.

$$\begin{aligned}\Gamma_J(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\Big[\theta\Big(\sum_{\tilde{\tau}_i\le t} Y_i h(T-\tilde{\tau}_i) + \sum_{\tilde{\tau}_i\in(t,T]} Y_i h(T-\tilde{\tau}_i)\Big)e^{-\int_t^T \theta J_s\,ds}\Big|\mathcal{F}_t^J\Big]\\
&= \theta J(t,T)\cdot S_J(\theta,t,T) + e^{\theta(\tilde{J}_t-\tilde{J}(t,T))}\mathbb{E}^{\mathbb{Q}}\Big[\theta\sum_{\tilde{\tau}_i\in(t,T]} Y_i h(T-\tilde{\tau}_i)e^{-\theta\int_t^T \sum_{\tilde{\tau}_i\in(t,s]} Y_i H(s-\tilde{\tau}_i)\,ds}\Big|\mathcal{F}_t^J\Big].\end{aligned}$$

Next, we consider the expectation more closely. The idea is to consider $\tilde{D}$ and derive w.r.t. T. We define

$$\tilde{J}^t(s) := \sum_{\tilde{\tau}_i\in(t,s]} Y_i h(s-\tilde{\tau}_i).$$

Then, the above expectation can be stated in a form suitable for our derivation:

$$\mathbb{E}^{\mathbb{Q}}\Big[\sum_{\tilde{\tau}_i\in(t,T]} Y_i h(T-\tilde{\tau}_i)e^{-\theta\int_t^T \sum_{\tilde{\tau}_i\in(t,s]} Y_i H(s-\tilde{\tau}_i)\,ds}\Big|\mathcal{F}_t^J\Big] = \mathbb{E}^{\mathbb{Q}}\Big[\tilde{J}^t(T)e^{-\theta\int_t^T \tilde{J}^t(s)\,ds}\Big|\mathcal{F}_t^J\Big] \quad (94)$$

Note that H is continuous and recall (93). So, if $D(\theta,x)$ exists in a neighborhood of x, we can derive the following expression w.r.t. x and obtain

$$\begin{aligned}\frac{\partial}{\partial x}e^{lx\left(\tilde{D}(\theta,x)-1\right)} &= \frac{\partial}{\partial x}\mathbb{E}^{\mathbb{Q}}\Big(e^{-\int_t^T \theta\tilde{J}^t(s)\,ds}\big|\mathcal{F}_t^J\Big)\\
&= -\mathbb{E}^{\mathbb{Q}}\Big(\theta\tilde{J}_T^t\cdot e^{-\int_t^T \theta\tilde{J}^t(s)\,ds}\big|\mathcal{F}_t^J\Big).\end{aligned}$$

The last equation follows if D is bounded in a neighbourhood of x. This follows if the Laplace transform is continuous around x. So we found a nice expression of (94). With

$$\begin{aligned}\frac{\partial}{\partial x}D(\theta,x) &= \int_0^1 \varphi_Y'\big(\theta H(xu)\big)\cdot\theta h(xu)\cdot u\,du\\
&= \varphi_Y\big(\theta H(xu)\big)u\Big|_0^1 - \int_0^1 \varphi_Y\big(\theta H(xu)\big)\,du\\
&= \varphi_Y\big(\theta H(x)\big) - D(\theta,x).\end{aligned}$$

we obtain

$$\frac{\partial}{\partial x}e^{lx\left(D(\theta,x)-1\right)} = e^{lx\left(D(\theta,x)-1\right)}\cdot l\cdot\Big[D(\theta,x)(1-x)-1+x\varphi_Y\big(\theta H(x)\big)\Big] \quad (95)$$

Noticing that $e^{\{\theta(\tilde{J}_t-\tilde{J}(t,T)\}}e^{\{lx(\tilde{D}(\theta,x)-1)\}} = S_J(\theta,t,T)$, we conclude.

