Evolution and refinement with endogenous mistake probabilities

Eric van Damme*
Center for Economic Research
Tilburg University
and
Jörgen W. Weibull
Department of Economics
Stockholm School of Economics, and
the Research Institute of Industrial Economics

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Abstract. Bergin and Lipman (1996) show that the refinement effect from the random mutations in the adaptive population dynamics in Kandori, Mailath and Rob (1993) and Young (1993) is due to restrictions on how these mutation rates vary across population states. We here model mutation rates as endogenously determined mistake probabilities, by assuming that players with some effort can control the probability of implementing the intended strategy. This is shown to corroborate the results in Kandori-Mailath-Rob (1993) and, under certain regularity conditions, those in Young (1993). The approach also yields a new refinement of the Nash equilibrium concept that is logically independent of Selten's (1975) perfection concept and Myerson's (1978) properness concept.

1. Introduction

It has been shown by Kandori, Mailath and Rob (1993), henceforth "KMR," and Young (1993), henceforth "Young," that adding small noise to certain adaptive dynamics in games can lead to rejection of strict Nash equilibria. Specifically, in 2×2 coordination games these dynamics allow one to conclude that in the long run the risk-dominant equilibrium (Harsanyi and Selten, 1988) will result. This surprisingly strong result has recently been challenged by Bergin and Lipman (1996) who show that it depends on specific assumptions about the mutation process, namely that the

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mutation rate does not vary "too much" across the different states of the adaptive process. They show that, if mutation rates at different states are not taken to zero at the same rate, then many different outcomes are possible. Indeed, any stationary state in the noise-free dynamics can be approximated by a stationary state in the noisy process, by choosing the mutation rates appropriately. In particular, any of the two strict Nash equilibria in a 2×2 coordination game may be selected in the long run.

Bergin and Lipman conclude from this lack of robustness that the nature of the mutation process must be scrutinized more carefully if one is to derive economically meaningful predictions, and they offer two suggestions for doing so. As suggested already in the original KMR and Young papers, there are two ways of interpreting the mutations in these models: Mutations may be thought of as arising from individuals' experiments or from their mistakes. In the first case, it is natural to expect the mutation rate to depend on the state - individuals may be expected to experiment less in states with higher payoffs. Also in the second case state-dependent mutation rates appear reasonable - exploring an idea proposed in Myerson (1978) one might argue that mistakes associated with larger payoff losses are less likely.

While Bergin and Lipman are right to point out that these considerations might lead to state-dependent mutation rates, they do not elaborate or formalize these ideas. Hence, it is not clear whether their concerns really matter for the conclusions drawn by KMR and Young. The aim of this study is to shed light on this issue by means of a model of mutations as mistakes. The model is based on the assumption that players "rationally choose to make mistakes" because it is too costly to avoid them completely. It turns out that our model produces mistake probabilities that under mild regularity conditions do not vary "too much" with the state of the system. Hence, the concerns of Bergin and Lipman are irrelevant in this case.

Indeed, in the case of a symmetric 2×2-coordination game, it is straightforward to see why such a result should come about in the KMR model. Namely, for $a > c$ and $d > b$, equilibrium (A, A) is risk dominant in the game

	A	B
A	a, a	b, c
B	c, b	d, d

(1)

if and only if A is the unique best reply to the mixed strategy $\frac{1}{2}A + \frac{1}{2}B$. Hence, (A, A) is risk dominant if and only if $a + b > c + d$, a condition which is equivalent to $a - c > d - b$. This latter condition amounts to saying that a mistake at (A, A) involves a larger payoff loss than a mistake at (B, B). Hence, any reasonable theory of endogenous mistakes should imply that mistakes at (A, A) are less likely than

mistakes at (B, B). In other words, the basin of attraction of population state "A" should not only be "larger" than that of state "B", it should also be "deeper" - thus making it even more difficult to upset this equilibrium. This intuition is not available, however, in other dynamics and in asymmetric games.

The formal model we develop in this paper concerns arbitrary finite games in normal form, and is similar to the control-cost model in van Damme (1987, Chapter 4). In essence, players are assumed to have a trembling hand, and by controlling it more carefully, which involves some disutility, the amount of trembles can be reduced. Since a rational player will try harder to avoid more serious mistakes, i.e. mistakes that lead to larger payoff losses, such mistakes will be less likely. However, they will still occur with positive probability since, by assumption, it is infinitely costly to avoid mistakes completely. Although mistake probabilities thus depend on the associated payoff losses, and therefore also on the state of the process, we show that, under mild regularity conditions, they all go to zero at the same rate when control costs become vanishingly small in comparison with the payoffs in the game. Consequently, the results established by KMR and Young then are valid also in this model of endogenous mistake probabilities.

As a by-product, we obtain a new refinement of the Nash equilibrium concept, which we call robustness with respect to endogenous trembles. This concept is similar in spirit to Selten's (1975) notion of trembling hand perfect equilibrium. In Selten's model both the mistake probabilities and the conditional error distributions (in case of a mistake) are exogenous, i.e. are chosen by the analyst. In our approach, the conditional error distributions are exogenous but the mistake probabilities are determined endogenously. We establish existence of strategy profiles which are robust to endogenous trembles, and show that these are Nash equilibria. We also show that this form of robustness neither implies nor is implied by perfection. It also differs from Myerson's (1978) concept of proper equilibrium in that in our model, all mistake probabilities may be of the same order of magnitude where properness prescribes that these be of different order of magnitude.

From a somewhat philosophical viewpoint one could say that, while robustness with respect to some slight probabilities of mistakes has cutting power in Selten's (1975) approach, Bergin and Lipman (1996) showed that such robustness has no cutting power in Young's (1993) approach. The present model of endogenous mistake probabilities has cutting power in Selten's rationalistic setting, and, under mild regularity conditions, also in Young's setting. While we restrict the analysis in this paper to the adaptive dynamics proposed in KMR and Young, the methodology can of course be applied also to other models which allow for mistakes in decision making, such as, to name just one example, Robson and Vega-Redondo (1996), where the long-run outcome in 2×2 coordination games is the Pareto dominant equilibrium,

even if this is different from the risk dominant equilibrium.

We do not know of any earlier work along the above lines. Blume (1993) studies strategic interaction between individuals who are located on a lattice and recurrently interact with their neighbors. These individuals play stochastically perturbed myopic best responses, in the sense that the choice probability for each pure strategy is an increasing positive function of its current payoff. As a consequence, more costly mistakes are assigned lower choice probabilities. Blume (1994) elaborates and extends this model to a more conventional random-matching setting in which choice probability ratios are an increasing function of the associated payoff differences. Maruta (1997) generalizes Blume's models by letting choice probability ratios be a function of both payoffs - not necessarily only of their difference. While these studies take random choice behavior as a starting point for the analysis, we here derive such behavior from an explicit decision-theoretic model in which individuals take account of their own mistake probabilities. Robles (1998) extends the KMR and Young models by letting mutation rates decline to zero over time, in one part of the study also allowing for state-dependent mutation rates. Also his work is complementary to ours in the sense that while he takes the state-dependence for given and analyzes implications thereof, we suggest a model that explains why and how mutation rates vary across population states.

