Comparison of Mean-Variance and Exact Utility Maximization in Stock Portfolio Selection

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Abstract

In the early 50's, Markowitz introduced the modern portfolio selection theory which, to this very day, constitutes the basis of many investment decisions. Given different correlated assets, how does an investor create a portfolio maximizing the expected utility? Markowitz's contribution was to show that an investor might do very well, relying only on the means and variances/covariances of the assets, which simplifies the portfolio selection tremendously. The validity of the mean-variance approximation to exact utility maximization has been verified, but only in the unrealistic case of choosing among 10-20 securities. This paper examines how well the quadratic approximation works in a larger allocation problem, where investors characterized by different utility functions can choose among nearly 120 securities. The effects of more aggressive investment strategies are also investigated, allowing for limited short selling and the inclusion of synthetic options in the security set.

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1 Introduction

The mean-variance approximation to expected utility maximization has been subject to much controversy, ever since introduced by Markowitz (1952) in the 50's. As is well known, the mean-variance approximation (a second-order Taylor expansion of the utility function around the mean portfolio return) is exact if returns are jointly normally distributed and the utility function is a negative exponential, or if returns belong to an arbitrary probabilistic distribution and the utility function is quadratic. However, there is neither theoretical nor empirical support for these assumptions. For arbitrary utility functions and normality in returns, the mean-variance model is no longer precise, but if the utility function can be approximated by a higher-order Taylor expansion, the portfolio maximizing expected utility will still belong to the efficient frontier, since the expected utility is then approximated by a function of the first two moments only. Naturally, a Taylor approximation is supposed to be valid only in the neighborhood of the expansion point. For other cases, all bets are off.

Levy and Markowitz (1979) showed the accuracy of the quadratic approximation over a wide range of portfolio returns numerically. They also "constructed" an empirical distribution of yearly and monthly returns of 149 mutual funds, and found that picking the portfolio (fund) on the basis of the mean-variance criteria meant that there was also a very high probability of maximizing the expected utility for a variety of utility functions. This put criticism on hold, but it still remained an open question how well relying on only mean and variance would work in a true asset allocation problem, that is, when choosing the portfolio weights maximizing the expected utility among correlated securities.

The issue was treated by Pulley (1981), Kallberg and Ziemba (1983), and Kroll et al. (1984), who all showed the excellency of the mean-variance approximation. However, they only examined cases with a very small number of securities (10-20) because of computational considerations. One of the purposes is to investigate whether these findings result from the small number of assets involved, or whether the mean-variance approximation still holds in a larger portfolio selection problem, namely maximizing expected utility over the historical distribution of monthly returns of 119 Swedish stocks from February 1984 to December 1990.

Furthermore, the articles above are concerned with the restriction of non-negative portfolio weights, although Kroll et al. explored the effects of leverage by allowing limited borrowing at a constant yearly rate of 10%. Therefore, I will examine the effects of limited short selling of risky assets. This is a greater challenge for the mean-variance approximation as it is more likely to fail when opportunity sets show increased variability. Since the mean-variance approximation only concerns an optimal trade-off of the first two first moments of the securities, I investigate how the inclusion of options, that is, securities with high levels of skewness and kurtosis, influences its performance. Monthly returns are considered, although a mean-variance approximation is expected to be less accurate for longer holding periods. The reason for this is that monthly monitoring of stock portfolios is believed to be more reasonable from a practical point of view.

Recently, Adcock and Shutes (1999) showed how skewness can be incorporated in portfolio selection using the multivariate skew normal distribution of Azzalini and Dalla Valle (1996), and assuming a negative exponential utility function. The possibility of an extension to a skewed multivariate t-distribution is mentioned. The approach here is different, however. I do not make any distributional assumptions, but instead examine the welfare losses of not considering higher moments of the asset returns than the variance in the expected utility maximizations. Higher order moments will certainly influence the utility maximizations, but they may or may not be of any economic significance.

In section 6.2, the maximization problem is discussed in more detail. Section 6.3 describes the data used, and section 6.4 presents the utility functions studied. Section 6.5 contains the results of the empirical comparisons, and a summary and concluding remarks are found in section 6.6.

2 The Maximization Problem

Assume that investors with utility function U act myopically. Then, they seek to maximize the expected utility of each end-of-period wealth subject to a budget constraint and, in the standard case, with no short positions:

$$\begin{aligned} \max \quad & E[U(1+\sum_i x_i r_i)] \\ \text{subject to} \quad & x_i \geq 0,\ i = 1, ..., n \\ & \sum_i x_i = 1, \end{aligned} \tag{1}$$

where n is the number of risky assets, r_i is the net return of asset i, and x_i is the proportion of the ith asset. Initial wealth is assumed to equal one. Following the notation from Kroll et al. (1984), the expected utility from this maximization is denoted EU.