Proof of (*vi*)

Let us denote

$$y(t,T) = \mathbb{E}^{\mathbb{Q}}\Big[\theta\eta_T e^{-\int_t^T r_u+\theta\eta_u du}|\mathcal{F}_t^W\Big]$$

since both r and η are quadratic functions of our factors Z and setting $G(T,z)=\theta\eta(T,z)$ we are exactly under the conditions of Lemma A.1. Thus we know

$$\mathbb{E}^{\mathbb{Q}}\Big[G(Z_T,T)e^{-\int_t^T r_s+\mu_s ds}|\mathcal{G}_t\Big] = g(t,Z_t,T)\underbrace{e^{\bar{A}(t,T)+\bar{B}^\top(t,T)Z_t+Z_t^\top\bar{C}(t,T)Z_t}}_{\bar{S}_\eta(\theta,t,T)} \quad (96)$$

and g solves the following PDE

$$\left\{\begin{array}{rcl} \frac{\partial g}{\partial t} + \sum_i \frac{\partial g}{\partial z_i}\alpha_i + \frac{1}{2}\sum_{ij}\left(\frac{\partial^2 g}{\partial z_i \partial z_j} + \frac{\partial g}{\partial z_i}\frac{\partial h}{\partial z_j} + \frac{\partial g}{\partial z_j}\frac{\partial h}{\partial z_i}\right)\sigma_i\sigma_j & = & 0 \\ g(T,z,T) & = & \eta(T,z) \end{array}\right. \tag{97}$$

and it remains to show that

$$g(t,z,T) = \bar{a}(\theta,t,T) + \bar{b}^\top(\theta,t,T)z + z^\top\bar{c}(\theta,t,T)z \tag{98}$$

with ($\bar{a},\ \bar{b},\ \bar{c},\ \bar{B}, \bar{C}, f+\theta\mathsf{f}, f+\theta\mathsf{f}, f+\theta\mathsf{f}$) solving the interlinked ODE system of Definition 3.17. We now compute

$$\frac{\partial g}{\partial t} = \frac{\partial \bar{a}}{\partial t} + \frac{\partial \bar{b}}{\partial t}z + z^\top \frac{\partial \bar{c}}{\partial t} z\ , \qquad \frac{\partial g}{\partial z_i} = \bar{b}_i + 2\bar{c}_i z\ , \qquad \frac{\partial^2 g}{\partial z_i \partial z_j} = 2\bar{c}_{ij}.$$

Replacing in the PDE (97), all these partial derivatives and η, α and $\sigma\sigma^\top$ using equations (11) and (4), we get an equivalent PDE, which in vector notation becomes[28]

$$\left\{\begin{array}{rcl} \frac{\partial \bar{a}}{\partial t} + \frac{\partial \bar{b}}{\partial t}z + z^\top\frac{\partial \bar{c}}{\partial t}z + d^\top\bar{b} + \left(E^*\bar{b}\right)z + (2\bar{c}d)z + z^\top(\bar{c}E)z & & \\ +z^\top(E^*\bar{c})z + \frac{1}{2}\left[\bar{B}^\top k_0 b + 2\,\mathrm{tr}\{\bar{c}K_0\} + \left(\tilde{\bar{B}}^\top K\bar{b}\right)z\right] & & \\ +\frac{1}{2}\left[\left(2\bar{C}k_0\bar{b} + 2\bar{c}k_0\bar{B}\right)z + z^\top\left(4\bar{C}k_0\bar{c}\right)z + z^\top\left(\tilde{\bar{B}}G\tilde{\bar{b}}\right)z\right] & = & 0 \\ g(T,z,T) & = & \theta\eta(T,z) \end{array}\right. \tag{99}$$

From the analysis of the PDE equation one soon realizes it is separable, in terms independent of z, linear in z and quadratic in z equivalent to the interlinked ODE system of Definition 3.17. To check the boundary conditions, note

$$\begin{array}{rcl} g(T,z,T) & = & \theta\eta(T,z) \\ & \Leftrightarrow & \\ \bar{a}(\theta,T,T) + \bar{b}^\top(\theta,T,T)z + z^\top\bar{c}(\theta,T,T)z & = & z^\top\mathsf{Q}(T)z + \mathsf{g}^\top(T)z + \mathsf{f}(T) \\ & \Downarrow & \\ \bar{a}(\theta,T,T) = \theta\mathsf{f}(T) \quad \bar{b}(\theta,T,T) & = & \theta\mathsf{g}(T) \quad \bar{c}(\theta,T,T) = \theta\mathsf{Q}(t). \end{array}$$