The remainder of the paper is organized as follows. In Section 2 we introduce games with endogenous mistake probabilities, derive some properties of equilibria of such games, and introduce an accompanying refinement. In Section 3 we show that the upper limit of any ratio of mistake probabilities is bounded away from zero, and we introduce a regularity condition that guarantees that also the lower limit is positive. In Section 4 we consider the adaptive dynamics in Young (1993, 1998) with mutations as endogenously determined mistakes. We show that, if the regularity conditions are satisfied, then Young's results for finite n-player games hold, implying, in particular, that the risk dominant equilibrium is selected in generic 2×2 coordination games. Section 5 considers an example where the regularity condition is violated. We show that there need be no unique stochastically stable equilibrium in this case, but that the risk dominant equilibrium nevertheless is in the limit set. Section 6 concludes.

2. Games with endogenous mistake probabilities

Van Damme (1987, chapter 4) develops a model where mistakes arise in implementing pure strategies in games. The basic idea is that players make mistakes because it is too costly to prevent these completely. Each player has a trembling hand, and by making effort to control it more carefully, which involves disutility, the amount of trembles can be reduced. It is assumed that the disutility of eliminating trembles completely is prohibitive.

We here elaborate a closely related model of mistake control. Consider a finite n-player normal-form game G, with pure strategy sets S_1, ..., S_n, and mixed-strategy sets $\Delta_i = \Delta(S_i)$. For any mixed-strategy profile $\sigma = (\sigma_1, ..., \sigma_n)$, let the payoff to player i be $\pi_i(\sigma) \in \mathbb{R}$, and let $\beta_i(\sigma) \subset \Delta_i$ be i's set of best replies to $\sigma \in \Delta = \times_i \Delta_i$. Let V denote the class of functions $v : (0, 1] \to \mathbb{R}_{++}$ which are twice differentiable with $v'(\varepsilon) < 0$ and $v''(\varepsilon) > 0$ for all $\varepsilon \in (0, 1)$, $\lim_{\varepsilon \to 0} v(\varepsilon) = +\infty$, and $v'(1) = 0$. Such a function v will be called a *(mistake-control) disutility function.*

Embed a game G as described above in an n-player game $\tilde{G}(\eta, \bar{v}, \delta)$, for $\bar{v} = (v_1, ...v_n) \in V^n$, $\eta \in int(\Delta)$, and $\delta > 0$, with strategy sets $X_i = \Delta_i \times (0, 1]$, $X = \times_i X_i$, and payoff functions $u_i : X \to \mathbb{R}$ defined by

$$u_i(x) = \pi_i(\omega) - \delta v_i(\varepsilon_i) \ , \tag{2}$$

where $x = (x_1, ..., x_n)$, $x_i = (\sigma_i, \varepsilon_i)$ for all i, and

$$\omega_i = (1 - \varepsilon_i)\, \sigma_i + \varepsilon_i \eta_i \ . \tag{3}$$

Our interpretation is that each player i chooses a pair $x_i = (\sigma_i, \varepsilon_i)$, where σ_i is a mixed strategy in G and ε_i is a mistake probability. Given this choice, strategy σ_i is implemented with probability $1 - \varepsilon_i$, otherwise the (exogenous) *error distribution* η_i is implemented. These random draws are statistically independent across player positions. Associated with each mistake-probability level ε_i is a disutility $v_i(\varepsilon_i)$ to player i, from the effort to keep his or her mistake probability at ε_i. The "disutility weight" δ measures the importance of this disutility of control effort relative to the payoffs in the underlying game G.

The assumptions made above concerning the disutility functions v_i imply that the marginal disutility of reducing one's mistake probability is increasing as the probability goes down, and that the disutility of reducing it to zero is prohibitive. The associated strategy profile $\omega \in int(\Delta)$, defined in equation (3), is the profile that will be played in G when the players choose strategies $x_i = (\sigma_i, \varepsilon_i)$ in $\tilde{G}(\eta, \bar{v}, \delta)$. We will say that the profile x in $\tilde{G}(\eta, \bar{v}, \delta)$ *induces* the profile ω in G, and we call $\tilde{G}(\eta, \bar{v}, \delta)$ a game *with endogenous mistake probabilities.* The limiting case $\delta = 0$ represents a situation in which all players are fully rational in the sense of being able to perfectly control their actions at no effort - as if they played game G.

2.1. Nash equilibrium. It is not difficult to characterize Nash equilibrium in $\tilde{G}(\eta, \bar{v}, \delta)$. First, it follows directly from equations (2) and (3) that a necessary condition is $\sigma_i \in \beta_i(\omega)$ for every player i who chooses $\varepsilon_i < 1$, while for players i with $\varepsilon_i = 1$ any $\sigma_i \in \Delta_i$ is optimal. To see this, note that

$$u_i(x) = (1 - \varepsilon_i)\,\pi_i(\sigma_i, \omega_{-i}) + \varepsilon_i \pi_i(\eta_i, \omega_{-i}) - \delta v_i(\varepsilon_i)\ . \tag{4}$$

In other words, irrespective of the chosen mistake probability ε_i, as long as this is less than one, player i will choose a best reply σ_i in G to the induced profile ω.[1] Thus, $\pi_i(\sigma_i, \omega_{-i}) = b_i(\omega)$, the *best payoff* that player i can obtain in G against a strategy profile $\omega \in \Delta$, where

$$b_i(\omega) = \max_{\sigma_i} \pi_i(\sigma_i, \omega_{-i})\ . \tag{5}$$

Let $l_i(\omega) \geq 0$ denote the *payoff loss* that player i incurs in G if his error-distribution η_i is played against $\omega \in \Delta$:

$$l_i(\omega) = b_i(\omega) - \pi_i(\eta_i, \omega_{-i})\ . \tag{6}$$

It follows from equation (4) that, in Nash equilibrium, each ε_i necessarily is the unique solution to the first-order condition

$$-v_i'(\varepsilon_i) = l_i(\omega)/\delta\ , \tag{7}$$

Note that $\varepsilon_i = 1$ if and only if $\eta_i \in \beta_i(\omega)$.[2] Equation (7) simply says that, unless η_i happens to be a best reply to ω, the mistake probability should be chosen such that the marginal disutility of a reduction of the mistake probability equal the payoff loss in case of a mistake. If η_i is a best reply to ω, then no effort to reduce the mistake probability is worthwhile. In sum: if $x = (x_1, ..., x_n)$, with $x_i = (\sigma_i, \varepsilon_i)$ for all i, is a Nash equilibrium of $\tilde{G}(\eta, \bar{v}, \delta)$, then each ε_i satisfies equation (7), and $\sigma_i \in \beta_i(\omega)$ if $\varepsilon_i < 1$. Conversely, if each ε_i satisfies equation (7), and if $\sigma_i \in \beta_i(\omega)$ if $\varepsilon_i < 1$, then $x = (x_1, ..., x_n)$, for $x_i = (\sigma_i, \varepsilon_i)$, is a Nash equilibrium of $\tilde{G}(\eta, \bar{v}, \delta)$.

