Appendix to "Testing for Volatility Interactions in the Constant Conditional Correlation GARCH Model"

Tomoaki Nakatani*† Timo Teräsvirta‡†

* *Department of Agricultural Economics, Hokkaido University, 060-8589, Sapporo, Japan*

† *Department of Economic Statistics, Stockholm School of Economics, P.O. Box 6501, SE-113 83 Stockholm, Sweden*

‡ *CREATES, School of Economics and Management, University of Aarhus, Building 1322, DK-8000 Aarhus C, Denmark*

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*E-mail addresses: sttn@hhs.se (T. Nakatani) and tterasvirta@econ.au.dk (T. Teräsvirta)

1 Preamble

This appendix includes mathematical derivations and proofs of lemmas as well as figures that are relevant to, but not reported in Nakatani and Teräsvirta (2007), which is refereed to as "the main paper" throughout.

2 Bivariate illustration of the test statistic

We illustrate the structure of the proposed test by a bivariate example. First we set up the model, then apply Theorem 1 to obtain the LM_{ECCC} statistic for the bivariate model.

2.1 The bivariate ECCC-GARCH$(1,1)$ model and the partial derivatives of the conditional variance equations

The bivariate ECCC-GARCH$(1,1)$ model has its conditional variance equation (3) in the main paper of the form

$$\begin{bmatrix} h_{1,t} \\ h_{2,t} \end{bmatrix} = \begin{bmatrix} a_{10} + a_{11}\varepsilon^2_{1,t-1} + a_{12}\varepsilon^2_{2,t-1} + b_{11}h_{1,t-1} + b_{12}h_{2,t-1} \\ a_{20} + a_{21}\varepsilon^2_{1,t-1} + a_{22}\varepsilon^2_{2,t-1} + b_{21}h_{1,t-1} + b_{22}h_{2,t-1} \end{bmatrix}. \tag{2.1}$$

To compute the LM_{ECCC} statistic, we require the partial derivatives of (2.1) with respect to parameters therein. Let $\boldsymbol{\omega} = [\boldsymbol{\omega}_1', \boldsymbol{\omega}_2']'$ where $\boldsymbol{\omega}_1 = [a_{10}\, a_{11}\, a_{12}\, b_{11}\, b_{12}]'$ and $\boldsymbol{\omega}_2 = [a_{20}\, a_{21}\, a_{22}\, b_{21}\, b_{22}]'$, and let $\mathbf{v}_t = [1\, \varepsilon^2_{1,t}\, \varepsilon^2_{2,t}\, h_{1,t}\, h_{2,t}]'$. The partial derivatives of $h_{1,t}$ with respect to $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are

$$\frac{\partial h_{1,t}}{\partial \boldsymbol{\omega}_1} = \mathbf{v}_{t-1} + b_{11}\frac{\partial h_{1,t-1}}{\partial \boldsymbol{\omega}_1} + b_{12}\frac{\partial h_{2,t-1}}{\partial \boldsymbol{\omega}_1} \tag{2.2}$$

and

$$\frac{\partial h_{1,t}}{\partial \boldsymbol{\omega}_2} = b_{11}\frac{\partial h_{1,t-1}}{\partial \boldsymbol{\omega}_2} + b_{12}\frac{\partial h_{2,t-1}}{\partial \boldsymbol{\omega}_2}. \tag{2.3}$$

The partial derivatives of $h_{2,t}$ with respect to $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ have a similar representation. Under the null hypothesis, $\mathbf{A}_1$ and $\mathbf{B}_1$ are jointly diagonal, *i.e.*,

$$H_0 : \boldsymbol{\omega}_1 = [a_{10}\ a_{11}\ 0\ b_{11}\ 0]' \quad \text{and} \quad \boldsymbol{\omega}_2 = [a_{20}\ 0\ a_{22}\ 0\ b_{22}]'. \tag{2.4}$$

Evaluating both (2.2) and (2.3) at $\boldsymbol{\omega}_1 = \widetilde{\boldsymbol{\omega}}_1$ and $\boldsymbol{\omega}_2 = \widetilde{\boldsymbol{\omega}}_2$, the ML estimator under H_0, yields

$$\frac{\partial \widetilde{h}_{1,t}}{\partial \boldsymbol{\omega}_1} = \widetilde{\mathbf{v}}_{t-1} + \tilde{b}_{11}\frac{\partial \widetilde{h}_{1,t-1}}{\partial \boldsymbol{\omega}_1} \quad \text{and} \quad \frac{\partial \widetilde{h}_{1,t}}{\partial \boldsymbol{\omega}_2} = \mathbf{0}. \tag{2.5}$$

Similarly,

$$\frac{\partial \widetilde{h}_{2,t}}{\partial \boldsymbol{\omega}_1} = \mathbf{0} \quad \text{and} \quad \frac{\partial \widetilde{h}_{2,t}}{\partial \boldsymbol{\omega}_2} = \widetilde{\mathbf{v}}_{t-1} + \tilde{b}_{22}\frac{\partial \widetilde{h}_{2,t-1}}{\partial \boldsymbol{\omega}_2}. \tag{2.6}$$

For these recursions to be tractable, initial values are needed. Therefore, we set $\widetilde{\boldsymbol{\varepsilon}}^{(2)}_0 = \widetilde{\mathbf{h}}_0 = (1/T)\sum_{t=1}^{T}\widetilde{\boldsymbol{\varepsilon}}^{(2)}_t$, and $\partial \widetilde{h}_{i,0}/\partial \boldsymbol{\omega}_j = \mathbf{0}$, $i,j = 1,2$, following the suggestion by Fiorentini, Calzolari, and Panattoni (1996). The recursions of the non-zero elements in (2.5) and (2.6) then proceed as

$$\frac{\partial \widetilde{h}_{i,1}}{\partial a_{ij}} = \widetilde{\varepsilon}_{j,0}^2, \quad \frac{\partial \widetilde{h}_{i,1}}{\partial b_{ij}} = \widetilde{h}_{j,0} \quad \text{for } t = 1, \tag{2.7}$$

and

$$\frac{\partial \widetilde{h}_{i,t}}{\partial a_{ij}} = \widetilde{\varepsilon}_{j,t-1}^2 + \tilde{b}_{ii}\widetilde{\varepsilon}_{j,t-2}^2, \quad \frac{\partial \widetilde{h}_{i,t}}{\partial b_{ij}} = \widetilde{h}_{j,t-1} + \tilde{b}_{ii}\widetilde{h}_{j,t-2} \quad \text{for } t > 1 \tag{2.8}$$

where $\tilde{b}_{ii}$, $\widetilde{\varepsilon}_{j,t-1}^2$ and $\widetilde{h}_{j,t-1}$ in (2.8) are estimated from the null model.