An approximation to the above would be to trace out the efficient frontier by solving the following quadratic programming problem:

$$\begin{aligned} \min \quad & x'\Omega x \\ \text{subject to} \quad & x_i \geq 0, \\ & \sum_i x_i = 1 \end{aligned} \tag{2}$$

$$\sum_i \mu_i x_i = e$$
$$\text{for all} \qquad e \in [e_{\min}, e_{\max}]$$

where Ω is the covariance matrix of the security returns, μ_i is the expected return of security i, $e_{\max}$ is the maximum feasible return of any portfolio, and $e_{\min}$ is the return of the minimum variance portfolio. For any utility function, it is now easy to calculate $E[U(1+\sum_i x_i r_i)]$. The maximum of the expected utility of the mean-variance efficient portfolios is denoted E^*U, and the portfolio weights x_i^*, $i = 1, ..., n$.

How is the accuracy of E^*U quantified compared to EU? Since an affine transformation of the utility function studied gives the same solution to the optimization problems, neither the difference nor the ratio of E^*U and EU could be used as a measure. Kroll et al. avoid this problem by using the following index:

$$I = \frac{E^*U - E_N U}{EU - E_N U} \tag{3}$$

where $E_N U$ is the expected utility of a "naive" portfolio. Virtually any inefficient portfolio could be used as a reference point. Kroll et al. choose a portfolio consisting of $1/n$th of each security. The index $I \in [0, 1]$, where a value close to zero indicates that the approximation performs badly, while if I is close to one, the approximation works well.

Although the I-measure is invariant to linear transformations of the utility function, it is not invariant to the choice of reference point, as commented by Pulley (1985), and Reid and Tew (1986) respectively. If $E_N U$ is high compared to E^*U and EU, the index I will be lower than if $E_N U$ is low, which is, of course, an inappropriate property of an efficiency criterion. To circumvent this problem, Reid, Tew, and Pulley suggested the use of another index, that is, the one also used in Kallberg and Ziemba (1983):

$$C = \frac{CE(E^*U)}{CE(EU)}. \tag{4}$$

where $CE(\cdot)$ is the certainty (or cash) equivalent of a risky portfolio that gives the investor the same utility as holding such a portfolio, that is $CE(\cdot) = U^{-1}(EU(\cdot))$.

Also the measure $C \in [0, 1]$, and it is indeed invariant to affine transformations of the utility function. In our empirical investigations we report both.

3 The Data

In this analysis, I use monthly percentage returns of 119 Swedish stocks, ranging from February 1984 to December 1990, yielding a total of 83 monthly observations for each stock. Table 1 displays some statistics of the data sample.

Table 1: Sample statistics for return data. The entry marked with an asterisk corresponds to the mean of the absolute values of the skewnesses.

Return Data	Minimum	Maximum	Mean
Means (%)	-0.966	3.227	1.309
Variances (%)	0.066	2.306	0.935
Skewnesses	-1.620	1.991	0.487*
Kurtoses	2.631	11.89	4.920

Clarifying whether normally distributed returns are the key to a potential success for the mean-variance approximation is of great interest. A frequently used test is the sum of the squares of the standardized sample skewness and kurtosis[1], but it is unsuitable except in very large samples. Instead, we use the omnibus test of Doornik and Hansen (1994) which can be used in univariate and multivariate normality tests, and transforms the data to a χ^2-distributed test statistic.

First, we test for univariate normality. In 50 cases (out of 119), normality cannot be rejected at the 5% significance level. Further, we test if these 50 stocks are jointly normally distributed. The null hypothesis of multivariate normality is strongly rejected with a p-value of $1.04 \cdot 10^{-3}$. We cannot exclude the fact that subsamples are jointly normal, but it is unlikely that these securities would be the only ones picked by our hypothetical investors[2].

4 The Utility Functions

I use five utility functions for the empirical analyses, each with two different parameter sets, giving a total of ten optimization problems. The utility functions, which all appear in Pratt (1964), are listed in Table 2 together with some properties regarding the Arrow-Pratt absolute and relative risk measures:

$$R_A(w) = -\frac{U''(w)}{U'(w)} \qquad \text{and} \qquad R_R(w) = -w\frac{U''(w)}{U'(w)} \tag{5}$$

where $w = 1 + \sum_i x_i r_i$ denotes wealth.