It follows from the assumed properties of the marginal-utility function $-v_i'$ that it has an inverse, which we denote $f_i : \mathbb{R}_+ \to (0, 1]$. Clearly f_i is differentiable with $f_i' < 0$, $f_i(0) = 1$, and $\lim_{y\to\infty} f_i(y) = 0$. Equation (7) can be re-written in terms of this inverse function as $\varepsilon_i = \hat{\varepsilon}_i(\delta, \omega)$, where

$$\hat{\varepsilon}_i(\delta, \omega) = f_i\left[l_i(\omega)/\delta\right]\ . \tag{8}$$

It follows immediately that a player chooses a smaller mistake probability if the expected loss in case of a mistake is larger:

$$l_i(\omega) > l_i(\omega') \qquad \Rightarrow \qquad \hat{\varepsilon}_i(\delta, \omega) < \hat{\varepsilon}_i(\delta, \omega')\ . \tag{9}$$

[1] Since ω is interior, this implies that σ_i is undominated.

[2] Since η_i is interior, this happens if and only if *all* pure strategies available to player i are best replies to ω.

Moreover, if the disutility weight δ is reduced, then each player i chooses a smaller mistake probability against every profile ω to which η_i is not a best reply:

$$\delta' < \delta \text{ and } \eta_i \notin \beta_i(\omega) \qquad \Rightarrow \qquad \hat{\varepsilon}_i(\delta', \omega) < \hat{\varepsilon}_i(\delta, \omega) \ . \tag{10}$$

After having characterized Nash equilibria of games with endogenous mistake probabilities, we now show that each such game has at least one Nash equilibrium.

Proposition 1. *For every $\eta \in int(\Delta)$, $\bar{v} \in V^n$ and $\delta > 0$, the game $\tilde{G}(\eta, \bar{v}, \delta)$ has at least one Nash equilibrium.*

Proof: Define the correspondence $\alpha : \Delta \twoheadrightarrow \Delta$ by

$$\alpha_i(\omega) = \{\omega_i' \in \Delta_i : \omega_i' = (1 - f_i[l_i(\omega)/\delta])\,\sigma_i + f_i[l_i(\omega)/\delta]\,\eta_i \ \text{ for some } \ \sigma_i \in \beta_i(\omega)\}\,. \tag{11}$$

It follows from Berge's Maximum Theorem that each loss function $l_i : \Delta \to \mathbb{R}_+$ is continuous. Since each inverse function f_i is continuous, and each correspondence β_i is upper hemi-continuous, α_i is upper hemi-continuous too. Moreover, since $\beta_i(\omega)$ is non-empty, convex and compact for all $\omega \in \Delta$, so is $\alpha_i(\omega)$. The polyhedron Δ of mixed-strategy profiles being convex and compact, there exists at least one fixed point under α by Kakutani's Fixed-Point Theorem. Given such a fixed point ω, let $\varepsilon_i = f_i[l_i(\omega)/\delta]$ and let $\sigma_i \in \beta_i(\omega)$ satisfy equation (3), for all i. Then $x = ((\sigma_i, \varepsilon_i))_{i \in I}$ induces ω, and x is a Nash equilibrium of $\tilde{G}(\eta, \bar{v}, \delta)$. **End of proof**.

2.2. Robustness against endogenous trembles. In the spirit of Selten's (1975) perfection of the Nash equilibrium concept, one could require that a strategy profile in a given finite game G be robust to slight disutilities of mistake control in the sense that some game with endogenous mistakes, and with a small weight attached to the disutility of mistake control, has some nearby Nash equilibrium. The above ideas can be used as a refinement of the Nash equilibrium concept. Formally:

Definition 1. *A strategy profile σ^* in G is* **robust to endogenous trembles** *if there exists a sequence $\langle \tilde{G}(\eta, \bar{v}, \delta_t) \rangle_{t \in \mathbb{N}}$ of games with endogenous mistake probabilities with $\delta_t \to 0$, and an accompanying convergent sequence of Nash equilibria $x^t = (\sigma^t, \varepsilon^t)$ such that $\omega^t \to \sigma^*$.*

Note that this concept differs from Selten's (1975) notion of trembling hand perfect equilibria in that in Selten's model both the mistake probability and the conditional error distribution, once a mistake is made, are exogenous, i.e. they can be freely chosen by the analyst. According to the above definition, the analyst can

choose the conditional error distribution, but the mistake probabilities are determined endogenously. We also note that, as with perfection, every completely mixed Nash equilibrium σ^* in G is robust to endogenous trembles in this sense. Just set $\eta = \sigma^* \in int(\Delta)$. Then $x^t = (\sigma, \varepsilon)$, with $\sigma_i = \sigma_i^*$ and $\varepsilon_i = 1$ for all i, constitutes a Nash equilibrium of $\tilde{G}(\eta, \bar{v}, \delta_t)$, for every t. Likewise, it is easily verified that strict equilibria are robust to endogenous trembles.

More generally, it follows from standard arguments that every finite game has at least one strategy profile which is robust to endogenous trembles, and that every such strategy profile constitutes a Nash equilibrium.

Proposition 2. *Every finite game G has at least one strategy profile which is robust to endogenous trembles, and every such profile is a Nash equilibrium of G.*

Proof: First, let $\eta \in int(\Delta)$, $\bar{v} \in V^n$ and $\langle \delta_t \rangle_{t=1}^{\infty} \to 0$. Then each game $\tilde{G}(\eta, \bar{v}, \delta_t)$ has at least one Nash equilibrium $x^t = ((\sigma_i^t, \varepsilon_i^t))_{i \in I} \in X$, by proposition 1. Let ω^t be the strategy profile induced by x^t. By the Bolzano-Weierstrass Theorem, there exists a convergent subsequence of $\langle x^t \rangle_{t=1}^{\infty}$, with limit in the closure of X. Along such a subsequence, let $\sigma_i = \lim_{t\to\infty} \sigma_i^t$, $\varepsilon_i = \lim_{t\to\infty} \varepsilon_i^t$, and $\theta_i = \lim_{t\to\infty} \omega_i^t$. By definition, θ is robust against endogenous trembles.

Secondly, let $\theta \in \Delta$ be robust to endogenous trembles. Then there exist $\eta \in int(\Delta)$, $\bar{v} \in V^n$, and sequences $\delta_t \to 0$ and $x^t \to x$, such that x^t is a Nash equilibrium of $\tilde{G}(\eta, \bar{v}, \delta_t)$ for each t, and $\omega^t \to \theta$. Suppose η_i is not a best reply to θ. Then $l_i(\theta) > 0$, and thus $l_i(\omega^t) > 0$ for all large t, by continuity of l_i. Thus $\varepsilon_i^t = 0$ and $\sigma_i^t \in \beta_i(\omega^t)$ for all i and all large t. Consequently, $\varepsilon_i = 0$, and $\sigma_i \in \beta_i(\theta)$ since the graph of β_i is closed. Hence, $\theta_i = \sigma_i \in \beta_i(\theta)$ in this case. Next, suppose η_i is a best reply to θ. Since η_i is interior, all pure strategies in S_i are best replies to θ, and thus $\theta_i \in \beta_i(\theta)$ also in this case. In sum: $\theta_i \in \beta_i(\theta)$ for all $i \in I$. **End of proof**.

Remark: The above definition of robustness is invariant under positive affine transformations of payoffs: If a game G' is obtained from a game G by a positive affine transformation of a player's payoffs, then a strategy profile is robust to endogenous trembles in G iff it is robust to endogenous trembles in G'. For although losses $l_i(\omega)$ are affected by linear re-scaling of payoffs, the analyst can compensate such a transformation by multiplying that player's disutility function by the same scalar, and thus leave equation (7) unchanged.