2.2 Components of the test statistic

We shall now provide analytical expressions for the components of LM_{ECCC} in a bivariate framework and begin by for $\bar{\mathbf{S}}(\boldsymbol{\theta})$ and $\mathcal{J}(\boldsymbol{\theta})$. To simplify notation, set $\mathbf{k}_{ij,t} = h_{i,t}^{-1}\partial h_{i,t}/\partial \boldsymbol{\omega}_j$ and $\tilde{\mathbf{k}}_{ij,t} = \tilde{h}_{i,t}^{-1}\partial \widetilde{h}_{i,t}/\partial \boldsymbol{\omega}_j$, $i, j = 1, 2$. Then we have the following corollaries regarding the score and the relevant part of the Hessian.

Corollary 1. *In the bivariate case, the average score has the form*

$$\begin{bmatrix} \bar{\mathbf{S}}_{\boldsymbol{\omega}_1}(\boldsymbol{\theta}) \\ \bar{\mathbf{S}}_{\boldsymbol{\omega}_2}(\boldsymbol{\theta}) \\ \bar{\mathbf{S}}_{\boldsymbol{\rho}}(\rho) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2T}\sum \left[\mathbf{k}_{11,t}\{1 - \frac{1}{(1-\rho^2)}z_{1,t}(z_{1,t} - \rho z_{2,t})\} + \mathbf{k}_{21,t}\{1 - \frac{1}{(1-\rho^2)}z_{2,t}(z_{2,t} - \rho z_{1,t})\}\right] \\ -\frac{1}{2T}\sum \left[\mathbf{k}_{12,t}\{1 - \frac{1}{(1-\rho^2)}z_{1,t}(z_{1,t} - \rho z_{2,t})\} + \mathbf{k}_{22,t}\{1 - \frac{1}{(1-\rho^2)}z_{2,t}(z_{2,t} - \rho z_{1,t})\}\right] \\ \frac{\rho}{(1-\rho^2)} - \frac{\rho}{T(1-\rho^2)^2}\sum(z_{1,t}^2 + z_{2,t}^2) + \frac{(1+\rho^2)}{T(1-\rho^2)^2}\sum z_{1,t}z_{2,t} \end{bmatrix} \tag{2.9}$$

where $\sum$ *denotes the summation from* $t = 1$ *to* T. *Under* H_0, $\bar{\mathbf{S}}_{\boldsymbol{\rho}}(\tilde{\rho}) = 0$ *so that*

$$\bar{\mathbf{S}}(\widetilde{\boldsymbol{\theta}}) = \begin{bmatrix} \bar{\mathbf{S}}_{\boldsymbol{\omega}_1}(\widetilde{\boldsymbol{\theta}}) \\ \bar{\mathbf{S}}_{\boldsymbol{\omega}_2}(\widetilde{\boldsymbol{\theta}}) \\ 0 \end{bmatrix} = -\frac{1}{2T}\begin{bmatrix} \sum \tilde{\mathbf{k}}_{11,t}\{1 - \frac{1}{(1-\tilde{\rho}^2)}\tilde{z}_{1,t}(\tilde{z}_{1,t} - \tilde{\rho}\tilde{z}_{2,t})\} \\ \sum \tilde{\mathbf{k}}_{22,t}\{1 - \frac{1}{(1-\tilde{\rho}^2)}\tilde{z}_{2,t}(\tilde{z}_{2,t} - \tilde{\rho}\tilde{z}_{1,t})\} \\ 0 \end{bmatrix}. \tag{2.10}$$

Remark. In (2.10), only its third, fifth, seventh and ninth elements do not equal zero. These non-zero elements correspond to the zero restriction in (2.4). Accounting explicitly for this would complicate the notation and is therefore not done here.

Corollary 2. *Denote by* $\mathcal{J}_{11}^{-1}(\boldsymbol{\theta})$ *the relevant upper left block of* $\mathcal{J}^{-1}(\boldsymbol{\theta})$. *Then*

$$\mathcal{J}_{11}(\boldsymbol{\theta}) = \frac{1}{4T}\begin{bmatrix} \frac{2-\rho^2}{(1-\rho^2)}\left(\sum \mathbf{k}_{11,t}\mathbf{k}_{11,t}'\right) & -\frac{\rho^2}{(1-\rho^2)}\left(\sum \mathbf{k}_{11,t}\mathbf{k}_{22,t}'\right) \\ \quad -\frac{\rho}{T(1+\rho^2)}\left(\sum \mathbf{k}_{11,t}\right)\left(\sum \mathbf{k}_{11,t}'\right) & \quad -\frac{\rho}{T(1+\rho^2)}\left(\sum \mathbf{k}_{11,t}\right)\left(\sum \mathbf{k}_{11,t}'\right) \\ -\frac{\rho^2}{(1-\rho^2)}\left(\sum \mathbf{k}_{22,t}\mathbf{k}_{11,t}'\right) & \frac{2-\rho^2}{(1-\rho^2)}\left(\sum \mathbf{k}_{22,t}\mathbf{k}_{22,t}'\right) \\ \quad -\frac{\rho}{T(1+\rho^2)}\left(\sum \mathbf{k}_{22,t}\right)\left(\sum \mathbf{k}_{11,t}'\right) & \quad -\frac{\rho}{T(1+\rho^2)}\left(\sum \mathbf{k}_{22,t}\right)\left(\sum \mathbf{k}_{22,t}'\right) \end{bmatrix}. \tag{2.11}$$

Finally, replacing the true parameters and relevant terms with the ML estimators under the null, namely $\boldsymbol{\theta}$ with $\widetilde{\boldsymbol{\theta}}$, $\mathbf{z}_t$ with $\tilde{\mathbf{z}}_t$, and $\mathbf{k}_{ij,t}$ with $\tilde{\mathbf{k}}_{ij,t}$, $i, j = 1, 2$, gives the LM_{ECCC} statistic (19) in the main paper. The statistic in the bivariate case has an asymptotic χ^2 distribution with four degrees of freedom when the null hypothesis is valid. In practice, we are able to compute the test statistic numerically through the relevant expressions.

3 Mathematical derivations and proofs

Vector and matrix derivatives, and some properties of special matrices used in this section can be found in Lütkepohl (1996). See also Eklund and Teräsvirta (2007).

3.1 Proof of Lemma 1

Since $\boldsymbol{\theta}' = [\boldsymbol{\omega}', \boldsymbol{\rho}']$, the partial derivative of the log-likelihood function for observation t with respect to $\boldsymbol{\theta}$ is partitioned into

$$\frac{\partial \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \begin{bmatrix} \frac{\partial \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\omega}} \\ \frac{\partial \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\rho}} \end{bmatrix}. \tag{3.1}$$

The upper block in (3.1) has $N(2N+1)$ entries while the lower one has $N(N-1)/2$ elements.