In the right-hand column, 'Unclear' means that the relative risk measure is not necessarily strictly increasing or decreasing and that its properties must be determined on a case-by-case basis. Not all utility functions are defined on $w > 0$. In all cases, the functions are defined

[1] Skewness is defined as the third central moment divided by the cube of the standard deviation. Kurtosis is defined as the fourth central moment divided by the fourth power of the standard deviation.

[2] The securities of the solution portfolios, to appear later, were tested for joint normality. The null hypotheses were strongly rejected in all cases, with p-values between 0 and $4.8{\cdot}10^{-13}$.

Table 2: Utility functions and parameter sets. The properties of $R_A(w)$ and $R_R(w)$ are also presented when w is within its respective definition range. The notion 'Unclear' means that the properties of the relative risk measure must be determined on a case-by-case basis.

$U(w)$	Parameter values	$R_A(w)$	$R_R(w)$
$\ln(\alpha + w)$	$\alpha = 0$	Decr.	Incr. if $\alpha > 0$
	$\alpha = -0.5$		Const. if $\alpha = 0$
			Decr. if $\alpha < 0$
$-e^{-\alpha w}, \alpha > 0$	$\alpha = 2$	Const.	Incr.
	$\alpha = 5$		
$(\alpha + w)^\beta, 0 < \beta < 1$	$\alpha = -0.8, \beta = 0.1$	Decr.	Incr. if $\alpha > 0$
	$\alpha = -0.5, \beta = 0.2$		Const. if $\alpha = 0$
			Decr. if $\alpha < 0$
$\ln(\beta + \ln(\alpha + w))$	$\alpha = 0.5, \beta = 0.7$	Decr.	Decr. if $\alpha \leq 0$
	$\alpha = -0.3, \beta = 1.5$		Unclear if $\alpha > 0$
$-\alpha e^{-\beta w} - \gamma e^{-\delta w}$,	$\alpha = 0.5, \beta = 3, \gamma = 1, \delta = 5$	Decr.	Unclear
$\alpha > 0, \beta > 0, \gamma > 0, \delta > 0$	$\alpha = 2.5, \beta = 3, \gamma = 3, \delta = 1.2$		

on $w > \max(0, -\alpha)$, except $U(w) = \ln(\beta + \ln(\alpha + w))$, the definition range of which is $w > \max(0, -\alpha + e^{-\beta})$. Furthermore, the utility functions satisfy $U^{(2k-1)} > 0$, and $U^{(2k)} < 0, k = 1, 2, ...,$ on the respective definition range.

5 Empirical Results

In the first section, I present the results from the case of no short-selling. The second section treats a more aggressive holding strategy, where the portfolio weights are allowed to vary between -0.5 and 1.5. Finally, in the last section, the effects of including synthetic options in the security set are examined.

5.1 Portfolio Selection with No Short Sales

First, we trace out the efficient frontier by solving the optimization problem (2) for 600 points between $e_{\min} = 0.73\%$ and $e_{\max} = 3.22\%$ giving a Δe of 0.0042%. For each utility function and parameter set, we choose the frontier portfolio maximizing the expected utility. With our historical return data, it takes the form of $E^*U = \frac{1}{T}\sum_t U(1 + \sum_i x_i^* r_{it})$ with portfolio weights x_i^*. Then, we solve the ten maximization problems described by (1). Henceforth, these solutions are denoted as *optimal*, as opposed to the former, which are referred to as *approximative*. Table 3 reports the means and variances of the solution portfolios together with the maximum absolute distance of the two weight vectors. The means and variances of

Table 3: Means and variances of the approximative and optimal portfolio returns. The maximum of the absolute distances between the corresponding weight vectors, and the properties of R_A and R_R implied by the parameter choices are also shown.

	Approximative		Optimal				
$U(w)$	$E(\%)$	$V(\%)$	$E(\%)$	$V(\%)$	Distance	R_A	R_R
1. $\ln(w)$	3.046	0.657	3.046	0.655	0.011	Decr.	Const.
2. $\ln(w-0.5)$	2.929	0.452	2.931	0.456	0.041	Decr.	Decr.
3. $-e^{-2w}$	2.917	0.437	2.918	0.440	0.010	Decr.	Incr.
4. $-e^{-5w}$	2.674	0.288	2.685	0.295	0.026	Const.	Incr.
5. $(w-0.8)^{0.1}$	2.753	0.326	2.787	0.351	0.058	Decr.	Decr.
6. $(w-0.5)^{0.2}$	2.967	0.504	2.969	0.509	0.017	Decr.	Decr.
7. $\ln(0.7+\ln(w+0.5))$	3.009	0.576	3.002	0.565	0.022	Decr.	Incr. if $w \leq 2.57$
							Decr. if $w > 2.57$
8. $\ln(1.5+\ln(w-0.3))$	2.896	0.416	2.896	0.418	0.029	Decr.	Decr.
9. $-0.5e^{-3w}-e^{-5w}$	2.845	0.377	2.851	0.382	0.015	Decr.	Incr.
10. $-2.5e^{-3w}-3e^{-1.2w}$	2.946	0.474	2.947	0.476	0.015	Decr.	Incr.