Finally, this implies

$$\bar{\Gamma}(\theta,t,T) = \left(\bar{a}(\theta,t,T) + \bar{b}^\top(\theta,t,T)z + z^\top\bar{c}(\theta,t,T)z\right)\cdot\bar{S}_\eta(\theta,t,T)\ .$$

■

Proof of Proposition 3.28

[28] Terms of order higher than two are omitted from the equation since the final solution must set those terms equal to zero and they are hard to write in vector notation.

Proof. (missing details)

It remains to show that

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_1} r_s+\eta_s ds}p(T_1,T_2)|\mathcal{F}_t^W\right] = e^{\alpha(t,T_1,T_2)+\beta^\top(t,T_1,T_2)Z_t+Z_t^\top\gamma(t,T_1,T_2)Z_t} \cdot e^{\bar{A}(t,T_1)+\bar{B}^\top(t,T_1)Z_t+Z_t^\top\bar{C}(t,T_1)Z_t}$$

where α, β and γ are deterministic and solve the system stated in the Proposition.

This is the case because the expectation is under the conditions of Lemma A.1 where the risk-free bond price $p(T_1,T_2)$ works like any other function known at time T_1 and concretely we know from Result 2.5 that

$$p(T_1,T_2) = \exp\left(A(T_1,T_2) + B^\top(T_1,T_2)Z_{T_1} + Z_{T_1}^\top C(T_1,T_2)\right)$$

thus,

$$G(T_1,Z_{T_1}) = \exp\left(A(T_1,T_2) + B^\top(T_1,T_2)Z_{T_1} + Z_{T_1}^\top C(T_1,T_2)\right).$$

and $y(t,z,T_1) = g(t,z,T_1)e^{h(t,z,T_1)}$.

As we see T_2 is just a parameter. To be easier to identify the role of T_2 we write $G(T_1,Z_{T_1},T_2)$, $g(t,Z_t,T_1,T_2)$ instead of just $G(T_1,Z_{T_1})$, $g(t,Z_t,T_1)$ but we should not forget that T_2 is just a parameter and will play no important role in the PDE we have to solve (note the boundary at T_1). Finally we know that g solves

$$\begin{cases} \dfrac{\partial g}{\partial t} + \displaystyle\sum_i \frac{\partial g}{\partial z_i}\alpha_i + \frac{1}{2}\sum_{ij}\left(\frac{\partial^2 g}{\partial z_i \partial z_j} + \frac{\partial g}{\partial z_i}\frac{\partial h}{\partial z_j} + \frac{\partial g}{\partial z_j}\frac{\partial h}{\partial z_i}\right)\sigma_i\sigma_j & = 0 \\ g(T_1,z,T_1,T_2) & = G(T_1,Z_{T_1},T_2) \end{cases}$$

Finally we note that for $g(T_1,z,T_2) = \exp\left(\alpha(T_1,T_2) + \beta^\top(T_1,T_2)Z_t + Z_t^\top\gamma(T_1,T_2)Z_t\right)$,

$$\begin{aligned} \frac{\partial g}{\partial t} &= \left(\frac{\partial\alpha}{\partial t} + \frac{\partial\beta}{\partial t}z + z^\top\frac{\partial\gamma}{\partial t}z\right)g \\ \frac{\partial g}{\partial z_i} &= (\beta_i + 2\gamma_i z)\,g \\ \frac{\partial^2 g}{\partial z_i\partial z_j} &= [2\gamma_{ij} + (\beta_i + 2\gamma_i x)(\beta_j + 2\gamma_j x)]\,g \end{aligned}$$

Replacing these in the above PDE, as well as α_i and $\sigma_i\sigma_j^\top$ from (4), leaves us with a PDE which is separable and solves the required ODEs given in the proposition. ∎

B Laplace Transform for Shot-Noise Processes

In this section we compute the conditional Laplace-transform for shot-noise processes. The conditional Fourier-transform follows similarly. We also comment on the conditional distribution function.