For the upper block in (3.1), we can use the chain rules of vector derivatives to have

$$\begin{aligned}
\frac{\partial \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\omega}} &= -\frac{\partial \ln|\mathbf{D}_t|}{\partial \boldsymbol{\omega}} - \frac{1}{2}\frac{\partial(\boldsymbol{\varepsilon}_t'\mathbf{H}_t^{-1}\boldsymbol{\varepsilon}_t)}{\partial \boldsymbol{\omega}} \\
&= -\frac{\partial \operatorname{vec}(\mathbf{D}_t)'}{\partial \boldsymbol{\omega}}\frac{\partial \ln|\mathbf{D}_t|}{\partial \operatorname{vec}(\mathbf{D}_t)} - \frac{1}{2}\frac{\partial \operatorname{vec}(\mathbf{D}_t)'}{\partial \boldsymbol{\omega}}\frac{\partial \operatorname{vec}(\mathbf{D}_t^{-1})'}{\partial \operatorname{vec}(\mathbf{D}_t)}\frac{\partial \operatorname{vec}(\mathbf{D}_t^{-1}\mathbf{P}^{-1}\mathbf{D}_t^{-1})'}{\partial \operatorname{vec}(\mathbf{D}_t^{-1})}\frac{\partial(\boldsymbol{\varepsilon}_t'\mathbf{H}_t^{-1}\boldsymbol{\varepsilon}_t)}{\partial \operatorname{vec}(\mathbf{H}_t^{-1})} \\
&= -\frac{\partial \operatorname{vec}(\mathbf{D}_t)'}{\partial \boldsymbol{\omega}}\operatorname{vec}(\mathbf{D}_t^{-1}) + \frac{1}{2}\frac{\partial \operatorname{vec}(\mathbf{D}_t)'}{\partial \boldsymbol{\omega}}\left(\mathbf{D}_t^{-1}\otimes\mathbf{D}_t^{-1}\right)\left(\mathbf{P}^{-1}\mathbf{D}_t^{-1}\otimes\mathbf{I}_N + \mathbf{I}_N\otimes\mathbf{P}^{-1}\mathbf{D}_t^{-1}\right)\operatorname{vec}(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t') \\
&= -\frac{\partial \operatorname{vec}(\mathbf{D}_t)'}{\partial \boldsymbol{\omega}}\operatorname{vec}\left(\mathbf{D}_t^{-1} - \frac{1}{2}\mathbf{z}_t\mathbf{z}_t'\mathbf{P}^{-1}\mathbf{D}_t^{-1} - \frac{1}{2}\mathbf{D}_t^{-1}\mathbf{P}^{-1}\mathbf{z}_t\mathbf{z}_t'\right).
\end{aligned} \tag{3.2}$$

By similar manipulations for the lower block in (3.1), we obtain

$$\begin{aligned}
\frac{\partial \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\rho}} &= -\frac{1}{2}\frac{\partial \ln|\mathbf{P}|}{\partial \boldsymbol{\rho}} - \frac{1}{2}\frac{\partial(\boldsymbol{\varepsilon}_t'\mathbf{D}_t^{-1}\mathbf{P}^{-1}\mathbf{D}_t^{-1}\boldsymbol{\varepsilon}_t)}{\partial \boldsymbol{\rho}} \\
&= -\frac{1}{2}\frac{\partial \operatorname{vec}(\mathbf{P})'}{\partial \boldsymbol{\rho}}\operatorname{vec}\left(\mathbf{P}^{-1} - \mathbf{P}^{-1}\mathbf{z}_t\mathbf{z}_t'\mathbf{P}^{-1}\right).
\end{aligned} \tag{3.3}$$

The equations (3.2) and (3.3) constitute (11) in the main paper. ∎

3.2 Proof of Lemma 2

The second partial derivative of the log-likelihood function for observation t with respect to $\boldsymbol{\theta}$, or the Hessian, can be partitioned in the following way:

$$\frac{\partial^2 \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}'\partial \boldsymbol{\theta}} = \begin{bmatrix} \frac{\partial^2 \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\omega}'\partial \boldsymbol{\omega}} & \frac{\partial^2 \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\rho}'\partial \boldsymbol{\omega}} \\ \frac{\partial^2 \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\omega}'\partial \boldsymbol{\rho}} & \frac{\partial^2 \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\rho}'\partial \boldsymbol{\rho}} \end{bmatrix}. \tag{3.4}$$

In the subsequent sections, we supply blockwise derivations of the Hessian (3.4). The final results (12) of the main paper is attained by taking the conditional expectations of derived blocks with the relations $\mathsf{E}[\mathbf{z}_t\mathbf{z}_t'] = \mathbf{P}$ and $\mathbf{H}_t = \mathbf{D}_t\mathbf{P}\mathbf{D}_t$.

3.2.1 The upper left block of the Hessian

The upper left block of (3.4) is given by

$$\frac{\partial}{\partial \boldsymbol{\omega}'}\left(\frac{\partial \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\omega}}\right) = -\frac{\partial}{\partial \boldsymbol{\omega}'}\left(\frac{\partial \operatorname{vec}(\mathbf{D}_t)'}{\partial \boldsymbol{\omega}} \operatorname{vec}\left(\mathbf{D}_t^{-1}\right)\right) + \frac{1}{2}\frac{\partial}{\partial \boldsymbol{\omega}'}\left(\frac{\partial \operatorname{vec}(\mathbf{D}_t)'}{\partial \boldsymbol{\omega}} \operatorname{vec}\left(\mathbf{D}_t^{-1}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\mathbf{D}_t^{-1}\mathbf{P}^{-1}\mathbf{D}_t^{-1}\right)\right) + \frac{1}{2}\frac{\partial}{\partial \boldsymbol{\omega}'}\left(\frac{\partial \operatorname{vec}(\mathbf{D}_t)'}{\partial \boldsymbol{\omega}} \operatorname{vec}\left(\mathbf{D}_t^{-1}\mathbf{P}^{-1}\mathbf{D}_t^{-1}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\mathbf{D}_t^{-1}\right)\right). \tag{3.5}$$