the approximative portfolio returns are almost identical to those of the optimal ones. The distances between the corresponding portfolio weights are also very small, indicating that the portfolios include almost the same securities. It should be noted that the approximative and optimal portfolios only involve a small number of assets. The portfolios with the largest number of securities (those with the lowest mean return) contain only nine securities (with portfolio weights larger than 0.1%) out of 119, while those with the smallest number (those with the highest mean) contain only four securities. This illustrates the effect of diversification; only a few stocks is needed to yield a favorable risk-return relationship. In Table 3, it is also worth noticing the properties of R_A and R_R implied by the parameter choices.

In the left-hand part of Figure 1, I plot the efficient frontier and the points corresponding to the optimal portfolios. Because of the closeness in mean-variance space, the approximative portfolios are only displayed in the magnified right-hand area. No points lie above the frontier as must be the case, since it follows from (2) that the frontier portfolios have the lowest variances, given their mean returns. Optimal portfolios far from the frontier would indicate other moments besides the mean and variance to have a great impact on maximizing the expected utility, but as can be seen in Figure 1, there is little support for this argument. The location of the naive portfolio, with $1/n$th invested in each security, is also plotted and is clearly inefficient.

Table 4 shows the evaluation indexes, C and I, for the various utility functions. For the sake of completeness, we also present EU, E^*U and E_NU. It can be noticed that C and I are virtually indistinguishable from 1, although the ranking of the ten comparisons of EU and E^*U do not

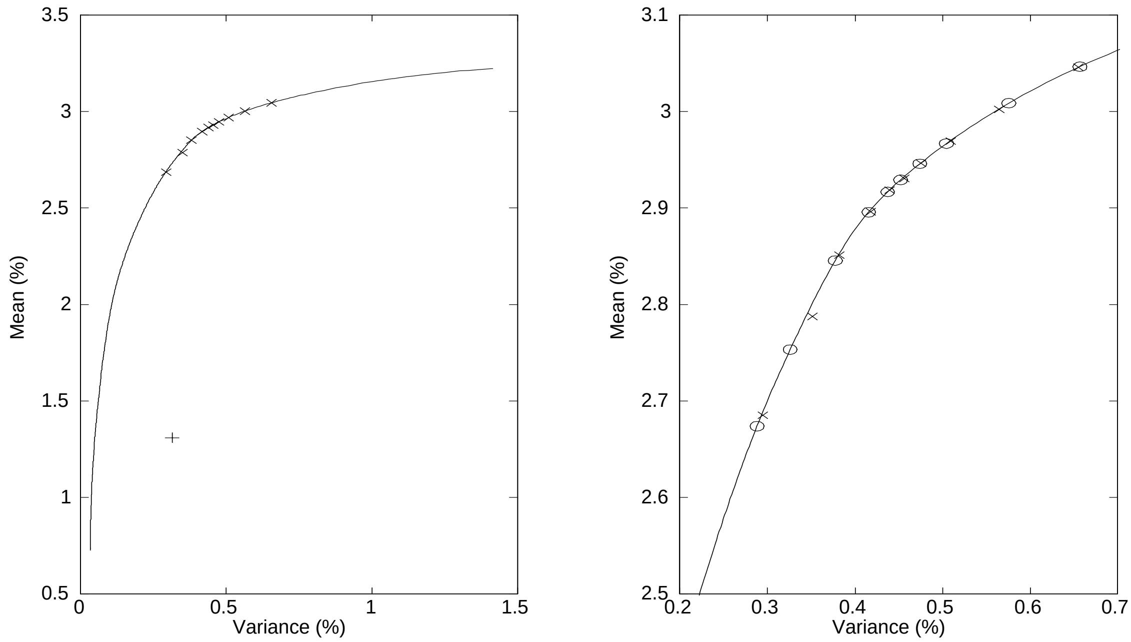

Figure 1: The efficient frontier and the location of the naive (+) portfolio, as well as the optimal (x) and approximative (o) portfolios. From the right to the left, the x's and the o's correspond to utility functions 1, 7, 6, 10, 2, 3, 8, 9, 5, and 4.