One might conjecture that robustness with respect to endogenous trembles implies robustness with respect to exogenous trembles, i.e. perfection in the sense of Selten (1975). However, the following example shows that this is not the case. Consider the two-player game (similar to Figure 4.3.1 in van Damme (1983)) with payoff bi-matrix

	L	R
T	$2,2$	$2,0$
B	$2,2$	$0,0$

(12)

This game has one perfect equilibrium, (T, L), and a continuum of imperfect Nash equilibria - all strategy profiles where player 2 plays L, and player 1 plays T with any probability less than one. Suppose $\eta_1 = \eta_2 = \left(\frac{1}{2}, \frac{1}{2}\right)$. Hence, each player randomizes 50/50 in case of a mistake. A mistake for player 2 is to play R, and, faced with 2's positive mistake probability, a mistake for player 1 is to play B. Let the common disutility function be $v(\varepsilon) = \varepsilon - \ln \varepsilon$. Then equation (7) gives $1/\varepsilon_1 - 1 = \varepsilon_2/\delta$ and $1/\varepsilon_2 - 1 = 1/\delta$. It follows that $\varepsilon_2/\delta \to 1$ as $\delta \to 0$, and thus $\varepsilon_1 \to 1/2$. Hence, the imperfect Nash equilibrium θ where player 2 plays L and player 1 plays T with probability $3/4$ is robust to endogenous trembles. Even in the limit, player 1 does not fully control his actions. The cost of doing so, near θ, exceeds the benefit that it brings. It may be noted that there is a similar finding in evolutionary game theory: evolutionary dynamics do not necessarily wipe out weakly dominated strategies, see Samuelson (1993) and Weibull (1995).

Similarly, a perfect equilibrium need not be robust to endogenous trembles. For example, add to the above game a third strategy, M, for player 2, as in the bi-matrix

	L	M	R
T	$2,2$	$0,1$	$2,0$
B	$2,2$	$2,1$	$0,0$

(13)

Because of the monotonicity property (9) of endogenous mistake probabilities, the choice M is more likely than R. Hence, player 1 has to choose B in an equilibrium that is robust against endogenous trembles. Nevertheless, (T, L) is a perfect equilibrium of this game. Hence, the refinement concept introduced above is logically independent from Selten's perfectness concept.

The spirit of this model of endogenous mistake probabilities is close to Myerson's (1978) notion of proper equilibrium - both view mistakes that result in larger payoff losses as less likely than mistakes that result in smaller payoff losses. However, while Myerson proposes no explicit model of mistake probability choice, his properness concept prescribes that more costly mistakes are an order of magnitude less likely than less costly mistakes. This need not be the case in the present model.[3] In fact, we will here identify two regularity conditions on the disutility functions under which all mistake probabilities are of the *same* order of magnitude when the disutility weight

[3]The same conclusion was drawn in the model of control costs developed in van Damme (1983,1987).

δ is taken to zero. As a consequence, the concept of robustness proposed here is also logically independent of Myerson's properness concept: a robust equilibrium need not be proper and a proper equilibrium need not be robust.

3. Regular disutility function profiles

There are two questions that are relevant concerning the order of magnitude of mistake probabilities when the disutility weight δ is taken to zero:

i) Will the mistake probabilities of one player be of the same order of magnitude as the mistake probabilities of another player, at any given strategy profile?

ii) Will the mistake probabilities of one player be of the same order of magnitude at different strategy profiles?

We discuss both issues in turn. First, however, we provide a simple condition under which all mistake probabilities necessarily are of the same order of magnitude, for all players and mixed-strategy profiles. The condition is that $\lim_{\varepsilon\to 0} -\varepsilon v_i'(\varepsilon)$ exists, is finite and non-zero, for all players i. Essentially, this condition requires all disutility functions to be logarithmic at small mistake probabilities. Suppose the condition is met. Consider two mixed-strategy profiles ω and ω' where two players, say i and j, have positive payoff losses in case of a mistake, $l_i(\omega) > 0$ and $l_j(\omega') > 0$. Then condition (7) implies

$$-v_i'\left[\hat{\varepsilon}_i(\delta,\omega)\right]/l_i(\omega) = \delta = -v_j\left[\hat{\varepsilon}_j(\delta,\omega')\right]/l_j(\omega') \tag{14}$$

Hence

$$\frac{-\hat{\varepsilon}_i(\delta,\omega)v_i'\left[\hat{\varepsilon}_i(\delta,\omega)\right]}{\hat{\varepsilon}_i(\delta,\omega)l_i(\omega)} = \frac{-\hat{\varepsilon}_j(\delta,\omega')v_j'\left[\hat{\varepsilon}_i(\delta,\omega')\right]}{\hat{\varepsilon}_j(\delta,\omega')l_j(\omega')} \tag{15}$$

Since the losses $l_i(\omega)$ and $l_j(\omega')$ are positive, both $\hat{\varepsilon}_i(\delta,\omega)$ and $\hat{\varepsilon}_j(\delta,\omega')$ go to zero as $\delta \to 0$. Thus,

$$\lim_{\delta\to 0}\frac{\hat{\varepsilon}_i(\delta,\omega)}{\hat{\varepsilon}_j(\delta,\omega')} = \square\frac{l_j(\omega')}{l_i(\omega)} > 0\ , \tag{16}$$

where $\square > 0$ is the ratio of $\lim_{\varepsilon\to 0} -\varepsilon v_i'(\varepsilon)$ and $\lim_{\varepsilon\to 0} -\varepsilon v_j'(\varepsilon)$. Hence, under the conditions stated, the mistake probabilities associated with positive losses are of the same order of magnitude in the limit as $\delta \to 0$.

We now turn to more general, sufficient conditions for this conclusion.

3.1. Similarity. Our formulation assumes that different players assign the same weight δ to the disutility of control effort. Hence, already this assumption introduces some comparability between players, and, as we have seen above, it implies that all mistakes are of the same order of magnitude under a certain condition. Assuming such comparability makes sense, for example, if individuals are drawn from the same background population, in which case one might even assume that all player populations have the same disutility function. This strict assumption, however, is not needed for our results. We will assume that the disutility functions of different player populations are *similar* in the sense that

$$\liminf_{y\to\infty} f_i(y)/f_j(y) > 0 \qquad \forall i,j \in I \ , \tag{17}$$

(where f_i is the inverse of the marginal-utility function $-v_i'$, see section 2). This condition is for example met if one player's disutility function is proportional to another player's disutility function. More generally, a sufficient condition for similarity is that there for every pair (i,j) of individuals exist a continuously differentiable function $g_{ij} : \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that $v_j(\varepsilon) = g_{ij}\left[v_i(\varepsilon)\right]$ for all $\varepsilon \in (0,1)$, where $a_{ij} < g_{ij}^{'} < b_{ij}$, for some $a_{ij} > 0$ and $b_{ij} < +\infty$.

In Section 5 we will show that, without a similarity assumption of this kind, results for adaptive population processes of the kind studied in Young (1993,1998) may depend on differences between disutility functions. (It is intuitive that some assumption of this type is needed. Because otherwise one player population may be induced to makes mistakes with much larger probability, and, essentially, the stability of an equilibrium depends on the largest mistake probability at that equilibrium.)