The first term in (3.5) is decomposed to

$$\frac{\partial}{\partial \boldsymbol{\omega}'}\left(\frac{\partial \operatorname{vec}(\mathbf{D}_t)'}{\partial \boldsymbol{\omega}} \operatorname{vec}\left(\mathbf{D}_t^{-1}\right)\right) = -\frac{\partial \operatorname{vec}(\mathbf{D}_t)'}{\partial \boldsymbol{\omega}}\left(\mathbf{D}_t^{-1}\otimes\mathbf{D}_t^{-1}\right)\frac{\partial \operatorname{vec}(\mathbf{D}_t)}{\partial \boldsymbol{\omega}'} + \left[\operatorname{vec}\left(\mathbf{D}_t^{-1}\right)'\otimes\mathbf{I}_k\right]\frac{\partial^2 \operatorname{vec}(\mathbf{D}_t)'}{\partial \boldsymbol{\omega}'\partial \boldsymbol{\omega}}.$$

The second term in (3.5) is developed to

$$\begin{aligned}
&\frac{\partial}{\partial \boldsymbol{\omega}'}\left(\frac{\partial \operatorname{vec}(\mathbf{D}_t)'}{\partial \boldsymbol{\omega}} \operatorname{vec}\left(\mathbf{D}_t^{-1}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\mathbf{D}_t^{-1}\mathbf{P}^{-1}\mathbf{D}_t^{-1}\right)\right)\\
&= \left[\operatorname{vec}\left(\mathbf{D}_t^{-1}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\mathbf{D}_t^{-1}\mathbf{P}^{-1}\mathbf{D}_t^{-1}\right)'\otimes\mathbf{I}_k\right]\frac{\partial^2 \operatorname{vec}(\mathbf{D}_t)'}{\partial \boldsymbol{\omega}'\partial \boldsymbol{\omega}}\\
&\quad + \frac{\partial \operatorname{vec}(\mathbf{D}_t)'}{\partial \boldsymbol{\omega}}\frac{\partial \operatorname{vec}\left(\mathbf{D}_t^{-1}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\mathbf{D}_t^{-1}\mathbf{P}^{-1}\mathbf{D}_t^{-1}\right)}{\partial \operatorname{vec}(\mathbf{D}_t^{-1})'}\frac{\partial \operatorname{vec}(\mathbf{D}_t^{-1})}{\partial \operatorname{vec}(\mathbf{D}_t)'}\frac{\partial \operatorname{vec}(\mathbf{D}_t)}{\partial \boldsymbol{\omega}'}\\
&= \left[\operatorname{vec}\left(\mathbf{z}_t\mathbf{z}_t'\mathbf{P}^{-1}\mathbf{D}_t^{-1}\right)'\otimes\mathbf{I}_k\right]\frac{\partial^2 \operatorname{vec}(\mathbf{D}_t)'}{\partial \boldsymbol{\omega}'\partial \boldsymbol{\omega}} - \frac{\partial \operatorname{vec}(\mathbf{D}_t)'}{\partial \boldsymbol{\omega}}\big(\mathbf{D}_t^{-1}\mathbf{P}^{-1}\mathbf{z}_t\mathbf{z}_t'\otimes\mathbf{D}_t^{-1}\\
&\quad + \mathbf{D}_t^{-1}\mathbf{P}^{-1}\mathbf{D}_t^{-1}\otimes\mathbf{z}_t\mathbf{z}_t' + \mathbf{D}_t^{-1}\otimes\mathbf{z}_t\mathbf{z}_t'\mathbf{P}^{-1}\mathbf{D}_t^{-1}\big)\frac{\partial \operatorname{vec}(\mathbf{D}_t)}{\partial \boldsymbol{\omega}'}.
\end{aligned}$$

By similar operations, the third term in (3.5) is rewritten as

$$\begin{aligned}
&\frac{\partial}{\partial \boldsymbol{\omega}'}\left(\frac{\partial \operatorname{vec}(\mathbf{D}_t)'}{\partial \boldsymbol{\omega}} \operatorname{vec}\left(\mathbf{D}_t^{-1}\mathbf{P}^{-1}\mathbf{D}_t^{-1}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\mathbf{D}_t^{-1}\right)\right)\\
&= \left[\operatorname{vec}\left(\mathbf{D}_t^{-1}\mathbf{P}^{-1}\mathbf{z}_t\mathbf{z}_t'\right)'\otimes\mathbf{I}_k\right]\frac{\partial^2 \operatorname{vec}(\mathbf{D}_t)'}{\partial \boldsymbol{\omega}'\partial \boldsymbol{\omega}} - \frac{\partial \operatorname{vec}(\mathbf{D}_t)'}{\partial \boldsymbol{\omega}}\big(\mathbf{z}_t\mathbf{z}_t'\mathbf{P}^{-1}\mathbf{D}_t^{-1}\otimes\mathbf{D}_t^{-1}\\
&\quad + \mathbf{z}_t\mathbf{z}_t'\otimes\mathbf{D}_t^{-1}\mathbf{P}^{-1}\mathbf{D}_t^{-1} + \mathbf{D}_t^{-1}\otimes\mathbf{D}_t^{-1}\mathbf{P}^{-1}\mathbf{z}_t\mathbf{z}_t'\big)\frac{\partial \operatorname{vec}(\mathbf{D}_t)}{\partial \boldsymbol{\omega}'}.
\end{aligned}$$

Substituting back into (3.5) and rearranging yields

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{\omega}'}\left(\frac{\partial \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\omega}}\right) &= -\Big\{\left[\operatorname{vec}\left(\mathbf{D}_t^{-1}\right)'\otimes\mathbf{I}_k\right] - \frac{1}{2}\left[\operatorname{vec}\left(\mathbf{z}_t\mathbf{z}_t'\mathbf{P}^{-1}\mathbf{D}_t^{-1}\right)'\otimes\mathbf{I}_k\right] - \frac{1}{2}\left[\operatorname{vec}\left(\mathbf{D}_t^{-1}\mathbf{P}^{-1}\mathbf{z}_t\mathbf{z}_t'\right)'\otimes\mathbf{I}_k\right]\Big\}\\
&\times\frac{\partial^2 \operatorname{vec}(\mathbf{D}_t)'}{\partial \boldsymbol{\omega}'\partial \boldsymbol{\omega}} + \frac{1}{2}\frac{\partial \operatorname{vec}(\mathbf{D}_t)'}{\partial \boldsymbol{\omega}}\Big\{2\left(\mathbf{D}_t^{-1}\otimes\mathbf{D}_t^{-1}\right) - \big(\mathbf{D}_t^{-1}\mathbf{P}^{-1}\mathbf{z}_t\mathbf{z}_t'\otimes\mathbf{D}_t^{-1}\\
&+ \mathbf{D}_t^{-1}\mathbf{P}^{-1}\mathbf{D}_t^{-1}\otimes\mathbf{z}_t\mathbf{z}_t' + \mathbf{D}_t^{-1}\otimes\mathbf{z}_t\mathbf{z}_t'\mathbf{P}^{-1}\mathbf{D}_t^{-1}\big) - \big(\mathbf{z}_t\mathbf{z}_t'\mathbf{P}^{-1}\mathbf{D}_t^{-1}\otimes\mathbf{D}_t^{-1}\\
&+ \mathbf{z}_t\mathbf{z}_t'\otimes\mathbf{D}_t^{-1}\mathbf{P}^{-1}\mathbf{D}_t^{-1} + \mathbf{D}_t^{-1}\otimes\mathbf{D}_t^{-1}\mathbf{P}^{-1}\mathbf{z}_t\mathbf{z}_t'\big)\Big\}\frac{\partial \operatorname{vec}(\mathbf{D}_t)}{\partial \boldsymbol{\omega}'}.
\end{aligned} \tag{3.6}$$