Table 4: Performance indexes, C and I, in the case of no short selling. The expected utilities of the optimal, the approximative, and the naive portfolios are denoted EU, E^*U, and E_NU, respectively.

$U(w)$	EU	E^*U	E_NU	C	I
1. $\ln(w)$	0.027092	0.027091	0.011443	0.999949	0.999912
2. $\ln(w-0.5)$	-0.643952	-0.643984	-0.673708	0.999345	0.998943
3. $-e^{-2w}$	-0.128745	-0.128746	-0.132683	0.999996	0.999736
4. $-e^{-5w}$	-0.006103	-0.006104	-0.006580	0.999967	0.997876
5. $(w-0.8)^{0.1}$	0.859917	0.859859	0.851938	0.992906	0.992706
6. $(w-0.5)^{0.2}$	0.879439	0.879436	0.874175	0.999569	0.999287
7. $\ln(0.7+\ln(w+0.5))$	0.116111	0.116109	0.106897	0.999812	0.999681
8. $\ln(1.5+\ln(w-0.3))$	0.162891	0.162863	0.144603	0.999006	0.998474
9. $-0.5e^{-3w}-e^{-5w}$	-0.029345	-0.029347	-0.030867	0.999977	0.998465
10. $-2.5e^{-3w}-3e^{-1.2w}$	-0.991335	-0.991342	-1.012956	0.999995	0.999665

Table 5: The means, variances, skewnesses, and kurtoses of the optimal and approximative portfolios in the case of limited short selling, together with the maximum of the absolute distances between the corresponding weight vectors.

	Approximative				Optimal				
$U(w)$	$E(\%)$	$V(\%)$	S	K	$E(\%)$	$V(\%)$	S	K	Distance
1. $\ln(w)$	43.77	17.98	0.27	3.74	45.15	22.17	0.57	3.38	0.86
2. $\ln(w-0.5)$	38.23	9.10	0.25	3.52	41.43	14.71	0.80	3.71	0.80
3. $-e^{-2w}$	36.62	7.29	0.27	3.43	37.96	9.33	0.98	4.94	0.67
4. $-e^{-5w}$	29.72	2.34	0.32	3.32	30.65	3.15	1.33	5.24	0.53
5. $(w-0.8)^{0.1}$	32.99	4.20	0.33	3.36	36.90	8.59	0.94	4.07	0.84
6. $(w-0.5)^{0.2}$	39.66	10.96	0.26	3.60	42.94	17.47	0.69	3.36	1.04
7. $\ln(0.7+\ln(w+0.5))$	42.64	15.70	0.27	3.67	42.93	16.65	0.51	3.55	0.35
8. $\ln(1.5+\ln(w-0.3))$	36.03	6.69	0.28	3.39	38.82	10.58	0.85	3.88	0.67
9. $-0.5e^{-3w}-e^{-5w}$	32.34	3.77	0.34	3.36	33.14	4.59	1.06	4.50	0.34
10. $-2.5e^{-3w}-3e^{-1.2w}$	39.54	10.80	0.26	3.60	40.61	12.95	0.86	4.53	0.63

coincide. The decrease in an investor's cash equivalent from choosing an approximative portfolio instead of the optimal one is, with the data and utility functions used here, totally negligible. In other words, the level of certain income making the investor indifferent to investing in a risky portfolio is the same notwithstanding if this is the optimal or the approximative portfolio. The computationally easier I-measure, at least when U is not invertible, as is the case for $U(w) = -\alpha e^{-\beta w} - \gamma e^{-\delta w}$, also underscores the adequacy of the mean-variance approximation. It remains to be seen if this result still prevails when allowing for a more aggressive holding strategy.

5.2 Portfolio Selection with Limited Short Sales

We now repeat the analysis for the case of limited short selling, that is, the weights in optimization problems (1) and (2) are restricted to $[-0.5, 1.5]$. It must be emphasized that this is a very aggressive investment strategy, which allows for considerably larger fluctuations in the portfolio returns than previously. Once more, we derive the efficient frontier by solving (2) for 650 points from $e_{\text{min}} = 12.87\%$ to $e_{\text{max}} = 51.27\%$ by steps of 0.060%, and choose the ten portfolio solutions that maximizing the expected utilities. The maximum monthly portfolio return has increased from 3% to 51%, and as such, it constitutes a greater challenge to the mean-variance approximation. The global maximization problems described by (1) are also solved. In contrast to the former case, all assets are now included with absolute portfolio weights considerable larger than 0.1%.