3.2. Niceness. We now turn to the more interesting issue of whether mistake probabilities of the same player, associated with different strategy profiles, ω and ω', will be of the same order in the limit. We first claim that in a game with endogenous mistake probabilities, as defined in section 2, the mistake probability $\hat{\varepsilon}_i(\delta,\omega)$ cannot be of smaller order of magnitude than $\hat{\varepsilon}_i(\delta,\omega')$ for *all* small δ. More exactly, in the appendix we prove:

Lemma 1. *Suppose* $\eta_i \notin \beta_i(\omega')$. *Then*

$$\limsup_{\delta\to 0} \frac{\hat{\varepsilon}_i(\delta,\omega)}{\hat{\varepsilon}_i(\delta,\omega')} > 0$$

It follows from this observation that if the ratio of two mistake probabilities converges, the limit must be positive. However, one can construct artificial examples in which the ratio does not converge, and where even the limit inferior of the ratio

$\hat{\varepsilon}_i(\delta,\omega)/\hat{\varepsilon}_i(\delta,\omega')$ is zero. We wish to exclude such examples by imposing a regularity condition on the disutility functions. We call this condition "niceness." Formally:

Definition 2. $v_i \in V$ *is* **nice** *if* $\liminf_{y\to\infty} f_i(\lambda y)/f_i(y) > 0$ *for some* $\lambda > 1$.

It follows from the above lemma that a sufficient condition for niceness is that, for some scalar $\lambda > 1$, the limit of $f_i(\lambda y)/f_i(y)$, as $y \to \infty$, exists.[4] Moreover, it is easily verified (and formally proved in the appendix) that if v_i is nice, then

$$\liminf_{y\to\infty} f_i(\lambda y)/f_i(y) > 0 \qquad \text{for all } \lambda > 0. \tag{18}$$

Niceness is defined rather indirectly, in terms of the inverse to the derivative of the disutility function in question. As for more direct sufficient conditions, we have already seen one, namely

$$0 < \liminf_{\varepsilon\to 0} -\varepsilon v_i'(\varepsilon) \;\square\; \limsup_{\varepsilon\to 0} -\varepsilon v_i'(\varepsilon) < +\infty\ . \tag{19}$$

Yet another sufficient condition is $\lim_{\varepsilon\to 0} -\varepsilon v_i'(\varepsilon) = +\infty$ (see van Damme (1983, Theorem 4.4.2) for a proof). The next result (of which the proof is again in the appendix) provides a third sufficient condition for niceness. It essentially says that the positive second derivative of the disutility function should not be increasing with ε, and that its relative risk aversion should be positive at small mistake probabilities.

Lemma 2. $v_i \in V$ *is nice if it is trice differentiable with* $v_i''' \;\square\; 0$, *and*

$$\liminf_{\varepsilon\to 0} -\varepsilon v_i''(\varepsilon)/v_i'(\varepsilon) > 0.$$

An example of a nice disutility function is $v_i(\varepsilon) = \varepsilon - \ln\varepsilon$. Then $v_i'(\varepsilon) = 1 - 1/\varepsilon$, $v_i''(\varepsilon) = 1/\varepsilon^2$, and $v_i'''(\varepsilon) = -2/\varepsilon^3$. Thus $-\varepsilon v_i'(\varepsilon) \to 1$ and $-\varepsilon v_i''(\varepsilon)/v_i'(\varepsilon) \to 1$ as $\varepsilon \to 0$.

3.3. Regularity. We are now in a position to state the advertised result. If the disutility function profile $\bar{v}$ is *regular* in the sense that all disutility functions are nice, and they are pairwise similar, then all mistake probabilities are of the same order of magnitude at all strategy profiles where all players have positive losses:

[4]By lemma 1, the limit superior of the ratio $f_i(\lambda y)/f_i(y)$, as $y \to \infty$, is positive, where $\lambda = l_i(\omega)/l_i(\omega')$. Hence, if the limit of this ratio exists, then it is positive and is identical with the limit inferior.

Proposition 3. *Consider a sequence of games $\tilde{G}(\eta, \bar{v}, \delta_t)$ with $\bar{v}$ regular, and $\delta_t \to 0$. Suppose $\omega, \omega' \in \Delta$, and $l_j(\omega') > 0$. Then*

$$\liminf_{t\to\infty} \frac{\hat{\varepsilon}_i(\delta_t, \omega)}{\hat{\varepsilon}_j(\delta_t, \omega')} > 0 \ .$$

Proof: If $l_i(\omega) \square\ l_j(\omega')$, then $\hat{\varepsilon}_i(\delta_t, \omega) \geq \hat{\varepsilon}_j(\delta_t, \omega')$ for all t, and thus the limit inferior is at least 1. If $l_i(\omega) > l_j(\omega') > 0$, then write $\lambda = l_i(\omega)/l_j(\omega') > 1$, and we have

$$\begin{aligned} \liminf_{\delta\to 0} \frac{\hat{\varepsilon}_i(\delta, \omega)}{\hat{\varepsilon}_j(\delta, \omega')} &= \liminf_{\delta\to 0} \frac{f_i\left[l_i(\omega)/\delta\right]}{f_j\left[l_j(\omega')/\delta\right]} = \liminf_{y\to\infty} \frac{f_i(\lambda y)}{f_j(y)} \\ &= \liminf_{y\to\infty} \frac{f_i(y)\, f_i(\lambda y)}{f_j(y)\, f_i(y)} > 0 \ , \end{aligned} \tag{20}$$

where the inequality follows from niceness and similarity: $f_i(\lambda y)/f_i(y) > a$ and $f_i(y)/f_j(y) > b$ for some $a, b > 0$, for all y sufficiently large. **End of proof**.

4. Boundedly rational adaptation

Kandori, Mailath and Rob (1993) analyze situations where a symmetric two-player game is recurrently played by a population of individuals. In each period, all pairs of individuals play against each other, and all individuals play pure strategies. Each individual plays a best reply to last period's strategy distribution with probability $1-\varepsilon$, for $\varepsilon > 0$ small. With the remaining probability, the individual plays a suboptimal strategy, with equal probability for all suboptimal strategies, and with statistical independence across individuals and periods. Hence, for any positive such mutation rate ε, this defines an ergodic Markov process. Kandori, Mailath and Rob (1993) study the limit as $\varepsilon \to 0$ of the associated unique stationary strategy distribution. In the case of a symmetric 2×2-coordination game, they show that this limiting distribution places all probability mass on the risk-dominant equilibrium.

It is easy to see that this result holds also under the present model of endogenous mistake probabilities mistakes. In a game with payoff bi-matrix (1), equilibrium (A, A) is risk dominant if and only if A is the unique best reply to the mixed strategy $\frac{1}{2}A + \frac{1}{2}B$. In this case, the state in which all individuals play A has a larger basin of attraction, under the above-mentioned best-reply dynamics, than the state where all individuals play B. Hence, it takes more simultaneous mutations to disrupt the first stationary state than the second. In the limit as $\varepsilon \to 0$, the probability of the first event becomes infinitesimally smaller than the probability of the latter. The ergodic process thus places all probability on state "A" in the limit. For the equilibrium (A, A) is risk dominant if and only if $a - c > d - b$. This latter condition amounts

to saying that a mistake at (A, A) involves a larger payoff loss than a mistake at (B, B). Hence, if mutation rates are modelled as endogenously determined mistake probabilities, then mistakes in state "A" are less likely than mistakes in state "B". In other words, the basin of attraction of state "A" is not only "larger" than that of state "B", it is also "deeper" - thus making state "A" even more difficult to upset. This is true without any regularity conditions imposed on the disutility function; it only relies on the monotonicity property (9).