3.2.2 The lower right block of the Hessian

The lower right block of (3.4) can be written as

$$\frac{\partial^2 \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\rho}' \partial \boldsymbol{\rho}} = -\frac{1}{2}\frac{\partial}{\partial \boldsymbol{\rho}'}\left(\frac{\partial \operatorname{vec}(\mathbf{P})'}{\partial \boldsymbol{\rho}} \operatorname{vec}\left(\mathbf{P}^{-1}\right)\right) + \frac{1}{2}\frac{\partial}{\partial \boldsymbol{\rho}'}\left(\frac{\partial \operatorname{vec}(\mathbf{P})'}{\partial \boldsymbol{\rho}} \operatorname{vec}\left(\mathbf{P}^{-1}\mathbf{z}_t\mathbf{z}_t'\mathbf{P}^{-1}\right)\right). \tag{3.7}$$

Noticing that $\partial^2 \operatorname{vec}(\mathbf{P})'/\partial \boldsymbol{\rho}' \partial \boldsymbol{\rho} = \mathbf{0}$, the first term in (3.7) is reduced to

$$\begin{aligned}\frac{\partial}{\partial \boldsymbol{\rho}'}\left(\frac{\partial \operatorname{vec}(\mathbf{P})'}{\partial \boldsymbol{\rho}} \operatorname{vec}\left(\mathbf{P}^{-1}\right)\right) &= \frac{\partial \operatorname{vec}(\mathbf{P})'}{\partial \boldsymbol{\rho}} \frac{\partial \operatorname{vec}\left(\mathbf{P}^{-1}\right)}{\partial \operatorname{vec}(\mathbf{P})'} \frac{\partial \operatorname{vec}(\mathbf{P})}{\partial \boldsymbol{\rho}'} \\ &= -\frac{\partial \operatorname{vec}(\mathbf{P})'}{\partial \boldsymbol{\rho}}(\mathbf{P}^{-1} \otimes \mathbf{P}^{-1})\frac{\partial \operatorname{vec}(\mathbf{P})}{\partial \boldsymbol{\rho}'},\end{aligned}$$

and the second term is equal to

$$\begin{aligned}\frac{\partial}{\partial \boldsymbol{\rho}'}\left(\frac{\partial \operatorname{vec}(\mathbf{P})'}{\partial \boldsymbol{\rho}} \operatorname{vec}\left(\mathbf{P}^{-1}\mathbf{z}_t\mathbf{z}_t'\mathbf{P}^{-1}\right)\right) &= \frac{\partial \operatorname{vec}(\mathbf{P})'}{\partial \boldsymbol{\rho}} \frac{\partial \operatorname{vec}\left(\mathbf{P}^{-1}\mathbf{z}_t\mathbf{z}_t'\mathbf{P}^{-1}\right)}{\partial \operatorname{vec}(\mathbf{P}^{-1})'} \frac{\partial \operatorname{vec}(\mathbf{P}^{-1})}{\partial \operatorname{vec}(\mathbf{P})'} \frac{\partial \operatorname{vec}(\mathbf{P})}{\partial \boldsymbol{\rho}'} \\ = -\frac{\partial \operatorname{vec}(\mathbf{P})'}{\partial \boldsymbol{\rho}}(\mathbf{P}^{-1}\mathbf{z}_t\mathbf{z}_t'\mathbf{P}^{-1} \otimes \mathbf{P}^{-1} + \mathbf{P}^{-1} \otimes \mathbf{P}^{-1}\mathbf{z}_t\mathbf{z}_t'\mathbf{P}^{-1})\frac{\partial \operatorname{vec}(\mathbf{P})}{\partial \boldsymbol{\rho}'}.\end{aligned}$$

Combining all together produces

$$\frac{\partial^2 \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\rho}' \partial \boldsymbol{\rho}} = \frac{1}{2}\frac{\partial \operatorname{vec}(\mathbf{P})'}{\partial \boldsymbol{\rho}}\left\{(\mathbf{P}^{-1} \otimes \mathbf{P}^{-1}) - (\mathbf{P}^{-1}\mathbf{z}_t\mathbf{z}_t'\mathbf{P}^{-1} \otimes \mathbf{P}^{-1}) - (\mathbf{P}^{-1} \otimes \mathbf{P}^{-1}\mathbf{z}_t\mathbf{z}_t'\mathbf{P}^{-1})\right\}\frac{\partial \operatorname{vec}(\mathbf{P})}{\partial \boldsymbol{\rho}'}. \tag{3.8}$$

3.2.3 The lower left and upper right blocks of the Hessian

Next consider the lower left block that is the cross derivatives. Using the facts that

$$\frac{\partial}{\partial \boldsymbol{\omega}'}\left(\frac{\partial \operatorname{vec}(\mathbf{P})'}{\partial \boldsymbol{\rho}} \operatorname{vec}(\mathbf{P}^{-1})\right) = \mathbf{0} \quad \text{and} \quad \frac{\partial}{\partial \boldsymbol{\omega}'} \operatorname{vec}\left(\frac{\partial \operatorname{vec}(\mathbf{P})'}{\partial \boldsymbol{\rho}}\right) = \mathbf{0},$$