Comparing Table 5 to Table 3, it appears that the approximate portfolios are far from as

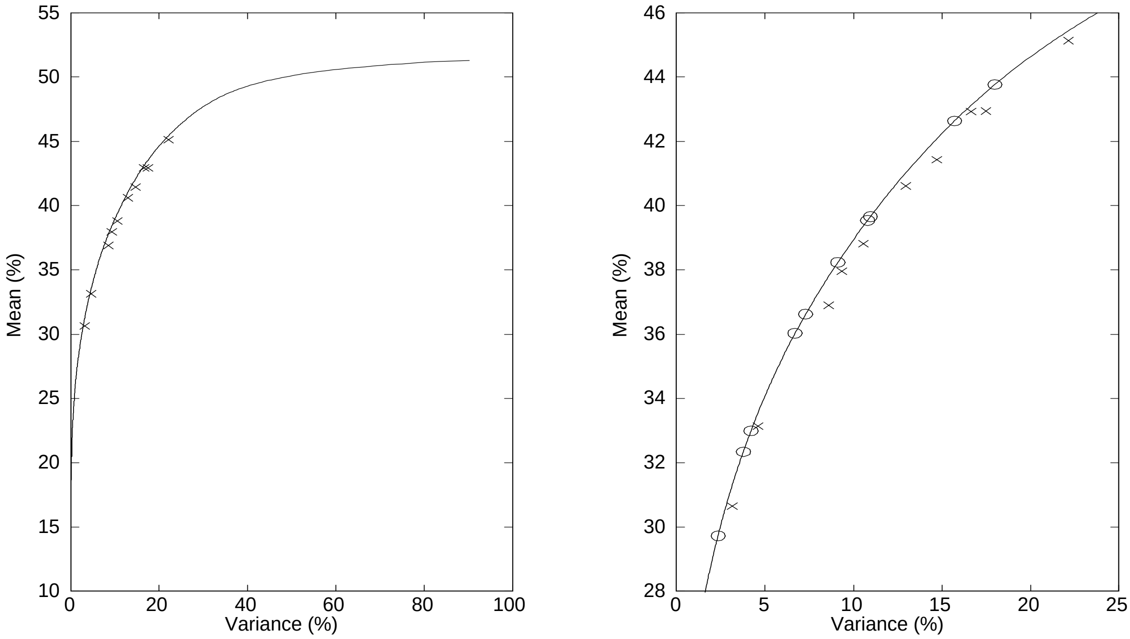

Figure 2: The efficient frontier and the location of the optimal (x) and approximative (o) portfolios in the case of limited short selling. From right to left, the x's correspond to utility functions 1, 6, 7, 2, 10, 8, 3, 5, 9, and 4, and the o's correspond to utility functions 1, 7, 6, 10, 2, 3, 8, 5, 9, and 4.

close to the optimal ones as previously. For example, the variance of the optimal portfolio for utility function 6 is 60% higher than the approximative, with a maximum (absolute) distance of the portfolio weights of more than 100%. This can also be seen in Figure 2, where the locations of the approximative and optimal portfolios are shown. The reason why an investor with, say, utility function 6 chooses an approximative portfolio (the third 'o' from the right) so far from the optimal one (the second 'x' from the right) is partly due to the fact that the definition range for $U(w)$ is restricted to $w > 0.5$. Mean-variance efficient portfolios, maximizing $E^*U = \frac{1}{T}\sum_t U(1 + \sum_i x_i^* r_{it})$, closer to the optimal one are therefore not feasible. The approximative portfolios for utility functions defined on $w > 0$ often lie closer to the optimal ones. It should be noticed that it is the positive skewnesses of the optimal portfolios that make them preferable to the approximative ones. The aversion to kurtosis is not as clear. Sometimes the optimal portfolios have higher kurtosis than the approximative ones, sometimes not.

The result from the utility maximizations are presented in Table 6 together with the performance indexes, C and I. In the earlier case of non-negative portfolio weights, both measures

Table 6: Performance indexes, C and I, in the case of limited short selling. The expected utilities of the optimal and the approximative portfolios are denoted EU and E^*U.