This simple intuition is not available in asymmetric games. In the risk-dominant equilibrium, one of the two deviation losses may be quite small, thus inducing relatively large mistake probabilities in that player position, as is shown in a counter-example in section 5. However, under the regularity conditions introduced in section 3 above, all mistake probabilities will be of the same order of magnitude and below we show that this implies that the results from Young (1993, 1998) carry over to this case.

4.1. Young's model. Let an n-person game G, as defined above, be given and, for each player position $i = 1, ...n$, let C_i be a finite population of individuals. In the Young model, the game is played recurrently between individuals, one from each population, who are randomly drawn from these populations. The state h of the system in his model is a full description of the pure-strategy profiles played in the last m such rounds ("the recent m-history"). Hence, $h \in H = S^m$. Each individual drawn to play in position i of the game is assumed to make a statistically independent sample of k of these m profiles, and plays a best reply to the opponent population's empirical frequency of actions (pure strategies) in the sample.

Noise is added to this selection process. Basically, an individual in player position i plays a best response with probability $1 - \lambda_i\varepsilon$. In case of multiple best replies, all are played with positive probability. A mutation occurs with positive probability $\lambda_i\varepsilon$, with statistical independence across time, states and individuals. Hence, all mutation rates $\lambda_i\varepsilon$ are positive, and the ratios between mutation rates across states and player positions are constant as $\varepsilon \to 0$. Once a mutation occurs, $q_i(\cdot \mid h) \in int[\Delta_i]$ is the conditional error distribution over player i's pure-strategy set.

With these mistakes as part of the process, each state of the system is reachable with positive probability from every other state. Hence, the full process is an irreducible Markov chain on the finite state space H. Consequently, there exists a unique stationary distribution μ^ε for each $\varepsilon > 0$, and Young establishes the existence of the limit distribution $\mu^* = \lim_{\varepsilon \to 0} \mu^\varepsilon$, and studies its properties. He calls an equilibrium of the underlying game G *stochastically stable* if μ^* places positive probability weight on the state in which this equilibrium is played (in the last m periods).

One noteworthy result is that, for a 2×2 game with a unique risk-dominant

equilibrium, this is the unique stochastically stable equilibrium. More generally, Young (1998) establishes that, for a generic class of finite n-player games, the limit distribution places all probability mass in one of the game's minimal *curb* sets, more exactly on a minimal *curb* set where the stochastic potential (in Young's model) is minimized (Theorem 7.2 in Young (1998)).[5] In the case of a 2×2 coordination game, there are two minimal curb sets, each corresponding to one of the strict equilibria, and such an equilibrium is risk dominant if and only if its stochastic potential is lower than that of the other equilibrium.

4.2. The Bergin-Lipman critique. Young derives his results under the assumption that the ratio between any pair of mutation probabilities is kept constant as $\varepsilon \to 0$. Bergin and Lipman (1996) note that the results continue to hold even with state-dependent mutation probabilities if the ratio between any pair of mutation probabilities, across all population states and player positions, has a non-zero limit when $\varepsilon \to 0$. However, Bergin and Lipman (1996) also show that if the mutation probabilities in different states are allowed to go to zero at different rates, then any stationary distribution in the mutation-free process can be turned into the unique limiting distribution $\mu^* = \lim_{\varepsilon \to 0} \mu_\varepsilon$ of the process with mutations. They conclude: "In other words, any refinement effect from adding mutations is solely due to restrictions of how mutation rates vary across states." (Bergin and Lipman, p. 944).

What is missing in this modelling approach, and this is Bergin's and Lipman's main message, is a theory of why and how mutations occur. One reason why this may be the case, suggested by Bergin and Lipman, is that mutation rates might be lower in high-payoff states than in low-payoff states, which might be expected if mutations are due to individuals' experimentation (see Bergin and Lipman, pp. 944, 945 and 947). Another reason why mutation probabilities may differ across population states, also suggested by Bergin and Lipman (pp. 945 and 955), is that mutations leading to larger payoff losses might have lower probabilities than mutations leading to smaller payoff losses. Bergin and Lipman do not investigate the consequences of either of these two ideas. We now follow up on their latter suggestion and show that it leads to a confirmation of the results of Young.

4.3. Endogenous mutation rates in Young's model. Let $h \in H$ denote the state of the population process in Young (1993). The state determines a range of possible probabilistic beliefs $\omega \in \Delta$ for each of the individuals drawn to play the game. More exactly, ω is one of the finitely many empirical statistically independent frequency distributions, over the other player positions' pure strategy sets, that cor-

[5] A *curb* set is a product set that is closed under rational behavior, i.e., that contains its best replies, see Basu and Weibull (1991).

respond to a sample of size k from the other populations' past m plays. Since the state space H is finite, the set $\Omega = \times_i \Omega_i \subset \Delta$ of possible beliefs ω is finite. Now suppose that, instead of an exogenous mistake probability ε, we take as exogenous the "disutility weight" δ, a regular profile $\bar{v} = (v_1, ..., v_n)$ of disutility functions, and, for each state h in the finite state space H, an interior strategy profile $\eta(h) = q(\cdot \mid h)$, the conditional (and potentially state dependent) error distribution in Young's (1993) model. Instead of taking ε to zero, we let δ go to zero.

Since all mistake probabilities $\hat{\varepsilon}_i(\delta, \omega)$ are positive, for any $\delta > 0$, and the error distribution $\eta(h)$ is interior in all states h, the resulting stochastic process is ergodic, just as in Young's (1993) model, and thus has a unique stationary distribution μ^δ. We are interested in the limit as $\delta \to 0$, i.e. when the disutility of effort becomes insignificant in comparison with the game payoffs.

Proposition 4. *If the disutility function profile is regular, then Theorems 2 and 3 in Young (1993), and Theorems 3.1 and 7.2 in Young (1998) hold. In particular, in a* 2×2 *coordination game with a unique risk-dominant equilibrium, this is the unique stochastically stable equilibrium.*

Proof: This follows from proposition 3 above, combined with the proofs in Young (1993,1998). While Young establishes the result formally only for the case where all mistakes are a positive multiple of ε (at each state h player i makes a mistake with probability $\lambda_i \varepsilon$), it is easily verified that his arguments remain valid in the more general case considered here. To see this, note that if $\omega, \omega' \in \Omega$ and $i, j \in I$ are such that $\beta_i(\omega) \neq \Delta_i$ and $\beta_j(\omega') \neq \Delta_j$, then $l_i(\omega), l_j(\omega') > 0$ (since $\eta(h)$ by hypothesis is interior for all $h \in H$). Proposition 3 implies that there for such $\omega, \omega' \in \Omega$ and $i, j \in I$ exist positive real numbers α, β and $\bar{\delta}$, such that $\alpha < \hat{\varepsilon}_i(\delta, \omega)/\hat{\varepsilon}_j(\delta, \omega') < \beta$ for all $\delta \in \left(0, \bar{\delta}\right)$. The sets Ω and I being finite, we can find α, β and $\bar{\delta}$ that work for all $\omega, \omega' \in \Omega$ and $i, j \in I$ such that $\beta_i(\omega) \neq \Delta_i$ and $\beta_j(\omega') \neq \Delta_j$. Hence, all mistake probabilities go to zero at the same rate, as $\delta \to 0$, in all states h which permit no sample that renders some individual indifferent. In the excluded states, however, there is a positive probability that some mistake probability will optimally be chosen to be 1. That individual will in effect play his conditional error distribution, which assigns positive probability to all his pure strategies. But this is just as in Young's model, where all best replies are played with positive probability, which, in this case, means that all pure strategies are played with positive probability. The proofs of Theorems 2 and 3 in Young (1993) and Theorems 3.1 and 7.2 in Young (1998) thus apply. **End of proof**.