we have

$$\begin{aligned}\frac{\partial^2 \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\omega}' \partial \boldsymbol{\rho}} &= \frac{1}{2}\frac{\partial \operatorname{vec}(\mathbf{P})'}{\partial \boldsymbol{\rho}} \frac{\partial(\mathbf{P}^{-1}\mathbf{D}_t^{-1}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\mathbf{D}_t^{-1}\mathbf{P}^{-1})}{\partial \operatorname{vec}(\mathbf{D}_t^{-1})'} \frac{\partial \operatorname{vec}(\mathbf{D}_t^{-1})}{\partial \operatorname{vec}(\mathbf{D}_t)'} \frac{\partial \operatorname{vec}(\mathbf{D}_t)}{\partial \boldsymbol{\omega}'} \\ &= -\frac{1}{2}\frac{\partial \operatorname{vec}(\mathbf{P})'}{\partial \boldsymbol{\rho}}(\mathbf{P}^{-1}\mathbf{D}_t^{-1}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t' \otimes \mathbf{P}^{-1} + \mathbf{P}^{-1} \otimes \mathbf{P}^{-1}\mathbf{D}_t^{-1}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t')(\mathbf{D}_t^{-1} \otimes \mathbf{D}_t^{-1})\frac{\partial \operatorname{vec}(\mathbf{D}_t)}{\partial \boldsymbol{\omega}'} \\ &= -\frac{1}{2}\frac{\partial \operatorname{vec}(\mathbf{P})'}{\partial \boldsymbol{\rho}}(\mathbf{P}^{-1}\mathbf{z}_t\mathbf{z}_t' \otimes \mathbf{P}^{-1}\mathbf{D}_t^{-1} + \mathbf{P}^{-1}\mathbf{D}_t^{-1} \otimes \mathbf{P}^{-1}\mathbf{z}_t\mathbf{z}_t')\frac{\partial \operatorname{vec}(\mathbf{D}_t)}{\partial \boldsymbol{\omega}'}. \end{aligned} \tag{3.9}$$

The upper right block of (3.4) is obtained by transposing (3.9), so that

$$\frac{\partial^2 \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\rho}' \partial \boldsymbol{\omega}} = -\frac{1}{2}\frac{\partial \operatorname{vec}(\mathbf{D}_t)'}{\partial \boldsymbol{\omega}}(\mathbf{z}_t\mathbf{z}_t'\mathbf{P}^{-1} \otimes \mathbf{D}_t^{-1}\mathbf{P}^{-1} + \mathbf{D}_t^{-1}\mathbf{P}^{-1} \otimes \mathbf{z}_t\mathbf{z}_t'\mathbf{P}^{-1})\frac{\partial \operatorname{vec}(\mathbf{P})}{\partial \boldsymbol{\rho}'}. \tag{3.10}$$

Finally, taking the conditional expectations of (3.6), (3.8), (3.9) and (3.10) with the relations $\mathsf{E}[\mathbf{z}_t\mathbf{z}_t'] = \mathbf{P}$ and $\mathbf{H}_t = \mathbf{D}_t\mathbf{P}\mathbf{D}_t$, and transposing them yields (12) of the main paper. ■

3.3 The fourth-order moment condition

As we mentioned in Section 3 of the main paper, the existence of the fourth- and the sixth-order moments of $\{\boldsymbol{\varepsilon}_t\}$ is necessary in developing the asymptotic theory for the quasi maximum likelihood estimator of of the parameters in the ECCC-GARCH model. However, finding these conditions, in particular those for the sixth-order moment, seems an involved task. It seems that the only available results are in Ling and McAleer (2003) and He and Teräsvirta (2004). Their results are general in the sense that the distribution of $\mathbf{z}_t$ need not be normal.

To introduce notation, let $\mathbf{a} = (a_1, \ldots, a_N)'$ be an $(N \times 1)$ vector and define the $(N \times N)$ diagonal matrix $\text{diagv}(\mathbf{a}) = \text{diag}(a_1, \ldots, a_N)$. Operator diagv creates the $(N \times N)$ diagonal matrix from an $(N \times 1)$ vector. Then under the assumption of normality we have from Corollary 2 in He and Teräsvirta (2004) the following sufficient condition for the existence of the unconditional fourth-moment matrix $\mathsf{E}[\boldsymbol{\varepsilon}_t^{(2)}\boldsymbol{\varepsilon}_t^{(2)'}]$ of $\{\boldsymbol{\varepsilon}_t\}$ generated by an ECCC-GARCH(p, q) process:

Lemma 1. *Assume that the stationarity condition (5) in the main paper holds, and that* $\mathbf{z}_t \sim \mathsf{N}(\mathbf{0}, \mathbf{P})$. *Then* $\mathsf{E}[\boldsymbol{\varepsilon}_t^{(2)}\boldsymbol{\varepsilon}_t^{(2)'}]$ *exists if*

$$\lambda(\boldsymbol{\Gamma}_{C\otimes C}) < 1 \tag{3.11}$$

where

$$\begin{aligned}\boldsymbol{\Gamma}_{C\otimes C} &= \mathsf{E}[\mathbf{C}_t \otimes \mathbf{C}_t] \\ &= (\mathbf{A}_1 + \mathbf{B}_1) \otimes (\mathbf{A}_1 + \mathbf{B}_1) + 2(\mathbf{A}_1 \otimes \mathbf{A}_1)\,\text{diagv}(\text{vec}(\mathbf{P}) \odot \text{vec}(\mathbf{P}))\end{aligned} \tag{3.12}$$

where $\odot$ *denotes the Hadamard (elementwise) product of two matrices or vectors.*

Proof. Since (3.11) is immediate from Corollary 2 in He and Teräsvirta (2004), we verify that (3.12) holds under the assumption of normality. First note that

$$\mathsf{E}[\mathbf{C}_t \otimes \mathbf{C}_t] = (\mathbf{A}_1 \otimes \mathbf{A}_1)\mathsf{E}[\mathbf{Z}_t^2 \otimes \mathbf{Z}_t^2] + (\mathbf{A}_1 + \mathbf{B}_1) \otimes (\mathbf{A}_1 + \mathbf{B}_1) - \mathbf{A}_1 \otimes \mathbf{A}_1. \tag{3.13}$$

Assume $\mathbf{z}_t \sim \mathsf{N}(\mathbf{0}, \mathbf{P})$ with $\mathbf{P} = [\boldsymbol{\rho}_1 \ldots \boldsymbol{\rho}_N]$ and let $\mathbf{M} = [\mathbf{M}_{ij}] = \mathbf{Z}_t^2 \otimes \mathbf{Z}_t^2$ where the blocks $\mathbf{M}_{ij}$ are of size $N \times N$. Then,

$$\begin{aligned}\mathbf{M}_{ii} &= z_{i,t}^2\,\text{diagv}(\mathbf{z}_t \odot \mathbf{z}_t) \\ &= z_{i,t}^2\,\text{diag}(z_{1,t}^2, \ldots, z_{N,t}^2),\ i = 1, \ldots, N\end{aligned}$$

and $\mathbf{M}_{ij} = \mathbf{0}$, $i \neq j$. Then (see, for instance, Nabeya, 1951),

$$\mathsf{E}z_{i,t}^2 z_{j,t}^2 = \begin{cases} 3 & i = j \\ 1 + 2\rho_{ij}^2 & \text{otherwise.} \end{cases}$$