Recall $x = T - t$ and that φ_Y denoted the Laplace transform of the jump heights $Y_1, Y_2, \ldots$.

Proposition B.1. *If*

$$D(\theta, x) := \int_0^1 \varphi_Y\Big(\theta \cdot h(xu)\Big)\, du$$

exists for all $v \geq 0$, then the conditional Laplace transform of J equals

$$\begin{aligned}\varphi_{J_{t+x}|\mathcal{F}_t^J} &= \mathbb{E}^{\mathbb{Q}}\Big(e^{-\theta J_{t+x}}|\mathcal{F}_t^J\Big) \\ &= \exp\Big[lx\big(D(\theta,x)-1\big) - \theta \sum_{\tilde{\tau}_i \leq t} Y_i h(t+x-\tilde{\tau}_i)\Big].\end{aligned}$$

Proof. First, we distinguish the measurable part from the future part:

$$\mathbb{E}^{\mathbb{Q}}\Big(e^{-\theta J_{t+x}}|\mathcal{F}_t^J\Big) = e^{-\theta \sum_{\tilde{\tau}_i \leq t} Y_i h(t+x-\tilde{\tau}_i)} \cdot \mathbb{E}^{\mathbb{Q}}\Big[e^{-\theta \sum_{\tilde{\tau}_i \in (t,t+x]} Y_i h(T-\tilde{\tau}_i)}\Big|\mathcal{F}_t^J\Big].$$

Second, we compute the expectation proceeding similarly as in the proof or Lemma 3.13. To this, denote by $\eta_1, \eta_2, \ldots$ i.i.d. $U[0,1]$ variables. Then

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}}\Big[e^{-\theta \sum_{\tilde{\tau}_i \in (t,t+x]} Y_i h(T-\tilde{\tau}_i)}\Big|\mathcal{F}_t^J\Big] \\ &\qquad = e^{-lx} + \sum_{k=1}^{\infty} e^{-lx}\frac{(lx)^k}{k!}\mathbb{E}^{\mathbb{Q}}\Big(e^{-\theta \sum_{i=1}^k Y_i h\big(x(1-\eta_i)\big)}\Big).\end{aligned}$$

The expectation can be computed using the Laplace transform of the Y_i, as

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\Big(e^{-\theta \sum_{i=1}^k Y_i h\big(x(1-\eta_i)\big)}\Big) &= \Big[\mathbb{E}^{\mathbb{Q}}\Big(e^{-\theta Y_1 h\big(x(1-\eta_1)\big)}\Big)\Big]^k \\ &= \Big[\int_0^1 \varphi_Y\Big(\theta h(xu)\Big)\, du\Big]^k = D(\theta,x)^k.\end{aligned}$$

Hence,

$$\mathbb{E}^{\mathbb{Q}}\Big[e^{-\theta \sum_{\tilde{\tau}_i \in (t,t+x]} Y_i h(T-\tilde{\tau}_i)}\Big|\mathcal{F}_t^J\Big] = \exp\Big[-lx + lx\, D(\theta,x)\Big]$$

and the conclusion follows. ∎

With the conditional Laplace transform at hand, one can use Raible's method to numerically derive prices for any European contingent claim, cf. Eberlein and Raible (1999).

Sometimes, it can be possible to obtain the distribution function quite explicitly. This is, for example, the case when the Y_i are normally distributed. Of course, in our framework, this is not suitable. But in the case of χ^2-distributions one can proceed similarly. Proceeding as above,

$$\mathbb{Q}\big(J_T \leq c|\mathcal{F}_t^J\big) = \mathbb{Q}\Big(\sum_{\tilde{\tau}_i \in (t,T]} Y_i h(T-\tilde{\tau}_i) \leq c - \sum_{\tilde{\tau}_i \leq t} Y_i h(T-\tilde{\tau}_i)\big|\mathcal{F}_t^J\Big)$$

and it becomes clear that for computing the conditional distribution one needs a nice expression for

$$\mathbb{Q}\Big(\sum_{i=1}^k Y_i h\big(x(1-\eta_i)\big) \leq c\Big).$$

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