5. Non-regular disutility function profiles

5.1. Non-similarity. As indicated in section 3, it is easy to see that without a similarity assumption on the disutility functions, the results may depend on differences between disutility functions. For example, consider the following "battle of the sexes" type game:

	L	R
T	$2,1$	$0,0$
B	$0,0$	$1,2$

(21)

First, assume that only individuals in player population 2 make mistakes, and let their mistake probability be fixed and independent of the state. Then two thirds or more of the sample taken from population 2 has to be mistakes, in order to upset (T, L), the equilibrium preferred by the error-free population 1. Similarly, it takes at least one third of mistakes to upset the other strict equilibrium. Hence, only the first equilibrium is stochastically stable. Second, note that this conclusion remains valid with a nice disutility function for each population, such that $\hat{\varepsilon}_1$ is of smaller order than $\hat{\varepsilon}_2$.

5.2. Non-niceness. We now briefly turn to the more interesting case of disutility functions that are similar, in fact identical, but non-nice. In this case, Young's results need not hold. What might happen is that no stochastically stable equilibrium exists. The intuition is simple. Consider the following asymmetric payoff bi-matrix for a 2×2 coordination game:

	A	B
A	$5,1$	$0,0$
B	$0,0$	$2,2$

(22)

With p_i denoting the probability that player i assigns to his pure strategy A in the unique mixed equilibrium of this game, we have $p_1 = \frac{2}{3}$ and $p_2 = \frac{2}{7}$. Hence $p_1+p_2 < 1$, so (A, A) is the risk-dominant equilibrium. With mistake probabilities that are of the same order of magnitude, we will, hence, select the equilibrium (A, A) in the limit. However, a mistake by player 2 at this equilibrium incurs the smallest payoff loss in the two equilibria - this player loses only 1 payoff unit, while all other equilibrium payoff losses are at least 2 units. Consequently, the largest mistake probability occurs in state "(A, A)". Now imagine a non-nice disutility function which is such that a mistake with a loss of only 1 is much more likely than a mistake resulting in a payoff loss of 2 or more. Then the limit outcome will be fully determined by the mistakes of player 2 at the state "(A, A)". Obviously, if the only mistakes that count occur at this equilibrium, then the limit outcome will be "(B, B)". Hence, for a non-nice disutility function, both "(B, B)" and "(A, A)" can be elements of the limit set.

In the remainder of this section, we formalize this observation. We first construct a non-nice disutility function.[6] For this purpose, consider the sequence $\langle (x_n, y_n) \rangle_{n=1}^{\infty}$ in $\mathbb{R}^2_+$,defined by $x_1 = 1$, $y_1 = \frac{1}{3}$, and for all integers $n > 1$:

$$\begin{cases} (x_n, y_n) = \left(2x_{n-1}, y_{n-1}^a\right) & \text{if } n \text{ is even} \\ (x_n, y_n) = (3x_{n-1} + 2/y_{n-1}, y_{n-1}/2) & \text{if } n \text{ is odd} \end{cases} \tag{23}$$

where $a > 1$. Clearly $\langle x_n \rangle$ is an increasing sequence, going to plus infinity, and $\langle y_n \rangle$ is a decreasing sequence, going to zero. Hence, there exist differentiable (to any order) and strictly decreasing functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(0) = \frac{1}{2}$, and $f(x_n) = y_n$ for all positive integers n. Any such function f decreases sufficiently on intervals (x_n, x_{n+1}) with n odd in order for $f(2x_n)/f(x_n) \to 0$ to hold for n odd, as $n \to \infty$. On the other intervals, however, f decreases sufficiently little in order for its total integral to be infinite (the integral of f over any interval (x_n, x_{n+1}) with n even is at least 1). The equation $v' = -f^{-1}$ defines a disutility function v that is non-nice, with f as the inverse to $-v'$.

For the bi-matrix game in (22), and with f as constructed above, we construct a sequence of δ's going to zero, such that the associated sequence of stationary distributions in the limit places all probability mass on state "(B, B)". To that end, for every odd integer n, let $\delta_n = 1/x_n$. By definition of f, and condition (9)[7],

$$\hat{\varepsilon}_2(\delta_n, A) = y_n, \;\; \hat{\varepsilon}_1(\delta_n, B) = \hat{\varepsilon}_2(\delta_n, B) = y_n^a \text{ , and } \;\; \hat{\varepsilon}_1(\delta_n, A) \square \; y_n^a \text{ ,} \tag{24}$$

By following arguments as in Young (1993, Sect. 7) we can now show that the equilibrium (A, A) is more easily upset than the equilibrium (B, B) when $\delta = \delta_n$ with n odd. Let m be the common memory size of the players, let k be the (fixed) sample size and, let "(A, A)" (resp. "(B, B)") be the stationary state in which (A, A) (resp. (B, B)) is played constantly. As in Young (1993), the path of least resistance from "(A, A)" to "(B, B)" either consists of player 1 making mistakes until player 2 has B as a best response to the sample of mistakes, or has player 2 making mistakes until player 1's best response is B. Let k' be the number of mistakes required by player 1 and let k'' be the number required by player 2. Then

$$k' = [k/3] \text{ and } k'' = [5k/7] \tag{25}$$

where $[x]$ denotes the smallest integer equal to or larger than x. The probability

[6]The following construction of a function f, that is the negative of the inverse of a non-nice disutility function, is an adaptation of a function suggested to us by Henk Norde, Tilburg University.

[7]To see this, note that if $\delta_n = 1/x_n$ then $\widehat{\varepsilon}_i(\delta_n, \omega) = f\left[l_i(\omega)x_n\right]$. Hence, $\widehat{\varepsilon}_2(\delta_n, A) = f\left(x_n\right) = y_n$, and $\widehat{\varepsilon}_1(\delta_n, B) = \widehat{\varepsilon}_2(\delta_n, B) = f\left(2x_n\right) = f\left(x_{n+1}\right) = y_n^a$.

to move from "(A, A)" to "(B, B)" is thus given by

$$\max \{\widehat{\varepsilon}_1(\delta, A)^{k'}, \widehat{\varepsilon}_2(\delta, A)^{k''}\} \tag{26}$$

If $\delta = \delta_n$ with n odd, and $a > 3$ then (24) implies that the easiest way to upset "(A, A)" is by player 2 making mistakes. Similarly, the easiest way to leave "(B, B)" is by player 2 making mistakes and the "resistance" of "(B, B)" is measured by

$$\widehat{\varepsilon}_2(\delta, B)^{k'''} \text{ with } k''' = [2k/7] \tag{27}$$

Now, with $\delta = \delta_n$ and n odd, we have

$$\frac{\widehat{\varepsilon}_2(\delta_n, B)^{k'''}}{\widehat{\varepsilon}_2(\delta_n, A)^{k''}} = \frac{y_n^{ak'''}}{y_n^{k''}} = y_n^{(ak''' - k'')}, \tag{28}$$

so that, if $a > 3$, the ratio goes to zero as $n \to \infty$. Hence, for large n (and associated δ_n) we have that "(A, A)" is much easier to upset than "(B, B)".