It thus follows that

$$\mathsf{E}\mathbf{M}_{ii} = \mathbf{I}_N + 2\,\text{diagv}(\boldsymbol{\rho}_i \odot \boldsymbol{\rho}_i),\ i = 1, \ldots, N$$

and

$$\mathsf{E}[\mathbf{Z}_t^2 \otimes \mathbf{Z}_t^2] = \mathbf{I}_N \otimes \mathbf{I}_N + 2\,\text{diagv}(\text{vec}(\mathbf{P}) \odot \text{vec}(\mathbf{P})). \tag{3.14}$$

Inserting (3.14) into (3.13) yields (3.12). ■

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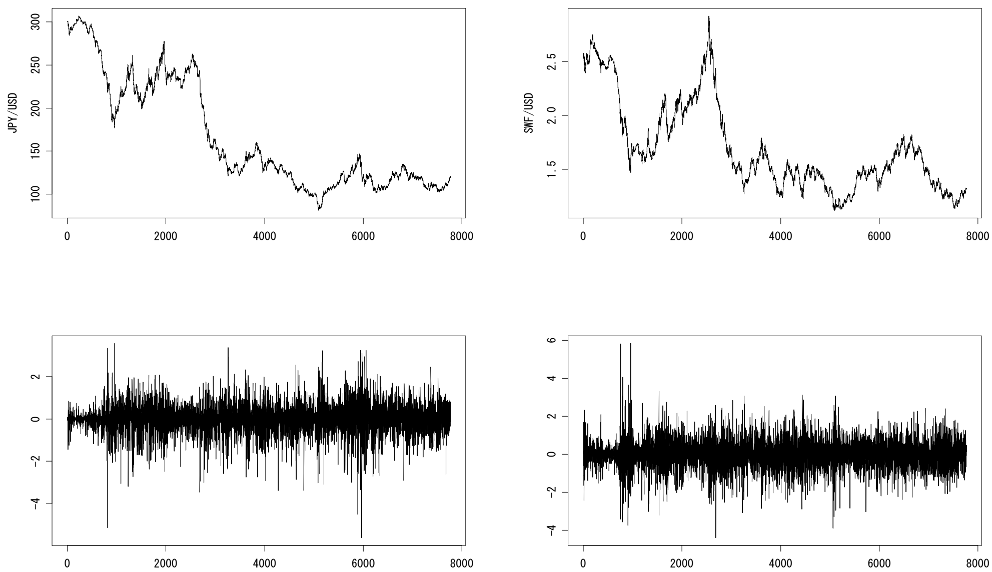
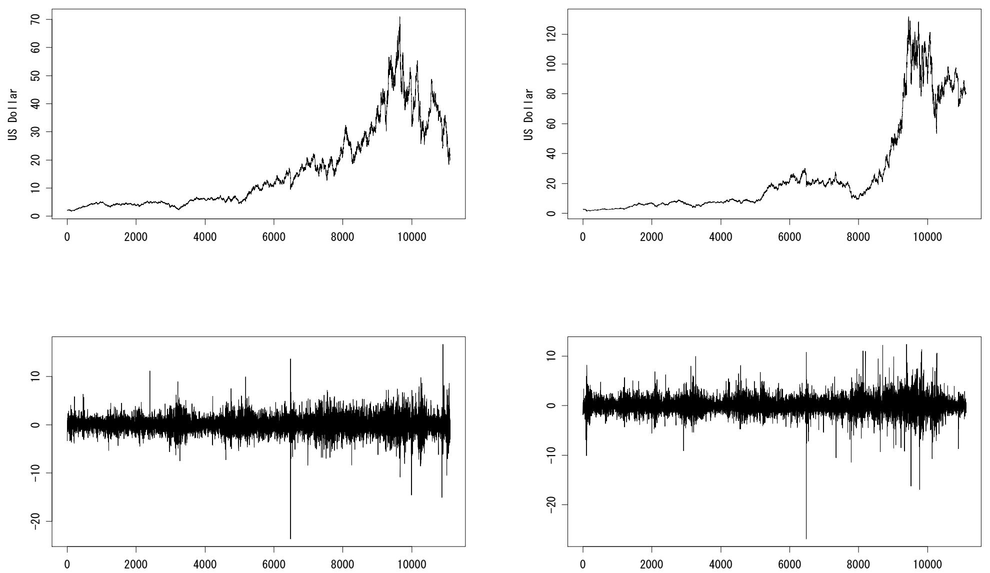


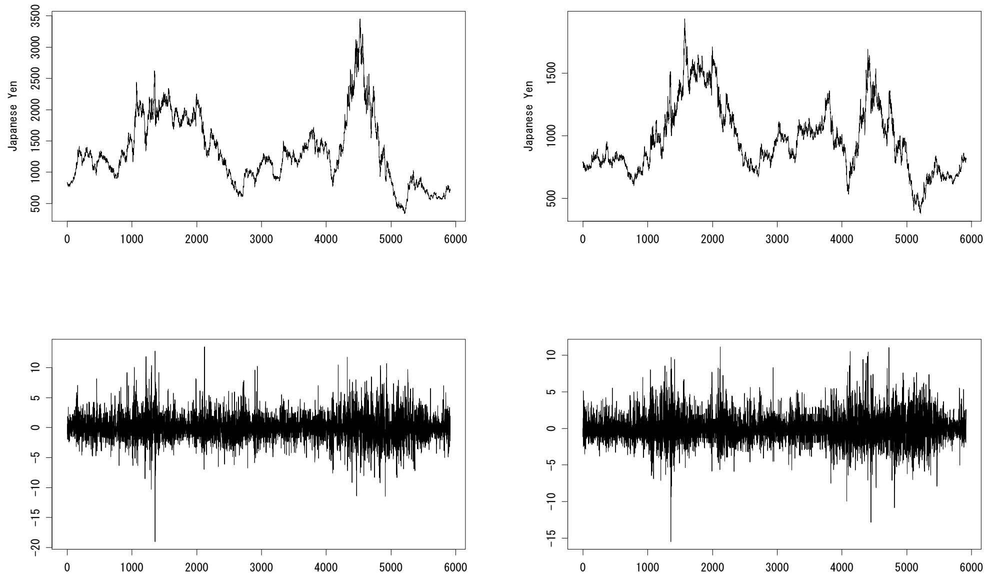
(a) Japanese Yen (JPY) (b) Swiss Franc (CHF)

Figure 1: Three foreign exchange rates against U.S. dollars between 2 Jan, 1975 and 2 Dec, 2005: the upper three panels depict the level series, and the lower ones are the mean subtracted return series.