$U(w)$	EU	E^*U	C	I
1. $\ln(w)$	0.319088	0.314200	0.982152	0.984112
2. $\ln(w-0.5)$	-0.177024	-0.196092	0.953154	0.961611
3. $-e^{-2w}$	-0.074045	-0.074712	0.996557	0.988631
4. $-e^{-5w}$	-0.001929	-0.001997	0.994428	0.985302
5. $(w-0.8)^{0.1}$	0.933913	0.930595	0.942088	0.959526
6. $(w-0.5)^{0.2}$	0.969109	0.965186	0.951630	0.958677
7. $\ln(0.7+\ln(w+0.5))$	0.275958	0.273954	0.986225	0.988144
8. $\ln(1.5+\ln(w-0.3))$	0.415316	0.407329	0.961395	0.970493
9. $-0.5e^{-3w}-e^{-5w}$	-0.012842	-0.013104	0.995133	0.985440
10. $-2.5e^{-3w}-3e^{-1.2w}$	-0.660605	-0.665734	0.995602	0.985442

were very close to one, indicating a negligible welfare loss from using the approximative solutions. This is no longer entirely true. The decrease in certain income now varies between 0.3% and 5.8% for investors with utility functions 3 and 5, respectively. It seems that the differences in portfolio composition do not affect the expected utility to such an extent as would be expected. As before, this is confirmed by the I-measure.

5.3 Portfolio Selection with Options Included

As shown above, it is mainly the preferences for skewness that explain the differences in composition of the approximative and optimal portfolios. Investigating how the inclusion of security returns with very high levels of skewness (and kurtosis) influences the utility maximizations might therefore be of interest. Highly realistic examples of such securities in portfolio selection are options. I do not have access to real world option prices in this period, but to a stock index, *Affärsvärldens Generalindex*[3], from which synthetical options can be constructed. This is done in the following way: We calculate the Black-Scholes prices of an at-the-money call and put option with one month to maturity (c_1 and p_1), using a yearly constant interest rate of 6% and the historical 30 day standard deviation of the index as a volatility estimate. After one month, when the options expire, we compute the returns and calculate new prices. The same roll-over strategy is also performed for a call and a put option with a time to maturity of three months (c_3 and p_3). Thus, we have 83 monthly returns of four synthetic options with the stock index as the underlying asset.

[3] Affärsvärldens Generalindex is a value weighted index of all stocks traded on the Stockholm Stock Exchange. It is not a value weighted index of the 119 stocks used here, since we have only used those stocks that exist at both the beginning and the end of the period.

Table 7: The means, variances, skewnesses, and kurtoses of the four synthetical options. The notion c_m is used for a call option with m months to maturity, and equivalently for a put option.

	$E(\%)$	$V(\%)$	S	K
c_1	75.05	425.5	1.326	4.101
p_1	53.43	1576	5.442	39.65
c_3	8.735	100.2	0.970	3.543
p_3	24.42	960.4	5.110	32.77

Table 8: The means, variances, skewnesses, and kurtoses of the optimal and approximative portfolios when options are included, together with the maximum of the absolute distances between the corresponding weight vectors.

	Approximative				Optimal				
$U(w)$	$E(\%)$	$V(\%)$	S	K	$E(\%)$	$V(\%)$	S	K	Distance
1. $\ln(w)$	11.67	3.77	0.15	6.26	11.84	4.10	0.27	5.79	0.16
2. $\ln(w-0.5)$	10.30	2.39	1.10	5.23	10.37	2.49	1.16	5.30	0.11
3. $-e^{-2w}$	10.61	2.65	0.89	5.26	10.64	2.68	0.89	5.26	0.03
4. $-e^{-5w}$	9.27	1.74	1.58	5.85	9.21	1.75	1.91	7.81	0.05
5. $(w-0.8)^{0.1}$	9.33	1.77	1.57	5.81	9.57	1.96	1.68	6.19	0.11
6. $(w-0.5)^{0.2}$	10.48	2.54	0.98	5.23	10.66	2.78	1.03	5.28	0.13
7. $\ln(0.7+\ln(w+0.5))$	11.27	3.29	0.42	5.74	11.38	3.49	0.46	5.55	0.16
8. $\ln(1.5+\ln(w-0.3))$	10.02	2.18	1.27	5.33	10.08	2.24	1.32	5.45	0.07
9. $-0.5e^{-3w}-e^{-5w}$	9.70	1.98	1.43	5.54	9.64	1.96	1.63	6.45	0.03
10. $-2.5e^{-3w}-3e^{-1.2w}$	10.95	2.97	0.64	5.45	10.99	3.02	0.68	5.41	0.06

The means, variances, skewnesses, and kurtoses of the option returns are presented in Table 7. As shown, all sample statistics are large. The mean returns of the put options are positive, although the market rose heavily in this period. The reason is that the stock market experienced some considerable declines, for example the October crash of '87, which compensated the holders of puts. This is also seen in the highly asymmetric distributions of the put option returns.