We conclude that for this sequence of δ's, converging to zero, the associated subsequence of stationary distributions in the limit indeed places all probability mass on state "(B, B)". At the same time, proposition 2 implies that there exists another sequence of δ's, also going to zero, such that the associated subsequence of stationary distributions places all probability mass on state "(A, A)" in the limit. Hence, the overall limit of stationary distributions does not exist for this disutility function.

Another way of formulating this observation is that, for every disutility function, the risk-dominant equilibrium belongs to the limit *set* of supports of stationary distributions, and that this limit set is a singleton if all disutility functions are nice.[8]

6. Conclusion

Bergin and Lipman (1996) showed that if mutation rates are state dependent, then the long-run equilibrium depends on exactly how these rates do vary with the state. They conclude that the causes of mutations need to be modeled in order to derive justifiable restrictions on how mutation rates depend on the state. In particular, they suggest that one might investigate the consequences of letting the probability of mistakes be related to the payoff losses resulting from these mistakes. This is exactly what we have done in this paper. We have developed a model in which mistakes are endogenously determined, and shown that this model vindicates the original results obtained by Kandori, Mailath and Rob (1993) and Young (1993).

[8]By the *limit set* of supports of stationary distributions, for a given control-cost function, we here mean the union of the supports of probability distributions μ, each of which is the limit to some subsequence of stationary distributions (i.e., associated with some sequence of positive δ's converging to zero).

The model analyzed in this paper, although allowing for mistakes, is based on strong rationality assumptions. Mistakes arise because players choose to make them, since it involves too much disutility to avoid them completely. Our individuals are hence unboundedly rational when it comes to decision making. Their lack of rationality is only procedural: At no disutility can they choose their own mistake probabilities in every population state. This is a very strong rationality assumption. However, we believe that our conclusions are robust in this respect, at least for generic 2×2 coordination games. For the effect of introducing control disutility was seen to only "deepen" the "basin of attraction" of the risk-dominant equilibrium, and hence speeding up the convergence to it. The limit result should therefore also be valid in intermediate cases of rationality.

As a by-product, we obtained a refinement of the Nash equilibrium concept - robustness to endogenous trembles. We showed that such robustness is logically independent from Selten's (1975) perfection concept and from Myerson's (1978) properness concept. Further study and elaboration of this new refinement might be worthwhile, but fall outside the scope of the present study.

7. Appendix

7.1. Proof of Lemma 1. Without loss of generality, assume $l_i(\omega) > l_i(\omega')$. (Otherwise the result is immediate by (9)). Suppose that the claimed inequality does not hold. Then $\lim_{\delta\to 0} \hat{\varepsilon}_i(\delta,\omega)/\hat{\varepsilon}_i(\delta,\omega') = 0$, or, equivalently (using (8)),

$$\lim_{\delta\to\infty} f_i\left[l_i(\omega)/\delta\right]/f_i\left[l_i(\omega')/\delta\right] = 0\ . \tag{29}$$

Writing $\lambda = l_i(\omega)/l_i(\omega')$, this in turn is equivalent to $\lim_{y\to\infty} f_i(\lambda y)/f_i(y) = 0$, since f_i is everywhere positive. Thus $f_i(\lambda y)/f_i(y) \leq \frac{1}{2}\lambda^{-1}$ for all y sufficiently large, say $y > y_0$. Hence,

$$\int_{y_0}^{\infty} f_i(y)dy \leq (\lambda-1)y_0 f_i(y_0)\sum_{t=0}^{\infty} 2^{-t} < \infty\ , \tag{30}$$

and thus $\int_0^\infty f_i(y)dy < \infty$ since f_i is decreasing with $f_i(0) = 1$. But we also have

$$\int_0^{\infty} f_i(y)dy = -\int_0^1 v_i'(p)dp = \lim_{\varepsilon\to 0} v_i(\varepsilon) - v_i(1)\ , \tag{31}$$

which contradicts the assumption that $\lim_{\varepsilon\to 0} v_i(\varepsilon) = +\infty$.

7.2. Proof of inequality (18). Assume that the condition in the definition holds for $\lambda_o > 1$. Then it holds for all $\lambda \in (0,\lambda_o)$, by monotonicity of f_i. And if $\lambda > \lambda_o$, then $\lambda_o^n > \lambda$ for some positive integer n. Again by monotonicity of f_i:

$$\begin{aligned}
\liminf_{y\to\infty} f_i(\lambda y)/f_i(y) &\geq \liminf_{y\to\infty} f_i(\lambda_o^n y)/f_i(y) && (32)\\
&= \liminf_{y\to\infty} \frac{f_i(\lambda_o^n y)}{f_i(\lambda_o^{n-1} y)} \cdot \frac{f_i(\lambda_o^{n-1} y)}{f_i(\lambda_o^{n-2} y)} \cdot \ldots \cdot \frac{f_i(\lambda_o y)}{f_i(y)} && (33)\\
&\geq \left[\liminf_{y\to\infty} f_i(\lambda_o y)/f_i(y) \right]^n > 0 && (34)
\end{aligned}$$

□

7.3. Proof of Lemma 2. Under the hypothesis of the lemma, the function $-v_i'$ is convex. Thus, for any $y > 0$, $\lambda > 1$, and $\varepsilon \in (0,1)$ such that $y = -v_i'(\varepsilon)$:

$$\frac{\lambda y - y}{f_i(y) - f_i(\lambda y)} \geq v_i''(\varepsilon) \ . \tag{35}$$

(To see this, draw a diagram with ε on the horizontal axis, and $-v_i'(\varepsilon)$ on the vertical axis.) Equivalently,

$$f_i(\lambda y) \geq f_i(y) - (\lambda - 1)\frac{y}{v_i''(\varepsilon)} \ , \tag{36}$$

or

$$\frac{f_i(\lambda y)}{f_i(y)} \geq 1 + (\lambda - 1)\frac{v_i'(\varepsilon)}{\varepsilon v_i''(\varepsilon)} \ .$$

Consequently,

$$\liminf_{y\to\infty} \frac{f_i(\lambda y)}{f_i(y)} \geq 1 + (\lambda - 1)\liminf_{\varepsilon\to 0} \frac{v_i'(\varepsilon)}{\varepsilon v_i''(\varepsilon)} \ . \tag{37}$$

But under the last hypothesis of the lemma, there exits a scalar $\lambda > 1$ such that

$$\limsup_{\varepsilon\to 0} \frac{-v_i'(\varepsilon)}{\varepsilon v_i''(\varepsilon)} < \frac{1}{\lambda - 1} \ . \tag{38}$$

Hence, for this λ,

$$1 + (\lambda - 1)\liminf_{\varepsilon\to 0} \frac{v_i'(\varepsilon)}{\varepsilon v_i''(\varepsilon)} > 0 \ . \tag{39}$$

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