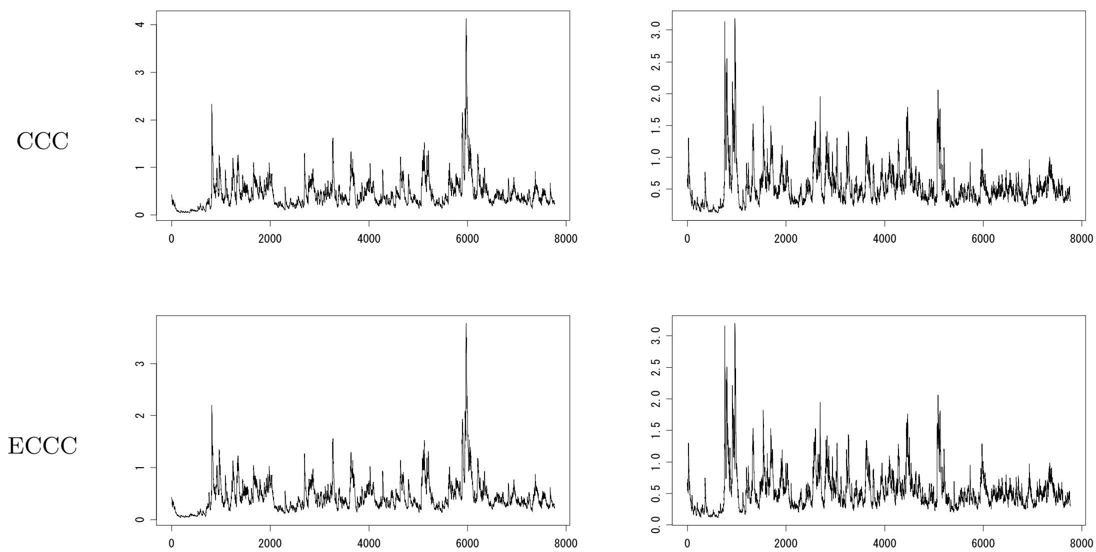
(a) GM (b) IBM

Figure 2: The three U.S. stocks between 2 Jan, 1962 and 28 Feb, 2006: the upper two panels depict the level series, and the lower ones are the mean subtracted return series.



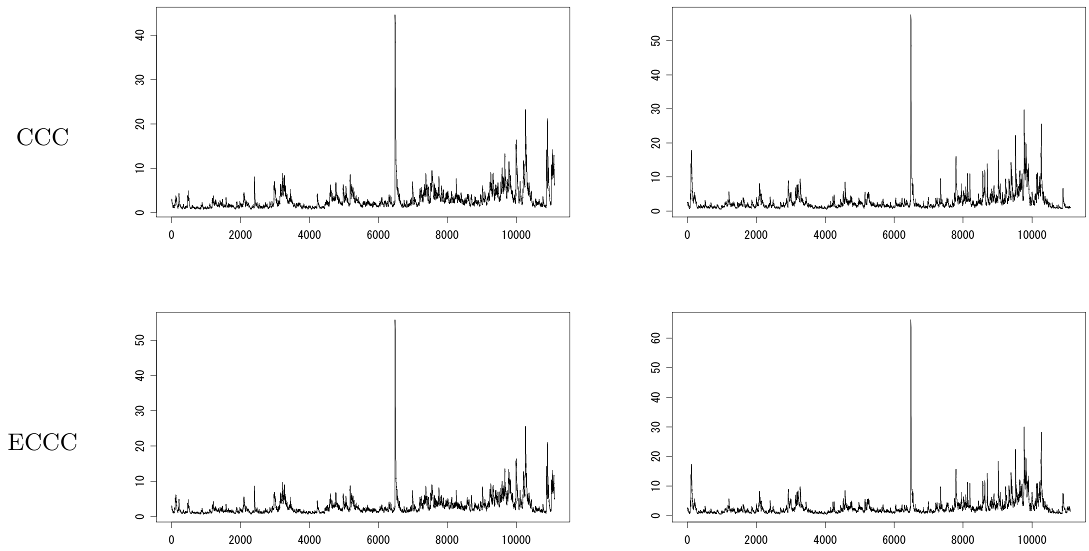
(a) NEC (b) Hitachi

Figure 3: The three Japanese stocks between 4 Jan, 1983 and 1 Mar, 2006: the upper three panels depict the level series, and the lower ones are the mean subtracted return series.



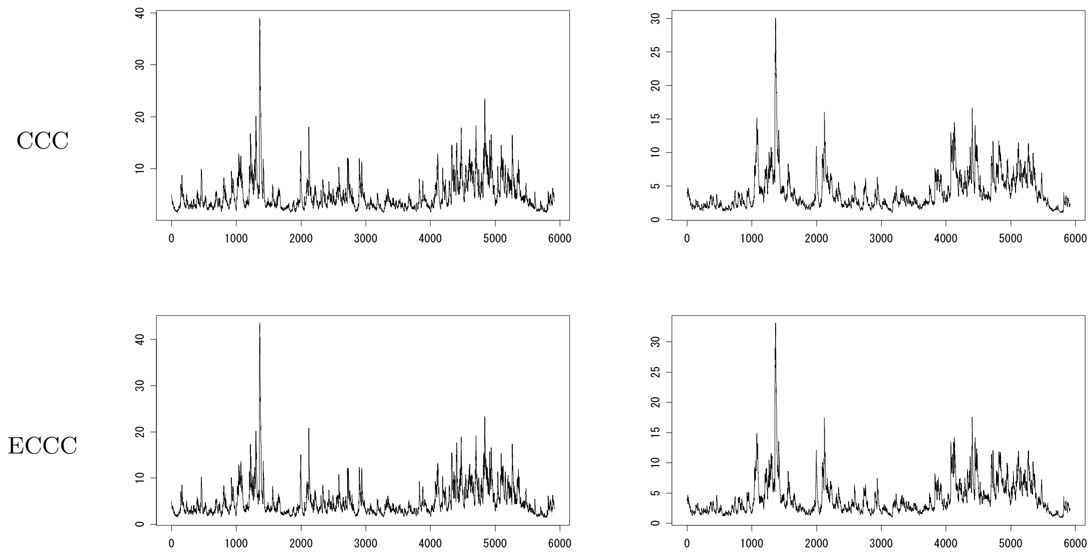
(a) Japanese Yen (JPY) (b) Swiss Franc (CHF)

Figure 4: Estimated conditional variances in the CCC-/ECCC-GARCH(1, 1) models.



(a) GM (b) IBM

Figure 5: Estimated conditional variances in the CCC-/ECCC-GARCH$(1,1)$ models.



(a) NEC (b) Hitachi

Figure 6: Estimated conditional variances in the CCC-/ECCC-GARCH$(1,1)$ models.