We calculate 1000 mean-variance efficient portfolios by solving (2) from 2% to 15% in steps of 0.013%, and choose the ten portfolios to compare to the optimal ones, given by solving (1). Again, we restrict the stock weights to be non-negative, while we impose the restrictions that the option weights are greater than -0.1, and sum to zero. The rationale behind this would be an investor who has fully invested in stocks, but who is allowed to take suitable positions in the option market, as long as these option positions are self financed and not too large.

In Table 8, I display the first four moments of the different solution portfolios. The options are always included, while only 4 (utility function 1) to 12 (utility function 4) of the stocks have weights $> 0.1\%$. The expectations and the variances of the portfolio returns are much

Table 9: Performance indexes, C and I, when options are included. The expected utilities of the optimal and the approximative portfolios are denoted EU and E^*U.

$U(w)$	EU	E^*U	C	I
1. $\ln(w)$	0.094035	0.093450	0.993476	0.992909
2. $\ln(w-0.5)$	-0.536717	-0.537177	0.996826	0.996644
3. $-e^{-2w}$	-0.114872	-0.114878	0.999975	0.999651
4. $-e^{-5w}$	-0.004958	-0.004973	0.999445	0.990987
5. $(w-0.8)^{0.1}$	0.877935	0.877580	0.984754	0.986341
6. $(w-0.5)^{0.2}$	0.899520	0.899360	0.994141	0.993719
7. $\ln(0.7+\ln(w+0.5))$	0.153378	0.153249	0.997437	0.997231
8. $\ln(1.5+\ln(w-0.3))$	0.223305	0.223007	0.996333	0.996221
9. $-0.5e^{-3w}-e^{-5w}$	-0.024993	-0.025022	0.999676	0.995018
10. $-2.5e^{-3w}-3e^{-1.2w}$	-0.910028	-0.910166	0.999900	0.998662

higher than in the former case with non-negative stock weights only. The mean returns of the approximative and optimal portfolios are quite close, and it seems as if, to some extent, the investors are willing to trade some increase in variance for higher skewness. The skewnesses of the optimal portfolios are all larger than those of the approximative ones, but the differences are not astounding. Again, the aversion of kurtosis is not as obvious. Furthermore, it can be seen that the maximum absolute distances of the portfolio weights range from 3% to 16% at the most. This is clearly higher than when the security set only contained stocks, but it is much lower than in the case of limited short selling of stocks.

Table 9 shows the performance indexes, C and I, which are very close to unity[4]. The decrease in certain income is limited to 1.5% at most for an investor characterized by utility function 5. These results are supported by the less intuitive I-measure.

6 Summary and Conclusions

The conclusion, with the utility functions and the return data used here, is that an investor wishing to maximize his expected utility, with small welfare losses, can choose a mean-variance efficient portfolio instead of the portfolio actually solving his maximization problem. In the case of non-negative portfolio weights, the decrease in utility is extremely small, much smaller than the precision of the raw data. This result has also been well documented in portfolio selections involving a small number of assets.

The main contribution is to show the validity of the mean-variance model in allocation

[4]The naive portfolio consists of $1/n$th of each stock, and no options. The expected utilities of this portfolio are therefore the same as in Table 4.

problems comparable to the size examined here. My solution portfolios involve a few assets only, as was the case in the earlier investigations where the security set consisted of 10-20 assets only. Choosing five securities out of 120 is, however, quite a different maximization problem than choosing five securities out of 20. Furthermore, I find that the validity of the mean-variance model is not due to any normality properties of the return data.

When allowing the portfolio weights to vary between -0.5 and 1.5, the returns of the solution portfolios are considerably more variable. Furthermore, the portfolio compositions of the approximative and exact portfolios differ to a much greater extent than in the former case. The welfare losses, measured as the decreases in the cash equivalents, are limited to 6% at most, however.

I further challenge the mean-variance approximation by incorporating synthetic options, with high levels of skewness and kurtosis in the security set. This increases the portfolio variabilities and the differences in asset allocations, but the decrease in certain income from choosing the best mean-variance efficient portfolio instead of the optimal one is less than 1.5%.

My personal opinion is that these welfare losses are of less importance, especially considering the fact that the hypothetical investors have been assumed to know their exact utility functions, which is hardly likely.

If the utility function is known, however, or an investor wishes to monitor his portfolio with different kinds of utility functions, the computational burden has been shown to be quite reasonable, at least when choosing among some hundred securities.

It should be pointed out that these results apply to a single-period setting, or when investors act myopically within a multiperiod framework. It remains to be settled if the excellency of the mean-variance approximation holds in a general multiperiod setting, when returns might exhibit serial dependences. This issue is left to future research.

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